On random attractors for mixing-type systems

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Abstract

The paper deals with infinite-dimensional random dynamical systems. Under the condition that a system in question is of mixing type and possesses a random compact attracting set, we show that the support of the unique invariant measure is a minimal random point attractor. The results obtained apply to the randomly forced 2D Navier–Stokes system.

0 Introduction

This paper deals with random dynamical systems (RDS) on a Polish[1] space H,

$$\varphi_k : H \to H, \quad H \ni u \mapsto \varphi_k u, \quad k \geq 0. \tag{0.1}$$

Here φ_k's are random transformations (so $\varphi_k = \varphi_k(\omega)$, where ω is a random parameter). As functions of k, the transformations are assumed to have independent increments. Usually the time will be discrete (i.e., $k \in \mathbb{Z}_+$); however, RDS with continuous time will also be briefly discussed in the context of stochastic PDE's.

Many features of long-time behaviour of trajectories for (0.1) are described by random attractors for this RDS. Among many possible definitions of random attractors (e.g., see [Cra91, CDF97, Arn98]), we choose the following: *a compact random set $\mathcal{A}_\omega$ is called a random attractor if all trajectories $\varphi_k(\omega)u$ of* (0.1) *converge to $\mathcal{A}_\omega$ in probability.* See Subsection 1.2 for a precise definition and its discussion.

The RDS (0.1) defines a Markov chain in H with the transition function

$$P_k(u, \Gamma) = \mathbb{P}\{\omega : \varphi_k(\omega)u \in \Gamma\}, \tag{0.2}$$

where Γ is a Borel subset in H. Long-time behaviour of this process is described, up to some extent, by its stationary measures. Recall that a probability Borel

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[1] A metric space is called *Polish* if it is complete and separable.

measure μ on H is said to be *stationary* if

$$\mu(\Gamma) = \int_H P_k(u, \Gamma)\mu(du)$$

for every $k \geq 0$ and every Borel set Γ. For systems in questions, every stationary measure μ admits a Markov disintegration:

$$\mu(\Gamma) = \mathbb{E}\, \mu_\omega(\Gamma).$$

Here $\omega \mapsto \mu_\omega$ is a measure-valued map measurable with respect to the past, i.e., the σ-algebra generated by the random transformations $\varphi_k(\theta_{-m}\omega)$, where $k \geq m \geq 0$ and θ_n is the corresponding measure-preserving shift in the probability space; see [Cra91, Arn98] and Subsection 1.1. It is known that

$$\operatorname{supp} \mu_\omega \subset \mathcal{A}_\omega \quad \text{a.s.,} \tag{0.3}$$

where $\mathcal{A}_\omega$ is an arbitrary random attractor; see [Cra01] and Subsection 1.2.

The main result of this paper is Theorem 2.4 which states that the support of disintegration of the unique stationary measure for the discrete time RDS (0.1) is its minimal random attractor if the system satisfies a non-restrictive compactness condition and is of mixing type in the sense that, for any bounded continuous function $f\colon H \to \mathbb{R}$,

$$\mathbb{E}\, f(\varphi_k(\omega)u) \to \int_H f(u)\mu(du) \quad \text{as} \quad k \to +\infty, \tag{0.4}$$

where $u \in H$ is an arbitrary initial point and μ is a stationary measure. In other words, under the above conditions we have the equality in (0.3), where $\mathcal{A}_\omega$ is the minimal random attractor. The proof is based on an ergodic-type theorem for the dynamical system $\{\Theta_k\}$ defined on the phase space $\Omega \times H$ as the skew-product of θ_k and φ_k:

$$\Theta_k(\omega, u) = (\theta_k\omega, \varphi_k(\omega)u), \quad k \geq 0,$$

see Theorem 2.3.

In Section 4 we consider the randomly forced 2D Navier–Stokes equations

$$\dot{u} - \nu\Delta u + (u, \nabla)u + \nabla p = \eta(t, x), \quad \operatorname{div} u = 0, \tag{0.5}$$

where $u = u(t, x)$ is the velocity field, $p = p(t, x)$ is the pressure, and $\eta(t, x)$ is a random external force. The equations are supplemented by the Dirichlet or the periodic boundary conditions. The random force η is smooth in x, while as a function of t it is either a kick-force (then (0.5) defines a discrete-time RDS), or a white-force (then it defines a continuous-time RDS). In both cases the RDS satisfies the compactness condition. Assuming certain non-restrictive non-degeneracy assumption and evoking the results from [KS01] or [KS02] respectively, we get that the RDS satisfies the mixing-type condition as well. Therefore the abstract Theorems 2.3 and 2.4 apply to system (0.5), both for the kick- and white-forces. Accordingly, the support of the Markov disintegration for the unique stationary measure defines a minimal random attractor for (0.5) (see Theorems 4.1 and 4.2), and functionals, depending both on solutions u and the corresponding forces, satisfy a theorem of ergodic type (see Theorem 4.3).

Notation. Let (H, d) be a Polish space, let $C_b(H)$ be the space of bounded continuous functions on H endowed with the norm $\sup_{u\in H}|f(u)|$, and let $L(H)$ be the space of functions $f: H \to \mathbb{R}$ such that

$$\|f\|_{L(H)} = \sup_{u\in H}|f(u)| + \sup_{u,v\in H}\frac{|f(u)-f(v)|}{|u-v|} < \infty.$$

If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathcal{F}'$ is a sub-σ-algebra, then we denote by $\mathbb{L}(H, \mathcal{F}')$ the set of functions $F(\omega, u): \Omega \times H \to \mathbb{R}$ that are $\mathcal{F}'$-measurable in ω for any fixed $u \in H$ and satisfy the condition

$$\operatorname*{ess\,sup}_{\omega\in\Omega}\|F(\omega,\cdot)\|_{L(H)} < \infty. \tag{0.6}$$

For $u \in H$ and $A \subset H$ we define the distance between u and A as

$$d(u, A) = \inf_{v\in A} d(u, v).$$

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1 Preliminaries

In this section, we recall some basic notions of the theory of random dynamical systems (RDS) and formulate a few results that will be used later. We mainly follow the book [Arn98]. To simplify the presentation, we confine ourselves to the case of discrete time.

