Controllability of three-dimensional Navier–Stokes equations and applications

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Abstract

We formulate two results on controllability properties of the 3D Navier–Stokes (NS) system. They concern the approximate controllability and exact controllability in finite-dimensional projections of the problem in question. As a consequence, we obtain the existence of a strong solution of the Cauchy problem for the 3D NS system with an arbitrary initial function and a large class of right-hand sides. We also discuss some qualitative properties of admissible weak solutions for randomly forced NS equations.

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1 Main results

Let $D \subset \mathbb{R}^3$ be a bounded domain with C^2-smooth boundary ∂D. Consider 3D Navier–Stokes (NS) equations

$$\dot{u} + (u, \nabla)u - \nu\Delta u + \nabla p = f(t,x), \quad \operatorname{div} u = 0, \quad x \in D, \tag{1}$$

where $u = (u_1, u_2, u_3)$ and p are unknown velocity and pressure fields, $\nu > 0$ is the viscosity, and $f(t,x)$ is an external force. We introduce the spaces

$$H = \{u \in L^2(D, \mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } D, \langle u, \boldsymbol{n}\rangle|_{\partial D} = 0\},$$
$$V = H_0^1(D, \mathbb{R}^3) \cap H, \quad U = H^2(D, \mathbb{R}^3) \cap V,$$

where $\boldsymbol{n}$ stands for the outward unit normal to ∂D and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^3$. It is well known (e.g., see [Tem79]) that H is a closed vector space in $L^2(D,\mathbb{R}^3)$, and we denote by Π the orthogonal projection in $L^2(D,\mathbb{R}^3)$ onto H. Equations (1) are equivalent to the following evolution equation in H:

$$\dot{u} + \nu L u + B(u) = f. \tag{2}$$

Here $L = -\Pi\Delta$, $B(u) = B(u,u)$, $B(u,v) = \Pi\{(u,\nabla)v\}$, and we use the same notation for the right-hand side of (1) and its projection to H. Equation (2) is supplemented with the initial condition

$$u(0) = u_0, \tag{3}$$

where $u_0 \in V$. Let us assume that the right-hand side of (2) is represented in the form

$$f(t,x) = h(t,x) + \eta(t,x), \tag{4}$$

where $h \in L^2_{\text{loc}}(\mathbb{R}_+, H)$ is a given function and η is a control taking on values in a finite-dimensional subspace. To formulate the main results, we introduce some notation.

Define the space $\mathcal{X}_T = C(J_T, V) \cap L^2(J_T, U)$, where $J_T = [0,T]$. For any $T > 0$, $h \in L^2(J_T, H)$, and $u_0 \in V$, we denote by $\Theta_T(h, u_0)$ the set of functions $\eta \in L^2(J_T, H)$ for which problem (2) – (4) has a unique solution $u \in \mathcal{X}_T$. It follows from the implicit function theorem that

$$\mathcal{D}_T := \{(u_0, \eta) \in V \times L^2(J_T, H) : \eta \in \Theta_T(h, u_0)\} \tag{5}$$

is an open subset of $V \times L^2(J_T, H)$, and the operator $\mathcal{R}$ taking $(u_0, \eta) \in \mathcal{D}_T$ to the solution $u \in \mathcal{X}_T$ of (2) – (4) is locally Lipschitz continuous. We denote by $\mathcal{R}_t$ the restriction of $\mathcal{R}$ to the time $t \in J_T$. Let $E \subset U$ and $F \subset H$ be finite-dimensional subspaces, let $\mathsf{P}_F : H \to H$ be the orthogonal projection onto F, and let $X \subset L^2(J_T, E)$ be a vector space, not necessarily closed. We denote by $B_F(R)$ the closed ball in F of radius R centred at origin.

