Coupling approach to white-forced nonlinear PDE's.

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Abstract

We consider the 2D Navier–Stokes system, perturbed by a white in time random force, such that sufficiently many of its Fourier modes are excited (e.g., all of them are). It is proved that the system has a unique stationary measure and that all solutions exponentially fast converge in distribution to this measure. The proof is based on the same ideas as in our previous works on equations perturbed by random kicks. It applies to a large class of randomly forced PDE's with linear dissipation.

0 Introduction

We consider the 2D Navier–Stokes (NS) system with random right-hand side:

$$\dot{u} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \eta(t,x), \quad \operatorname{div} u = 0, \tag{0.1}$$

where x belongs to either a smooth bounded domain, and then the Dirichlet boundary conditions are imposed, or to the two-dimensional torus $\mathbb{T}^2$, and then we assume that $\int u\,dx \equiv \int \eta\,dx \equiv 0$. We denote by H the corresponding L^2-space of divergence free vector fields and by $\{e_j\}$ the Hilbert basis of H formed by eigenvectors of the operator $L = -\nu\Pi\Delta$, where Π is the orthogonal projector to the space H (see e.g. [CF88, Lio69]). We denote by α_j the eigenvalues of L and by $|\cdot|$ the norm in H. Concerning the right-hand side, we assume that either η is a kick-force

$$\eta(t,x) = \sum_{k=-\infty}^{\infty} \eta_k(x)\delta(t-Tk), \quad \eta_k(x) = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j(x), \tag{0.2}$$

where $b_j \ge 0$ are some constants such that $\sum b_j^2 < \infty$ and $\{\xi_{jk}\}$ are independent random variables with k-independent distributions; or that the random force η is white in time:

$$\eta(t,x) = \frac{d}{dt}\sum_{j=1}^{N'} b_j\beta_j(t)e_j(x), \quad N' \le \infty, \tag{0.3}$$

where $\{\beta_j\}$ are independent standard Wiener processes, defined for $t \in \mathbb{R}$.

In the kick case (0.1), (0.2), the long-time behaviour of solutions $u(t) \in H$ is determined by the values they take in points of the lattice $T\mathbb{Z}$, and

$$u((k+1)T) = S\big(u(kT)\big) + \eta_{k+1}, \tag{0.4}$$

where the operator $S : H \to H$ is the time-T shift along trajectories of the free NS system. The random dynamical system (RDS) (0.4) defines a Markov chain in H. A probability Borel measure μ on H is called a *stationary measure* for (0.1), (0.2) if it is a stationary measure for the Markov chain (0.4). Similarly, the white-forced equation (0.1), (0.3) defines a Markov process in H, and a stationary measure of this process is called a *stationary measure* of the NS system.

In [KS00], we assumed that the random variables ξ_{jk} in (0.2) are uniformly bounded, [1] their distributions satisfy some mild regularity assumptions, and

$$b_j \neq 0, \quad 1 \leq j \leq N, \tag{0.5}$$

for a sufficiently large N. Under these assumptions, we used a Foias–Prodi type reduction [FP67] of the NS system (0.1), (0.2) to a finite-dimensional random system with delay to prove that the former has a unique stationary measure μ. This measure is isomorphic to a 1D Gibbs measure, and

$$\mathbb{E} f(u(t)) \to \int_H f(u)\, d\mu(u) \quad \text{as} \quad t \to \infty, \tag{0.6}$$

$t \in T\,\mathbb{Z}$, for any bounded continuous function f and for any solution u of (0.1), (0.2). That is, distributions of all solutions weakly converge to μ. So this measure comprises asymptotic in time stochastic properties of solutions.

E, Mattingly, Sinai [EMS01] and Bricmont, Kupiainen, Lefevere [BKL00] used later the Foias–Prodi reduction to prove that the NS system (0.1), (0.3), (0.5), where $N \leq N' < \infty$, has a unique stationary measure μ. Moreover, it is proved in [BKL00] that the convergence (0.6) holds and is exponentially fast, provided that $u(0)$ is a deterministic vector belonging to a subset of H of full μ-measure. We note that Flandoli and Maslowski [FM95] and Mattingly [Mat99] proved earlier the uniqueness of a stationary measure for (0.1), (0.3) for the cases when the force η is singular in x (namely, $c\,j^{-\frac{1}{2}} \leq b_j \leq C\,j^{-\frac{3}{8}-\varepsilon}$, $\varepsilon > 0$) and is sufficiently small, respectively. (These restrictions on η are different from what we are interested in our work.)