1.1 Random dynamical systems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\theta_k : \Omega \to \Omega$, $k \in \mathbb{Z}$, be a group of measure preserving transformations of Ω, and let H be a Polish space endowed with a metric d and the Borel σ-algebra $\mathcal{B}_H$.

Definition 1.1. A *(continuous) random dynamical system over* θ_k is defined as a family of mappings $\varphi_k(\omega): H \to H$, where $k \in \mathbb{Z}_+$ and $\omega \in \Omega$, that possesses the following properties:

(i) *Measurability.* The mapping $(\omega, u) \mapsto \varphi_k(\omega)u$ from the space $\Omega \times H$ endowed with the σ-algebra $\mathcal{F} \otimes \mathcal{B}_H$ to H is measurable for any $k \geq 0$;

(ii) *Continuity.* For any $\omega \in \Omega$ and $k \geq 0$, the mapping $\varphi_k(\omega)$ is continuous;

(iii) *Cocycle property.* For any $\omega \in \Omega$, we have

$$\varphi_0(\omega) = \mathrm{Id}_H, \quad \varphi_{k+l}(\omega) = \varphi_k(\theta_l\omega) \circ \varphi_l(\omega) \quad \text{for all} \quad k, l \geq 0. \tag{1.1}$$

For any integers $m \leq n$, we denote by $\mathcal{F}_{[m,n]}$ the σ-algebra generated by the family of H-valued random variables $\varphi_k(\theta_{m-1}\omega)u$, where $u \in H$ and $k = 0, \dots, n-m+1$. We extend this notation to the case when $m = -\infty$ and/or $n = +\infty$ by setting $\mathcal{F}_{[-\infty,n]} = \sigma\{\mathcal{F}_{[m,n]}, m \leq n\}$ and similarly for $\mathcal{F}_{[m,+\infty]}$ and $\mathcal{F}_{[-\infty,+\infty]}$. It is a matter of direct verification to show that

$$\theta_k^{-1}\mathcal{F}_{[m,n]} = \mathcal{F}_{[m+k,n+k]} \quad \text{for all } m \leq n \text{ and } k. \tag{1.2}$$

The σ-algebras $\mathcal{F}^- = \mathcal{F}_{[-\infty,0]}$ and $\mathcal{F}^+ = \mathcal{F}_{[1,+\infty]}$ are called the *past* and the *future* of $\varphi_k(\omega)$.

Let $\mathcal{P}_\mathbb{P}$ be the set of probability measures on $(\Omega \times H, \mathcal{F} \otimes \mathcal{B}_H)$ whose projections to Ω coincide with $\mathbb{P}$. It is well known (see [Arn98, Section 1.4]) that any measure $\mathfrak{M} \in \mathcal{P}_\mathbb{P}$ admits a unique *disintegration* $\omega \mapsto \mu_\omega$, which is a random variable valued in the space of measures such that

$$\mathfrak{M}(\Gamma) = \int_\Omega \int_H I_\Gamma(\omega, u)\mu_\omega(du)\mathbb{P}(d\omega) \quad \forall\, \Gamma \in \mathcal{F} \otimes \mathcal{B}_H,$$

where I_Γ is the indicator function of Γ.

Given an RDS $\varphi_k(\omega)$ over θ_k, we introduce the following semigroup of measurable mappings on $\Omega \times H$:

$$\Theta_k(\omega, u) = \big(\theta_k\omega, \varphi_k(\omega)u\big), \quad k \geq 0.$$

The semigroup Θ_k is called the *skew-product* of θ_k and $\varphi_k(\omega)$. A measure $\mathfrak{M} \in \mathcal{P}_\mathbb{P}$ is said to be *invariant for* Θ_k if $\Theta_k(\mathfrak{M}) = \mathfrak{M}$ (that is, $\mathfrak{M}(\Theta_k^{-1}(\Gamma)) = \mathfrak{M}(\Gamma)$ for any $\Gamma \in \mathcal{F} \otimes \mathcal{B}_H$). By Theorem 1.4.5 in [Arn98], a measure $\mathfrak{M} \in \mathcal{P}_\mathbb{P}$ is invariant if and only if its disintegration μ_ω satisfies the relation below for $\mathbb{P}$-a.a. $\omega \in \Omega$:

$$\varphi_k(\omega)\mu_\omega = \mu_{\theta_k\omega} \quad \text{for all} \quad k \geq 0.$$

The set of all invariant measures for Θ_k will be denoted by $\mathcal{I}_\mathbb{P}(\varphi)$.

Definition 1.2. An invariant measure $\mathfrak{M} \in \mathcal{I}_\mathbb{P}(\varphi)$ is said to be *Markov* if its disintegration μ_ω is measurable with respect to the past $\mathcal{F}^-$. The set of such measures will be denoted by $\mathcal{I}_{\mathbb{P},\mathcal{F}^-}(\varphi)$.

We now turn to the important class of *RDS with independent increments* (also called *white noise RDS*).

Definition 1.3. We shall say that an RDS $\varphi_k(\omega)$ has independent increments if its past and future are independent.