Definition 1. Equations (2), (4) with $\eta \in X$ are said to be *approximately controllable in time* T if for any $u_0, \hat{u} \in V$ and any $\varepsilon > 0$ there is a control $\eta \in \Theta_T(h, u_0) \cap X$ such that

$$\|\mathcal{R}_T(u_0, \eta) - \hat{u}\|_V < \varepsilon. \tag{6}$$

Equations (2), (4) with $\eta \in X$ are said to be F*-controllable in time* T if for any $u_0 \in V$ and $\hat{u} \in F$ there is $\eta \in \Theta_T(h, u_0) \cap X$ such that

$$\mathsf{P}_F \mathcal{R}_T(u_0, \eta) = \hat{u}. \tag{7}$$

Equations (2), (4) with $\eta \in X$ are said to be *solidly* F*-controllable in time* T if for any $u_0 \in V$ and any $R > 0$ there is a constant $\delta > 0$ and a compact set $\mathcal{C}$ in a finite-dimensional subspace $Y \subset X$ such that $\mathcal{C} \subset \Theta_T(h, u_0)$, and for any continuous mapping $\Phi : \mathcal{C} \to F$ satisfying the inequality

$$\sup_{\eta \in \mathcal{C}} \|\Phi(\eta) - \mathsf{P}_F \mathcal{R}_T(u_0, \eta)\|_F \le \delta, \tag{8}$$

we have $\Phi(\mathcal{C}) \supset B_F(R)$.

For any finite-dimensional subspace $G \subset U$, we denote by $\mathcal{F}(G)$ the largest vector space $G_1 \subset U$ such that any element $\eta_1 \in G_1$ is representable in the form

$$\eta_1 = \eta - \sum_{j=1}^{k} \lambda_j B(\zeta^j),$$

where $\eta, \zeta^1, \dots, \zeta^k \in G$ are some vectors and $\lambda_1, \dots, \lambda_k$ are non-negative constants. Since B is a quadratic operator continuous from U to V, we see that $\mathcal{F}(G) \subset U$ is a well-defined vector space of finite dimension. Also note that $\mathcal{F}(G) \supset G$.

We now define a sequence of subspaces $E_k \subset U$ by the rule

$$E_0 = E, \quad E_k = \mathcal{F}(E_{k-1}) \quad \text{for } k \geq 1, \quad E_\infty = \bigcup_{k=1}^{\infty} E_k. \tag{9}$$

The following theorem established in [Shi06a, Shi06b].

Theorem 2. *Let $E \subset U$ be a finite-dimensional subspace such that E_∞ is dense in H. Then the following assertions take place for any $T > 0$, $\nu > 0$, and $h \in L^2(J_T, H)$.*

- **(i)** *Equations* (2), (4) *with $\eta \in C^\infty(J_T, E)$ are approximately controllable in time T.*
- **(ii)** *Equations* (2), (4) *with $\eta \in C^\infty(J_T, E)$ are solidly F-controllable in time T for any finite-dimensional subspace $F \subset H$.*

In the general case, it is difficult to verify whether a subspace $E \subset U$ satisfies the conditions of Theorem 2. However, if D is a torus in $\mathbb{R}^3$, then one can obtain a sufficient condition under which E_∞ is dense in H.

2 Case of a torus

In this subsection, we study controlled Navier–Stokes equations with periodic boundary conditions. More precisely, let us fix a vector $q = (q_1, q_2, q_3)$ with positive components and set $\mathbb{T}_q^3 = \mathbb{R}^3 / 2\pi\mathbb{Z}_q^3$, where

$$\mathbb{Z}_q^3 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_i/q_i \in \mathbb{Z} \text{ for } i = 1, 2, 3\}.$$

Consider the Navier–Stokes system on $\mathbb{T}_q^3$. In other words, we consider Eqs. (1) with $D = \mathbb{R}^3$ and assume that all functions are periodic of period $2\pi q_i$ with respect to x_i, $i = 1, 2, 3$. To simplify notation, we shall assume, without loss of generality, that the mean values of u, h, and η with respect to $x \in \mathbb{T}_q^3$ are zero. As in the case of a bounded domain with Dirichlet boundary condition, one can reduce (1) to an evolution equation in an appropriate Hilbert space. Namely, we set

$$H = \left\{ u \in L^2(\mathbb{T}_q^3, \mathbb{R}^3) : \operatorname{div} u \equiv 0, \int_{\mathbb{T}_q^3} u(x)\, dx = 0 \right\}$$

and denote by $\Pi : L^2(\mathbb{T}_q^3, \mathbb{R}^3) \to H$ the orthogonal projection in $L^2(\mathbb{T}_q^3, \mathbb{R}^3)$ onto the closed subspace H. Define the spaces

$$V = H^1(\mathbb{T}_q^3, \mathbb{R}^3) \cap H, \quad U = H^2(\mathbb{T}_q^3, \mathbb{R}^3) \cap H.$$