Next in [KS01a] and [KPS02] the authors and A. Piatnitski developed a coupling approach to study the RDS (0.4) which allows to get a much shorter proof of the uniqueness and to show that the convergence (0.6) is exponentially fast for all solutions. Independently similar results were obtained by N. Masmoudi and L.-S. Young in [MY02].

In [Kuk02] the first author used some ideas of L. Kantorovich to get a shorter version of the coupling approach. Namely, it was shown in [Kuk02] that the transfer-operator of the RDS (0.4), which sends $\mathcal{D}\big(u(kT)\big)$ to $\mathcal{D}\big(u((k+1)T)\big)$

[1] Equations with unbounded kick-forces were studied later in [KS01b].

($\mathcal{D}$ signifies distribution), determines a contraction of a suitable Kantorovich type functional defined on pairs of measures. Therefore the transfer-operator determines a contraction of the space of measures; so it has a unique fixed point (the stationary measure), and the distributions of all solutions converge to this measure exponentially fast.

In [Mat02], J. Mattingly applied a coupling to (0.1), (0.3) with $N' < \infty$ and proved that convergence (0.6) is exponential for all $u(0)$. Unfortunately, we found it very difficult to follow his arguments.

We also mention the papers [EH01, Hai02], which are devoted to studying a class of randomly perturbed parabolic problems with strong nonlinear dissipation, including the Ginzburg–Landau equation.

In this work we show that the coupling approach from the works [KS01a, KPS02, Kuk02] applies to the white-forced NS system. It implies the uniqueness of a stationary measure and the exponentially fast convergence (0.6). More specifically, we fix a sufficiently large T and replace (0.1), (0.3) by the embedded Markov chain

$$u((k+1)T) = S_T\big(u(kT)\big), \tag{0.7}$$

where the random operator $S_T : H \to H$ is the time-T shift along trajectories of (0.1), (0.3). It turns out that the RDS (0.7) is quite similar to (0.4), and it is possible to apply the coupling approach in the form proposed in [Kuk02] to prove the uniqueness of a stationary measure and convergence (0.6). Finally, we easily go back from (0.7) to (0.1), (0.3) and obtain the following result:

MAIN THEOREM. *Suppose that, in* (0.3), $N' = \infty$ *and* $\sum \alpha_j b_j^2 < \infty$. *Then for any* $\nu > 0$ *and* $B > 0$ *there is an integer* $N \geq 0$ *such that if* $\sum b_j^2 \leq B$ *and* (0.5) *holds, then the NS system* (0.1), (0.3) *has a unique stationary measure* μ. *Moreover, there are positive constants* C *and* σ (*depending on* ν *and* $\{b_j\}$) *such that, if* u_0 *is any vector in* H, $u(t)$ *is a solution such that* $u(0) = u_0$, *and* f *is a bounded Lipschitz function on* H, *satisfying* $\sup|f| \leq 1$ *and* $\operatorname{Lip}(f) \leq 1$, *then*

$$\left|\mathbb{E}f(u(t)) - \int_H f(u)\,d\mu(u)\right| \leq C(1 + |u_0|^2)e^{-\sigma t}.$$

The theorem means that, for any $u_0 \in H$, the distribution $\mathcal{D}(u(t))$ converges to μ exponentially fast in the Lipschitz–dual norm (see Subsection 2.3). As convergence in this norm is equivalent to the weak convergence [Dud02], for each u_0 we have $\mathcal{D}(u(t)) \rightharpoonup \mu$ as $t \to \infty$.

Since our approach to the randomly forced 2D NS system is heavily based on the Foias–Prodi reduction, then we use essentially the assumption (0.5) (same is true for all other works on the randomly forced NS system, written after [KS00] up to now). In this assumption the number N grows as a negative degree of ν as $\nu \to 0$. Fortunately, since we allow $N' = \infty$ in (0.3), the assumption is met for any $\nu > 0$ if all b_j's are non-zero. Because of that, our theorem can be used to propose the following mathematical interpretation of the problem of

2D-turbulence. Let us consider the equation (0.1), (0.3) such that $b_j \neq 0$ for all j. Due to the Main Theorem, for any positive ν the equation has a unique stationary measure μ_ν.