It follows from (1.2) that $\varphi_k(\omega)$ has independent increments if and only if the σ-algebras $\mathcal{F}_{[m,n]}$ and $\mathcal{F}_{[m',n']}$ are independent for any non-intersecting (finite or infinite) intervals $[m, n]$ and $[m', n']$.

For any RDS $\varphi_k(\omega)$ with independent increments, the set of random sequences $\{\varphi_k(\cdot)u, k \geq 0\}$, $u \in H$, is a family of Markov chains with respect to the filtration $\mathcal{F}_k = \theta_k^{-1}\mathcal{F}^-$. The corresponding transition function $P_k(u,\Gamma)$ has the form (0.2) and the Markov operators associated with P_k are given by the formulas

$$\mathfrak{P}_k f(u) = \int_H P_k(u,dv)f(v), \quad \mathfrak{P}_k^*\mu(\Gamma) = \int_H P_k(u,\Gamma)\mu(du),$$

where $f \in C_b(H)$ and $\mu \in \mathcal{P}(H)$. We recall that $\mu \in \mathcal{P}(H)$ is called a *stationary measure* for the Markov family if $\mathfrak{P}_1^*\mu = \mu$. The set of such measures will be denoted by $\mathcal{S}_\varphi$. The following important result is established (for different situations) in [Led86, Le 87, Cra91].

Proposition 1.4. *Let $\varphi_k(\omega)$ be an RDS with independent increments. Then there is a one-to-one correspondence between Markov invariant measures $\mathcal{I}_{\mathbb{P},\mathcal{F}^-}(\varphi)$ for the skew-product Θ_k and the stationary measures $\mathcal{S}_\varphi$ for the associated Markov family. Namely, if $\mu \in \mathcal{S}_\varphi$, then the limit*

$$\mu_\omega = \lim_{k\to+\infty} \varphi_k(\theta_{-k}\omega)\mu \tag{1.3}$$

exists in the weak topology almost surely and gives the disintegration of a Markov invariant measure $\mathfrak{M}$. Conversely, if $\mathfrak{M} \in \mathcal{I}_{\mathbb{P},\mathcal{F}^-}$ is a Markov invariant measure and μ_ω is its disintegration, then $\mu = \mathbb{E}\,\mu_\omega$ is a stationary measure for the Markov family.*

1.2 Point attractors

Let $\{\varphi_k(\omega)\}$ be an RDS in a Polish space H over $\{\theta_k\}$ as above. A family of subsets $\mathcal{A}_\omega$, $\omega \in \Omega$, is called a *random compact* (*closed*) *set* if $\mathcal{A}_\omega$ is compact (closed) for a.a. ω and $\Omega_U := \{\omega \in \Omega : \mathcal{A}_\omega \cap U \neq \varnothing\} \in \mathcal{F}$ for any open set $U \subset H$. A random compact set $\mathcal{A}_\omega$ is said to be measurable with respect to a sub-σ-algebra $\mathcal{F}' \subset \mathcal{F}$ if $\Omega_U \in \mathcal{F}'$ for any open set $U \subset H$.

Definition 1.5. A random compact set $\mathcal{A}_\omega$ is called a *random point attractor* (*in the sense of convergence in probability*) if for any $u \in H$ the sequence of random variables $d(\varphi_k(\omega)u, \mathcal{A}_{\theta_k\omega})$ converges to zero in probability, i.e., for any $\delta > 0$,

$$\lim_{k\to+\infty} \mathbb{P}\big\{d(\varphi_k(\omega)u, \mathcal{A}_{\theta_k\omega}) > \delta\big\} = 0. \tag{1.4}$$

A random point attractor $\mathcal{A}_\omega$ is said to be *minimal* if for any other random point attractor $\mathcal{A}'_\omega$ we have $\mathcal{A}_\omega \subset \mathcal{A}'_\omega$ for a.a. ω.

It is clear that a minimal random point attractor is unique (if it exists), i.e., if $\mathcal{A}_\omega$ and $\mathcal{A}'_\omega$ are two minimal random attractors, then $\mathcal{A}_\omega = \mathcal{A}'_\omega$ a.s. Since θ_k is a measure-preserving transformation, (1.4) is equivalent to

$$\lim_{k\to+\infty} d(\varphi_k(\theta_{-k}\omega)u, \mathcal{A}_\omega) = 0, \tag{1.5}$$

where the limit is understood in the sense of convergence in probability. This type of convergence of a trajectory to a random set—when the initial data are specified at time $-k$, $k \to \infty$, and the distance is evaluated at time zero—is normally used to define random attractors. We prefer the "forward" definition (1.4), which seems to be more natural. We note that among various types of random attractors, considered now in mathematical literature, that in Definition 1.5 is the smallest, cf. [Cra91, CDF97, Arn98].

If we replace relation (1.4) in Definition 1.5 by the condition that (1.5) holds for all $u \in H$ and $\omega \in \Omega_0$, where $\Omega_0 \in \mathcal{F}$ is a set of full measure not depending on u, then we obtain the definition of a *random point attractor in the sense of almost sure convergence*). Since the a.s. convergence implies the convergence in probability, the resulting attractor also satisfies (1.4). In what follows, we mainly deal with random point attractors in the sense of convergence in probability, therefore they will be simply called *random attractors*.

The following proposition is a straightforward consequence of Theorems 3.4 and 4.3 and Remark 3.5 (iii) in [Cra01].

Proposition 1.6. (i) *Let $\varphi_k(\omega)$ be an RDS with independent increments. Suppose that there is a random compact set $\mathcal{K}_\omega$ attracting trajectories of $\varphi_k(\omega)$, i.e., there is a set $\Omega_0 \in \mathcal{F}$ such that $\mathbb{P}(\Omega_0) = 1$ and*

$$\lim_{k\to+\infty} d\big(\varphi_k(\theta_{-k}\omega)u, \mathcal{K}_\omega\big) = 0 \quad \textit{for any} \quad \omega \in \Omega_0, \quad u \in H. \tag{1.6}$$

Then $\varphi_k(\omega)$ possesses a random attractor $\mathcal{A}_\omega$ that is measurable with respect to the past $\mathcal{F}^-$.