Projecting (1) to the space H, we obtain Eq. (2) in which $L = -\Delta$ is the Stokes operator with the domain $D(L) = U$ and $B(u) = \Pi\{(u, \nabla)u\}$. Theorem 2, which was formulated for the Dirichlet boundary condition, remains valid in this case as well. Our aim is to describe explicitly a finite-dimensional subspace $E \subset U$ for which the hypothesis of Theorem 2 is fulfilled.

To this end, we first construct an orthogonal basis in H formed of the eigenfunctions of L. For $x, y \in \mathbb{R}^3$, let

$$\langle x, y \rangle_q = \sum_{i=1}^{3} q_i^{-1} x_i y_i, \quad \langle x, y \rangle = \sum_{i=1}^{3} x_i y_i, \quad |x| = \sum_{i=1}^{3} |x_i|.$$

We set $\mathbb{Z}_*^3 = \mathbb{Z}^3 \setminus \{0\}$ and $\mathbb{R}_*^3 = \mathbb{R}^3 \setminus \{0\}$. For $a \in \mathbb{R}_*^3$, denote by $a^\perp$ the two-dimensional subspace in $\mathbb{R}^3$ defined by the equation $\langle x, a \rangle_q = 0$. Note that $a^\perp = (-a)^\perp$. For any $m \in \mathbb{Z}_*^3$, let us choose a vector $\ell(m) \in m^\perp$ so that $\{\ell(m), \ell(-m)\}$ is an orthonormal basis in $m^\perp$ with respect to the scalar product $\langle \cdot, \cdot \rangle$. We now set

$$c_m(x) = \ell(m) \cos\langle m, x \rangle_q, \quad s_m(x) = \ell(m) \sin\langle m, x \rangle_q \quad \text{for } m \in \mathbb{Z}_*^3.$$

It is a matter of direct verification to show that c_m and s_m are eigenfunctions of L and that $\{c_m, s_m, m \in \mathbb{Z}_*^3\}$ is an orthogonal basis in H. For a finite family of functions $\mathcal{A}$, we denote by $\operatorname{span} \mathcal{A}$ the vector space spanned by $\mathcal{A}$.

Theorem 3. *For any vector $q = (q_1, q_2, q_3)$ with positive components there is an integer $d \geq 4$ such that if*

$$E = \operatorname{span}\{c_m, s_m, |m| \leq d\},$$

then the vector space E_∞ defined in (9) is dense in H.

Theorems 2 and 3 imply the following result on controllability of the NS system by a force of finite dimension.

Corollary 4. *Let $E \subset U$ be the subspace defined in Theorem 3. Then for any finite-dimensional subspace $F \subset H$ and arbitrary constants $T > 0$ and $\nu > 0$ the Navier–Stokes equations (2), (4) with $\eta \in C^\infty(J_T, E)$ are approximately controllable and solidly F-controllable in time T.*

The proofs of the above results are based on a development of a general approach introduced by Agrachev and Sarychev in the case of 2D Navier–Stokes equations (see [AS05, AS06]).

3 Applications

Our first application concerns the Cauchy problem for (2). Let $G \subset H$ be a closed vector space. For any $u_0 \in V$, $T > 0$, and $\nu > 0$, let $\Xi_{T,\nu}(G, u_0)$ be the set of functions $f \in L^2(J_T, G)$ for which problem (2), (3) has a unique solution $u \in \mathcal{X}_T$. If $E \subset G$ is a closed subspace, then we denote by $G \ominus E$ the orthogonal complement of E in G and by $Q(T, G, E)$ the orthogonal projection in $L^2(J_T, G)$ onto the subspace $L^2(J_T, G \ominus E)$. The following result is established in [Shi06a].