PROBLEM. *What are limiting properties of the measures μ_ν as $\nu \to 0$? In particular, do these measures converge (in some "reasonable" sense) to a limiting measure?*

See [EKMS00] and section 5 in [Kuk02] for some related results. For discussions see [Gal01].

Our proof of the Main Theorem does not use specifics of the NS system and apply to a large class of randomly forced nonlinear PDE's with linear dissipation. Roughly, the proof works if information, available on the equation, allows to prove that the equation, perturbed by a time-independent force, has a finite-dimensional attractor. For discussion of nonlinear PDE's with finite-dimensional attractors, see, e.g., [BV92].

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Notations

Let $\{e_j\}$ be an orthonormal basis in H that is formed of the eigenvectors of the operator L defined in Subsection 1.1 and let α_j be the corresponding eigenvalues. We assume that $\alpha_1 \le \alpha_2 \le \cdots$. For any integer $N \ge 1$, we denote by H_N the subspace in H generated by $e_1, \dots, e_N$ and by $H_N^\perp$ its orthogonal complement. Let P_N and Q_N be the orthogonal projections onto H_N and $H_N^\perp$, respectively.

We set $B_0 = \sum_j b_j^2$, $B_1 = \sum_j \alpha_j b_j^2$, $C_0 = B_0/\alpha_1$, $\gamma_0 = \alpha_1/2b_{\max}$, and denote by $T_{(1)}, T_{(2)}, \dots, C_{(1)}, C_{(2)}$, etc. various positive constants which depend only on $\{b_j\}$ and $\{\alpha_j\}$.

For a set A, A^c denotes its complement and I_A stands for its indicator function. For a random variable ξ, we denote by $\mathcal{D}(\xi)$ its distribution.

Let X be a Banach space and let $J \subset \mathbb{R}$ be a closed interval. We shall use the following functional spaces:

$C(J;X)$ is the space of continuous functions on J with range in X.

$D^T(J;X)$, $T > 0$, is the space of continuous from the right maps from J to X that are continuous outside the lattice $T\mathbb{Z}$ and have limits from the left at points of $T\mathbb{Z}$.

$L^2(J;X)$ is the space of Bochner-measurable functions $f\colon J \to X$ such that $\int_J \|f(t)\|_X^2 dt < \infty$.

1 Preliminaries

In this section, we compile some known results on strong and weak solutions for the Navier–Stokes (NS) equations (0.1). In what follows, to simplify the notations, we shall assume that $\nu = 1$.

1.1 Strong and weak solutions

We rewrite the NS system (0.1) in the form

$$\dot{u} + Lu + B(u,u) = \eta(t). \tag{1.1}$$

Here $u = u(t) \in H$, $L = -\Pi\Delta$ and $B(u,u) = \Pi(u,\nabla)u$, where Π is the orthogonal projection onto the space H. The right-hand side η is a white-noise force in H:

$$\eta(t) = \frac{\partial}{\partial t}\zeta(t), \quad \zeta(t,x) = \sum_{j=1}^{\infty} b_j \beta_j(t) e_j(x).$$

Let us set $V = H^1 \cap H$, where H^1 is the Sobolev space of order 1, and denote by $\|\cdot\|$ the norm in V and by V^* the adjoint space for V.

Definition 1.1. A random process $u(t) = u(t,x;\omega)$ in H defined on the half-line $t \ge l$ and progressively measurable with respect to the σ-algebras $\mathcal{F}_t$ generated by $\zeta(s)$, $l \le s \le t$, is called a *strong solution* of Eq. (1.1) if the following two conditions hold with probability 1:

(i) For any $T > l$, the function $u(t,x)$ belongs to $L^2(l,T;V) \cap C(l,T;H)$.

(ii) For any $t > l$, we have

$$u(t) + \int_l^t \bigl(Lu + B(u,u)\bigr)\,ds = u(l) + \zeta(t) - \zeta(l),$$

where the left- and right-hand sides of this relation are regarded as elements of V^*.

If, in addition, the process satisfies the initial condition

$$u(l) = u^0 \in H, \tag{1.2}$$

then it is called a *strong solution* of the problem (1.1), (1.2).