(ii) *For any Markov invariant measure $\mathfrak{M} \in \mathcal{I}_{\varphi,\mathcal{F}^-}$ its disintegration μ_ω is supported by each random attractor $\mathcal{A}'_\omega$, i.e., $\mu_\omega(\mathcal{A}'_\omega) = 1$ a.s.*

Outline of proof. (i) As shown in [Cra01], under the conditions of proposition, the RDS $\varphi_k(\omega)$ possesses a random point attractor $\mathcal{A}_\omega$ in the sense of almost sure convergence. Since θ_k is a measure-preserving transformation, we conclude that (1.5) holds for $\mathcal{A}_\omega$. The construction implies that $\mathcal{A}_\omega$ is measurable with respect to the past.

(ii) Let $\mathcal{A}'_\omega$ be an arbitrary random attractor. To show that the disintegration of any invariant measure $\mathfrak{M} \in \mathcal{I}_{\varphi,\mathcal{F}^-}$ is supported by $\mathcal{A}'_\omega$, it suffices to observe that (1.4) implies the a.s.-convergence (1.5) along an appropriate subsequence $k = k_n$, and to repeat the argument in [Cra01, Theorem 4.3]. ☐

2 Main results

As before, we denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and by H a Polish space endowed with a metric d and Borel σ-algebra $\mathcal{B}_H$. Let $\{\varphi_k(\omega)\}$ be an RDS in H over a measure-preserving group of transformations θ_k. We introduce the two hypotheses below.

Condition 2.1. *Mixing:* The Markov family $\{\varphi_k(\omega)u\}$ is a system of mixing type in the following sense: it has a unique stationary measure μ, and for any $f \in L(H)$ and any initial point $u \in H$ we have

$$\mathfrak{P}_k f(u) = \mathbb{E}\, f\bigl(\varphi_k(\cdot)u\bigr) \to (\mu, f) = \int_H f(u)\mu(du) \quad \text{as} \quad k \to \infty. \tag{2.1}$$

Condition 2.2. *Compactness:* There is a random compact set attracting trajectories of $\varphi_k(\omega)$ (in the sense specified in Proposition 1.6). Moreover, for any $u \in H$ and $\varepsilon > 0$ there is $\Omega_\varepsilon \in \mathcal{F}$, a compact set $K_\varepsilon \subset H$, and an integer $k_\varepsilon = k_\varepsilon(u) \geq 1$ such that $\mathbb{P}(\Omega_\varepsilon) \geq 1 - \varepsilon$ and

$$\varphi_k(\theta_{-k}\omega)u \in K_\varepsilon \quad \text{for} \quad \omega \in \Omega_\varepsilon, \quad k \geq k_\varepsilon. \tag{2.2}$$

Let $\mu \in \mathcal{S}_\varphi$ be the unique stationary measure for the Markov semigroup $\mathfrak{P}_k^*$ and let $\mathfrak{M} \in \mathcal{I}_{\varphi,\mathcal{F}^-}$ be the corresponding Markov invariant measure for the skew-product Θ_k (see Proposition 1.4).

Theorem 2.3. *Suppose that Condition 2.1 is satisfied. Then for any function $F(\omega, u) \in \mathbb{L}(H, \mathcal{F}^-)$ we have*

$$\mathbb{E}\, F\bigl(\Theta_k(\cdot, u)\bigr) \to (\mathfrak{M}, F) = \int_\Omega \int_H F(\omega, u)\mu_\omega(du)\mathbb{P}(d\omega) \quad \textit{as} \quad k \to \infty, \tag{2.3}$$

where $u \in H$ is an arbitrary initial point.

We now discuss relationship between invariant measures and random attractors. Let us denote by μ_ω the disintegration of $\mathfrak{M}$ and set

$$\mathcal{A}_\omega = \begin{cases} \operatorname{supp} \mu_\omega, & \omega \in \Omega_0, \\ H, & \omega \notin \Omega_0, \end{cases} \tag{2.4}$$

where $\Omega_0 \in \mathcal{F}$ is a set of full measure on which the limit (1.3) exists. By Corollary 1.6.5 in [Arn98], $\mathcal{A}_\omega$ is a random closed set. Moreover, it follows from (1.3) that $\mathcal{A}_\omega$ is measurable with respect to $\mathcal{F}^-$.

Theorem 2.4. *Suppose that Conditions 2.1 and 2.2 are satisfied. Then $\mathcal{A}_\omega$ is a minimal random point attractor.*

The proofs of the above theorems are given in Section 3. We now discuss a class of RDS that satisfies Conditions 2.1 and 2.2.