Theorem 5. *Let $E \subset U$ be a finite-dimensional subspace such that E_∞ is dense in H and let $G \subset H$ be a closed subspace containing E. Then $\Xi_{T,\nu}(G, u_0)$ is a non-empty open subset of $L^2(J_T, G)$ such that*

$$Q(T, G, E)\Xi_{T,\nu}(G, u_0) = L^2(J_T, G \ominus E) \quad \textit{for any } T > 0,\ \nu > 0,\ u_0 \in V.$$

Our second application concerns the case in which Navier–Stokes equations are perturbed by a random force. Namely, suppose that

$$f(t, x) = h(x) + \eta(t, x), \tag{10}$$

where $h \in H$ is a deterministic function and η is an H-valued random process satisfying the following condition.

(C) There is an orthonormal basis $\{f_k\}$ in V and a sequence of standard independent Brownian motions $\{\beta_j(t), t \geq 0\}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ such that

$$\eta(t) = \frac{\partial}{\partial t}\zeta(t), \quad \zeta(t) = \sum_{j,k=1}^{\infty} b_{jk}\beta_j(t) f_k,$$

where $\{b_{jk}\}$ is a family of real constants satisfying the condition

$$B := \sum_{j,k=1}^{\infty} b_{jk}^2 < \infty.$$

Let us recall the concepts of an admissible weak solution and of a stationary measure for (2), (10). Define an Ornstein–Uhlenbeck process by the formula

$$z(t) = \int_0^t e^{-\nu(t-s)L} d\zeta(t).$$

It is well known that if Condition (C) is fulfilled, then z is a Gaussian process whose almost every trajectory belongs to the space $C(\mathbb{R}_+, V) \cap L^2_{\text{loc}}(\mathbb{R}_+, U)$ and satisfies the Stokes equation

$$\dot{u} + \nu L u = \eta(t).$$

Definition 6. An H-valued random process $u(t)$ is called an *admissible weak solution* for (2), (10) if it is representable in the form

$$u(t) = v(t) + z(t),$$

where $v(t)$ is an H-valued $\mathcal{F}_t$-adapted random process whose almost every trajectory belongs to the space $L^2_{\text{loc}}(\mathbb{R}_+, V) \cap L^\infty_{\text{loc}}(\mathbb{R}_+, H)$ and satisfies the equation

$$\dot{v} + \nu L v + B(v + z) = h$$

in the sense of distributions and the energy inequality

$$\frac{1}{2}\|v(t)\|^2 + \nu \int_0^t \|v(s)\|_V^2 ds + \int_0^t (B(v+z, z), v)\, ds \\ \leq \frac{1}{2}\|v(0)\|^2 + \int_0^t (h, v)\, ds, \quad t \geq 0,$$

where $(\cdot, \cdot)$ denotes the scalar product in H.

Definition 7. An admissible weak solution $u(t)$ for (2), (10) is said to be *stationary* if its distribution does not depend on t:

$$\mathcal{D}(u(t)) = \mu \quad \text{for all } t \geq 0.$$

In this case, μ is called a *stationary measure* for (2), (10).

Existence of admissible weak stationary solutions for 3D Navier–Stokes equations was established in [VF88, FG95]. Moreover, the construction of these works implies that

$$\int_H \|v\|_V^2 \mu(dv) < \infty. \tag{11}$$

Let us denote by Q the vector space of functions $v \in V$ that are representable in the form

$$v = \sum_{j,k=1}^{\infty} b_{jk} u_j f_k,$$

where $\{u_j\}$ is a sequence of real numbers such that $\sum_j u_j^2 < \infty$. Recall that the vector space E_∞ is defined in (9). For a finite-dimensional space F, denote by ℓ_F the Lebesgue measure on F. The following theorem established in [Shi06c] provides some qualitative properties of stationary measures for (2), (10) (see also [AKSS06]).

Theorem 8. *Let η be a stationary process satisfying Condition* (C)*, let $E \subset U$ be a finite-dimensional vector space for which E_∞ is dense in H, and let μ be a stationary measure for* (2), (10) *such that* (11) *holds. Suppose that $Q \supset E$. Then the following assertions take place.*

- **(i)** *The support of μ coincides with H.*
- **(ii)** *Let $F \subset H$ be a finite-dimensional subspace and let μ_F be the projection of μ to F. Then there is a function $\rho_F \in C(F)$ such that $\mu_F \geq \rho_F \ell_F$ and $\rho_F(x) > 0$ for ℓ_F-almost every $x \in F$.*

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