Example 2.5. *Randomly forced dynamical systems.* Let H be a Hilbert space with a norm $|\cdot|$ and an orthonormal basis $\{e_j\}$ and let P_N be the orthogonal projection onto the subspace $H_N \subset H$ generated by $e_1, \dots, e_N$. Suppose that a continuous operator $S\colon H \to H$ satisfies the following two conditions:

$$|S(u)| \leq q|u| \quad \text{for} \quad u \in H, \tag{2.5}$$

$$\bigl|\mathsf{P}_N(S(u) - S(v))\bigr| \leq \tfrac{1}{2}|u - v| \quad \text{for} \quad |u| \vee |v| \leq R, \tag{2.6}$$

where $q < 1$ is a constant not depending on u, $R > 0$ is an arbitrary constant, and $N \geq 1$ is an integer depending only on R. We consider the RDS generated by the equation

$$u_k = S(u_{k-1}) + \eta_k, \tag{2.7}$$

where $k \in \mathbb{Z}$ and η_k is a sequence of i.i.d. H-valued random variables. If the distribution χ of the random variables η_k has a compact support, then Condition 2.2 is satisfied. If, in addition, χ is sufficiently non-degenerate (in the sense of [KS01]), then the Markov family corresponding to (2.7) is of mixing type. Thus, under the above hypotheses, the support of the unique invariant measure is a random point attractor. We note that Equation (4.3) below, corresponding to the kick-forced NS system, satisfies (2.5) and (2.6); see [KS01].

Conditions 2.1 and 2.2 are also satisfied for a large class of unbounded kicks η_k (see (2.7)). We shall not dwell on that case.

Remark 2.6. Theorems 2.3 and 2.4 remain valid for RDS $\varphi_t(\omega)$ with continuous time $t \geq 0$. In this case, we assume that $\varphi_t(\omega)u$ is continuous with respect to (t, u) for any fixed $\omega \in \Omega$ and that Conditions 2.1 and 2.2 hold with k replaced by t. Reformulation of the above results for continuous time is rather obvious, and therefore we do not give detailed statements.

3 Proofs

3.1 Proof of Theorem 2.3

Step 1. We first assume that $F(\omega, u) \in \mathbb{L}(H, \mathcal{F}_{[-\ell,0]})$, where $\ell \geq 0$ is an integer. Since θ_k is a measure-preserving transformation, for any $m \geq 1$ we have

$$\begin{aligned} p_k(u) :=& \mathbb{E}\, F\big(\theta_k\omega, \varphi_k(\omega)u\big) = \mathbb{E}\, F\big(\omega, \varphi_k(\theta_{-k}\omega)u\big) \\ =& \mathbb{E}\,\mathbb{E}\big\{F\big(\omega, \varphi_k(\theta_{-k}\omega)u\big) \,\big|\, \mathcal{F}_{[1-m,0]}\big\}. \end{aligned}$$

By the cocycle property (see (1.1)),

$$\varphi_k(\theta_{-k}\omega) = \varphi_m(\theta_{-m}\omega)\varphi_{k-m}(\theta_{-k}\omega), \quad m \leq k.$$

Hence, setting $F_m(\omega, u) = F(\omega, \varphi_m(\theta_{-m}\omega)u)$, for any $m \leq k$ we derive

$$p_k(u) = \mathbb{E}\,\mathbb{E}\big\{F_m\big(\omega, \varphi_{k-m}(\theta_{-k}\omega)u\big) \,\big|\, \mathcal{F}_{[1-m,0]}\big\}. \tag{3.1}$$

We now note that $F_m \in \mathbb{L}(H, \mathcal{F}_{[1-m,0]})$ if $m \geq \ell+1$. Since $\varphi_{k-m}(\theta_{-k}\omega)u$ is measurable with respect to $\mathcal{F}_{[1-k,-m]}$ and since the σ-algebras $\mathcal{F}_{[1-m,0]}$ and $\mathcal{F}_{[1-k,-m]}$ are independent, it follows from (3.1) that

$$p_k(u) = \mathbb{E}\,\mathbb{E}'\big\{F_m\big(\omega, \varphi_{k-m}(\theta_{-k}\omega')u\big) = \mathbb{E}\,(\mathfrak{P}_{k-m}F_m)(\omega, u), \tag{3.2}$$

where $\ell+1 \leq m \leq k$ and $\mathbb{E}'$ denotes the expectation with respect to ω'. In view of Condition 2.1 and the Lebesgue theorem, for any $m \geq \ell + 1$, the right-hand side of (3.2) tends to $\mathbb{E}\,(\mu, F_m(\omega,\cdot))$ as $k\to+\infty$. Recalling the definition of F_m, we see that

$$\big(\mu, F_m(\omega,\cdot)\big)=\big(\varphi_m(\theta_{-m}\omega)\mu, F(\omega,\cdot)\big)\to\big(\mu_\omega, F(\omega,\cdot)\big)\quad\text{as}\quad m\to\infty,$$

where we used Proposition 1.4. What has been said implies that

$$\lim_{k\to+\infty} p_k(u)=\mathbb{E}\,\big(\mu_\omega, F(\omega,\cdot)\big),$$

which coincides with (2.3).

Step 2. We now show that (2.3) holds for functions of the form $F(\omega,u)=f(u)g(\omega)$, where $f\in L(H)$ and g is a bounded $\mathcal{F}^-$-measurable function. To this end, we use a version of the monotone class theorem (see [Rev84, Theorem 3.3]).

Let us fix $f\in L(H)$ and denote by $\mathcal{H}$ the set of those bounded $\mathcal{F}^-$-measurable functions g for which convergence (2.3) with $F=fg$ holds. It is clear that $\mathcal{H}$ is a linear space containing the constant functions. Moreover, as was shown in Step 1, it contains all bounded functions measurable with respect to $\mathcal{F}_{[-\ell,0]}$ for some $\ell\ge 0$. Since the union of $\mathcal{F}_{[-\ell,0]}$, $\ell\ge 0$, generates $\mathcal{F}^-$, the required assertion will be proved as soon as we establish the following property: if $g_n\in\mathcal{H}$ is an increasing sequence of non-negative functions such that $g=\sup g_n$ is bounded, then $g\in\mathcal{H}$.

Suppose that a sequence $\{g_n\}\subset\mathcal{H}$ satisfies the above conditions. Without loss of generality, we shall assume that $0\le g, g_n\le 1$. By Egorov's theorem, for any $\varepsilon>0$ there is $\Omega_\varepsilon\in\mathcal{F}$ such that $\mathbb{P}(\Omega_\varepsilon)\ge 1-\varepsilon$ and

$$\lim_{k\to+\infty}\sup_{\omega\in\Omega_\varepsilon}\big|g_n(\omega)-g(\omega)\big|=0.$$

It follows that for any $\varepsilon>0$ there is an integer $n_\varepsilon\ge 1$ such that $n_\varepsilon\to+\infty$ as $\varepsilon\to 0$ and

$$g_{n_\varepsilon}(\omega)\le g(\omega)\le g_{n_\varepsilon}(\omega)+\varepsilon+I_{\Omega_\varepsilon^c}(\omega)\quad\text{for all}\quad\omega\in\Omega.$$

Multiplying this inequality by $f(\varphi_k(\theta_{-k}\omega)u)$, taking the expectation, passing to the limit as $k\to+\infty$, and using the estimate $\mathbb{P}(\Omega_\varepsilon^c)\le\varepsilon$, we derive

$$\begin{aligned}\mathbb{E}\big\{(\mu_\omega,f)g_{n_\varepsilon}(\omega)\big\}&\le\liminf_{k\to+\infty}\mathbb{E}\big\{f(\varphi_k(\theta_{-k}\omega)u)g(\omega)\big\}\\&\le\limsup_{k\to+\infty}\mathbb{E}\big\{f(\varphi_k(\theta_{-k}\omega)u)g(\omega)\big\}\le\mathbb{E}\big\{(\mu_\omega,f)g_{n_\varepsilon}(\omega)\big\}+2\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary and $\mathbb{E}\big\{(\mu_\omega,f)g_{n_\varepsilon}(\omega)\big\}\to\mathbb{E}\big\{(\mu_\omega,f)g(\omega)\big\}$ as $\varepsilon\to 0$ (by the monotone convergence theorem), we conclude that

$$\mathbb{E}\big\{f(\varphi_k(\omega)u)g(\theta_k\omega)\big\}=\mathbb{E}\big\{f(\varphi_k(\theta_{-k}\omega)u)g(\omega)\big\}\overset{k\to+\infty}{\longrightarrow}\mathbb{E}\big\{(\mu_\omega,f)g(\omega)\big\},$$

which means that $g\in\mathcal{H}$. This completes the proof of (2.3) in the case when $F(\omega,u)=f(u)g(\omega)$.

Step 3. Now we consider the general case. Let $F \in \mathbb{L}(H)$ be an arbitrary function such that $\|F(\omega,\cdot)\|_{L(H)} \le 1$ for a.e. $\omega \in \Omega$. For any $u \in H$ and $\varepsilon > 0$, we choose an integer $k_\varepsilon(u) \ge 1$ and sets $\Omega_\varepsilon \in \mathcal{F}$ and $K_\varepsilon \Subset H$ for which (2.2) holds. By the Arzelà–Ascoli theorem, the unit ball $B_\varepsilon = \{f \in L(K_\varepsilon) : \|f\|_{L(H)} \le 1\}$ is compact in the space $C_b(K_\varepsilon)$, and therefore there is a finite set $\{h_j\} \subset B_\varepsilon$ whose ε-neighbourhood contains B_ε. It follows that B_ε can be covered by non-intersecting Borel sets $U_j \ni h_j$, $j = 1, \dots, N$, whose diameters do not exceed 2ε. Let us denote by $f_j \in L(H)$ arbitrary extensions of h_j to H such that $\|f_j\|_{L(H)} \le 2$. For instance, we can take

$$f_j(u) = \inf_{v \in K_\varepsilon} \bigl(h_j(v) + d(u,v) \wedge 1\bigr).$$

Let us consider the following approximation of F:

$$G_\varepsilon(\omega,u) = \sum_{j=1}^{N} f_j(u) g_j(\omega), \quad g_j(\omega) = I_{U_j}\bigl(F_{K_\varepsilon}(\omega,\cdot)\bigr),$$

where $F_{K_\varepsilon}(\omega,u)$ is the restriction of F to $\Omega \times K_\varepsilon$. Since only one of the functions g_j can be nonzero, we have $\|G_\varepsilon(\omega,\cdot)\|_\infty \le 2$. Therefore, for any $u \in H$ and a.e. $\omega \in \Omega$, we derive

$$\begin{aligned}\bigl|G_\varepsilon(\omega,u) - F(\omega,u)\bigr| &\le 2\varepsilon + I_{K_\varepsilon^c}(u)\bigl(\|G_\varepsilon(\omega,\cdot)\|_\infty + \|F(\omega,\cdot)\|_\infty\bigr)\\ &\le 2\varepsilon + 3I_{K_\varepsilon^c}(u),\end{aligned} \tag{3.3}$$

where we used the inequality $\|F(\omega,\cdot)\|_\infty \le 1$. Let us set

$$p_k(u) = \mathbb{E}\, F(\theta_k\omega, \varphi_k(\omega)u), \qquad p_k(u,\varepsilon) = \mathbb{E}\, G_\varepsilon(\theta_k\omega, \varphi_k(\omega)u).$$

It is clear that

$$\bigl|p_k(u) - (\mathfrak{M}, F)\bigr| \le \bigl|p_k(u) - p_k(u,\varepsilon)\bigr| + \bigl|p_k(u,\varepsilon) - (\mathfrak{M}, G_\varepsilon)\bigr| + \bigl|(\mathfrak{M}, G_\varepsilon - F)\bigr|. \tag{3.4}$$

Let us estimate each term on the right-hand side of (3.4). Combining (2.2) and (3.3), for $k \ge k_\varepsilon(u)$ we derive

$$\begin{aligned}\bigl|p_k(u) - p_k(u,\varepsilon)\bigr| &\le \bigl|\mathbb{E}\bigl\{F(\omega, \varphi_k(\theta_{-k}\omega)u) - G_\varepsilon(\omega, \varphi_k(\theta_{-k}\omega)u)\bigr\}\bigr|\\ &\le 2\varepsilon + 3\mathbb{P}\bigl\{\varphi_k(\theta_{-k}\omega)u) \notin K_\varepsilon\bigr\}\\ &\le 2\varepsilon + 3\mathbb{P}(\Omega_\varepsilon^c) \le 5\varepsilon.\end{aligned} \tag{3.5}$$

Furthermore, the functions g_j are $\mathcal{F}^-$-measurable, and hence, by Step 2, for any fixed $\varepsilon > 0$,

$$p_k(u,\varepsilon) \to (\mathfrak{M}, G_\varepsilon) \quad \text{as} \quad k \to +\infty. \tag{3.6}$$

Finally, inequality (3.3) implies that

$$\bigl|(\mathfrak{M}, G_\varepsilon - F)\bigr| \le 2\varepsilon + 3(\mathfrak{M}, I_{K_\varepsilon^c}) = 2\varepsilon + 3\mu(K_\varepsilon^c). \tag{3.7}$$

Since $\varepsilon > 0$ is arbitrary, it follows from (3.4) – (3.7) that the required convergence (2.3) will be established if we show that $\mu(K_\varepsilon^c) \to 0$ as $\varepsilon \to 0$.

To this end, we note that

$$\mu(K_\varepsilon^c) = \int_H \mathbb{P}\bigl\{\varphi_k(\omega)u \notin K_\varepsilon\bigr\}\mu(du). \tag{3.8}$$

It follows from Condition 2.1 that, for any fixed $u \in H$,

$$\limsup_{k\to+\infty} \mathbb{P}\bigl\{\varphi_k(\omega)u \notin K_\varepsilon\bigr\} = \limsup_{k\to+\infty} \mathbb{P}\bigl\{\varphi_k(\theta_{-k}\omega)u \notin K_\varepsilon\bigr\} \le \varepsilon.$$

Passing to the limit $k \to +\infty$ in (3.8), we conclude that $\mu(K_\varepsilon^c) \le \varepsilon$ for any $\varepsilon > 0$. This completes the proof of Theorem 2.3.

3.2 Proof of Theorem 2.4

We first show that the random compact set $\mathcal{A}_\omega$ is a random attractor. Let us fix $\delta \in (0,1)$ and consider the function

$$F(\omega,u) = 1 - \tfrac{d(u,\mathcal{A}_\omega)}{\delta} \wedge 1, \quad u \in H, \quad \omega \in \Omega.$$

We claim that $F \in \mathbb{L}(H, \mathcal{F}^-)$. Indeed, the definition implies that $F(\omega,u)$ is bounded and that

$$\bigl|F(\omega,u) - F(\omega,v)\bigr| \le \tfrac{d(u,v)}{\delta} \quad \text{for all} \quad u,v \in H, \quad \omega \in \Omega.$$

So F satisfies (0.6). Since $\mathcal{A}_\omega$ is a random compact set measurable with respect to $\mathcal{F}^-$, we conclude that the random variable $\omega \to d(u,\mathcal{A}_\omega)$ is measurable with respect to $\mathcal{F}^-$ for any $u \in H$ (see Section 6.1 in [Arn98]). This proves the required properties of F.

Since $F(\omega,u) = 1$ for $u \in \mathcal{A}_\omega$, then $(\mathfrak{M}, F) = 1$. So, applying Theorem 2.3, we get that

$$\mathbb{E}\, F(\theta_k\omega, \varphi_k(\omega)u) = 1 - \mathbb{E}\Bigl(\frac{d(\varphi_k(\omega)u, \mathcal{A}_{\theta_k\omega})}{\delta} \wedge 1\Bigr) \to (\mathfrak{M}, F) = 1.$$

That is,

$$p_k(u) := \mathbb{E}\Bigl(\frac{d(\varphi_k(\omega)u, \mathcal{A}_{\theta_k\omega})}{\delta} \wedge 1\Bigr) \to 0. \tag{3.9}$$

We now note that, by Chebyshev's inequality,

$$\mathbb{P}\bigl\{d(\varphi_k(\omega)u, \mathcal{A}_{\theta_k\omega}) > \delta\bigr\} \le \frac{p_k(u)}{\delta}.$$

In view of (3.9), the right-hand side of this inequality goes to zero as $k \to +\infty$. This completes the proof of the fact that $\mathcal{A}_\omega$ is a random attractor.

To show that $\mathcal{A}_\omega$ is a minimal random attractor, it suffices to note that, by Proposition 1.6, the invariant measure $\mathfrak{M} \in \mathcal{I}_{\mathbb{P},\mathcal{F}^-}$ is supported by any random attractor $\mathcal{A}'_\omega$, and therefore $\operatorname{supp}\mu_\omega \subset \mathcal{A}'_\omega$ for a.e. $\omega \in \Omega$.

4 Navier–Stokes equations

In this section we consider the randomly forced 2D Navier–Stokes (NS) system

$$\dot{u} - \nu\Delta u + (u, \nabla)u + \nabla p = \eta(t, x), \quad \operatorname{div} u = 0. \tag{4.1}$$

The space variable x belongs either to a smooth bounded domain D, and then the boundary condition $u\big|_{\partial D} = 0$ is imposed, or to the torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, and then we assume that $\int u\,dx = \int \eta\,dx \equiv 0$. We are interested in time-evolution of the velocity field u (not of the pressure p). Accordingly, we replace the force η by its divergence-free component (neglecting the gradient-component), and assume below that

$$\operatorname{div} \eta = 0.$$

We first consider the case when the right-hand side η is a random kick force of the form

$$\eta(t, x) = \sum_{k\in\mathbb{Z}} \delta(t - k)\eta_k(x), \tag{4.2}$$

where η_k are i.i.d. random fields as in [KS00, KS01]. Let us denote by H the Hilbert space of divergence-free vector fields on the domain in question that satisfy the boundary conditions in the usual sense (e.g., see [Lio69]). We normalize the solutions $u(t, x)$ for (4.1), (4.2), treated as random curves in H, to be continuous from the right. Then evaluating the solutions at integer times $t = k \in \mathbb{Z}_+$ and setting $u_k = u(k, \cdot)$, we obtain the equation

$$u_k = S(u_{k-1}) + \eta_k. \tag{4.3}$$

Here S is the time-one shift along trajectories of the free NS system (4.1) (with $\eta \equiv 0$); see [KS00, KS01] for details. Defining φ_k, $k \geq 0$, as a map sending $u \in H$ to a solution u_k of (4.3) equal to u at $t = 0$, we get an RDS of the form (0.1). One easily checks that it satisfies the required compactness condition (cf. [KS00, Section 2.2.1]). Moreover, if the distribution of the kicks η_k meets some non-degeneracy assumption specified in [KS01], then the corresponding Markov chain in H has a unique stationary measure μ and the condition (0.4) holds. Hence, Theorem 2.4 applies, and we get the following result:

Theorem 4.1. *If the kick force* (4.2) *satisfies the above conditions, then the support $\mathcal{A}_\omega$ of the Markov disintegration μ_ω of its unique stationary measure μ is a minimal random attractor for the RDS* (4.3)*. Moreover, there is a deterministic constant $D = D_\nu$ such that the Hausdorff dimension of the set $\mathcal{A}_\omega$ does not exceed D for a.e. ω.*

In Remark 2.6, we point out that Theorems 2.3 and 2.4 remain valid for a class of RDS with continuous time $t \geq 0$. This class includes the system describing the white-forced 2D NS equations, i.e., Equation (4.1) with

$$\eta(t, x) = \frac{\partial}{\partial t}\zeta(t, x), \qquad \zeta(t, x) = \sum_{j=1}^{\infty} b_j \beta_j(t) e_j(x). \tag{4.4}$$

Here $\{e_j\}$ is the L^2-normalised trigonometric basis in H and $\{\beta_j\,, t \in \mathbb{R}\}$ is a family of independent standard Wiener processes. It is assumed that the real coefficients b_j decay faster than any negative degree of j:

$$|b_j| \le C_m j^{-m} \qquad \text{for all} \quad j, m \ge 1,$$

so that $\eta(t,x)$ is a.s. smooth in x. Let us denote by $\mathcal{H}$ the space of continuous curves $\zeta\colon \mathbb{R} \to H$ such that $\zeta(0) = 0$ and endow it by the topology of the uniform convergence on bounded intervals. Let $\mathcal{B}$ be the σ-algebra of Borel subsets of $\mathcal{H}$, $\{\theta_t\}$ be the group of canonical shifts of $\mathcal{H}$, $\theta_t\zeta(s) = \zeta(s+t) - \zeta(t)$, and $\mathbb{P}$ be the distribution of the process ζ in $\mathcal{H}$. We take $(\mathcal{H}, \mathcal{B}, \mathbb{P})$ for the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the NS system (4.1), (4.4) defines a continuous-time RDS over θ_t (see [Arn98]) and a Markov process in H. The RDS meets the compactness condition, see [CDF97, Section 3.1]. Moreover, it is shown in [KS02] (see also [EMS01, BKL02]) that there is an integer $N = N_\nu$ such that, if

$$b_j \ne 0 \quad \text{for} \quad 1 \le j \le N, \tag{4.5}$$

then the corresponding Markov process in H has a unique stationary measure μ and satisfies (0.4). Thus, we get the following result:

Theorem 4.2. *If* (4.5) *holds, then the white-forced 2D NS system* (4.1), (4.4) *has a unique stationary measure μ. The supports $\mathcal{A}_\omega$ of its Markov disintegration μ_ω define a minimal random attractor for the corresponding RDS in H. Moreover, there is a deterministic constant $D = D_\nu$ such that the Hausdorff dimension of the set $\mathcal{A}_\omega$ does not exceed D for a.e. ω.*

The fact that $\operatorname{supp}\mu_\omega$ has finite Hausdorff dimension in both discrete and continuous cases follows from general results on upper bounds for the Hausdorff dimension of global random attractors (see [CV93, Deb98, CV02]), since the latter contain $\operatorname{supp}\mu_\omega$ (see Corollary 3.6 in [Cra01]).

Let us consider the skew-product system $\{\Theta_k\}$ corresponding to the RDS above. Applying a continuous-time version of Theorem 2.3, we get the following theorem:

Theorem 4.3. *Let G be a bounded measurable functional on $H \times \mathcal{H}$, uniformly Lipschitz in the first variable, and such that $G(u, \zeta(\cdot))$ depends only on $\{\zeta(s)\,, s \le 0\}$. Let $u(t)$ be a solution for* (4.1), (4.4) *equal to u_0 for $t = 0$. Then under the assumption* (4.5) *the following convergence holds for any $u_0 \in H$:*

$$\mathbb{E}G(u(t), \theta_t\zeta) \to \mathbb{E}\int_H F(v,\zeta)\,\mu_\zeta(dv) \quad \text{as} \quad t \to \infty.$$

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