Definition 1.2. A random process $u(t) = u(t;\omega') \in H$, $t \ge 0$, defined on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ is called a *weak solution* of Eq. (1.1) if there is a process $\zeta'(t)$ defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ and distributed as $\zeta(t)$ such that $u(t)$ is a strong solution of (1.1) with $\eta = \partial_t \zeta'$.

Weak and strong solutions for (1.1) and for (1.1), (1.2) with $t \in [l,T]$, $l < T < \infty$, are defined in a similar way.

It is well known that for any $u^0 \in H$ the problem (1.1), (1.2) has a unique strong solution, defined for $t \ge l$ (see [VF88, Chapter 10]).

If $J \subset \mathbb{R}$ is a finite or infinite interval and $u(t)$, $t \in J$, is a weak solution for (1.1), then it will be convenient for us to replace the process $\zeta'(t)$ (as in

Definition 1.2) by a process $\zeta'_T(t)$ such that its trajectories a.s. belong to the space $D^T(J;V)$ and

$$\partial_t\zeta'(t) = \partial_t\zeta'_T(t) \quad \text{for} \quad t \in J \setminus T\mathbb{Z} \quad \text{almost surely,}$$

where the derivatives of ζ' and ζ'_T are understood in the sense of distributions. Clearly, u is a solution for (1.1) with $\eta = \partial_t\zeta'_T$ on each interval $[(k-1)T, kT]\cap J$, and the process ζ' can be easily recovered from ζ'_T. Abusing language, we shall say that u solves (1.1) with $\eta = \partial_t\zeta'_T$, or that ζ' is a right-hand side corresponding to u.

1.2 An exponential estimate for the growth of solutions

In this subsection we apply the classical supermartingale inequality to get an exponential bound for the probability of super-linear growth of solutions of the NS system. Our arguments closely follow the proof of Lemma A.2 in [Mat02].

Let $u(t)$ be a weak solution for (1.1), satisfying the equation with η replaced by $\partial_t\zeta'$. Let us denote by α_1 the first eigenvalue of L and set $b_{\max} = \max_j b_j$ and

$$\mathcal{E}(t) = |u(t)|^2 + \int_0^t \|u(s)\|^2 ds, \quad B_0 = \sum_{j=1}^\infty b_j^2,$$

where $|\cdot|$ and $\|\cdot\|$ are the norms in the spaces H and V, respectively.

Lemma 1.3. *For any $T > 0$, any integer $k \geq 1$, and any $\rho > 0$, we have*

$$\mathbb{P}\Big\{\sup_{(k-1)T\leq t\leq kT}\big(\mathcal{E}(t) - B_0t\big) \geq |u(0)|^2 + \rho\Big\} \leq e^{-\gamma_0\rho}, \quad k \geq 1. \tag{1.3}$$

where $\gamma_0 = \frac{\alpha_1}{2b_{\max}}$.

Proof. By Itô's formula, we have

$$|u(t)|^2 + 2\int_0^t \|u(s)\|^2 ds = |u(0)|^2 + B_0t + 2\int_0^t (u, d\zeta'). \tag{1.4}$$

It follows that

$$\begin{aligned}\mathcal{E}(t) &= |u(0)|^2 + B_0t + \big(M_t - \gamma_0\langle M\rangle_t/2\big) - \Big(\int_0^t \|u(s)\|^2 ds - \gamma_0\langle M\rangle_t/2\Big)\\ &\leq |u(0)|^2 + B_0t + \big(M_t - \gamma_0\langle M\rangle_t/2\big),\end{aligned} \tag{1.5}$$

where we denoted by M_t the stochastic integral on the right-hand side of (1.4), by $\langle M\rangle_t$ its quadratic variation, and used the inequality

$$\gamma_0\langle M\rangle_t/2 = 2\gamma_0\sum_{j=1}^\infty b_j^2\int_0^t u_j^2(s)\,ds \leq 2\gamma_0 b_{\max}^2\int_0^t |u(s)|^2 ds \leq \int_0^t \|u(s)\|^2 ds.$$

Taking into account (1.5), we derive

$$
\begin{aligned}
\mathbb{P}\Big\{ & \sup_{(k-1)T\le t\le kT}\big(\mathcal{E}(t)-B_0t\big)-|u(0)|^2\ge\rho\Big\}\\
&\le \mathbb{P}\Big\{\sup_{(k-1)T\le t\le kT}\big(M_t-\gamma_0\langle M\rangle_t/2\big)\ge\rho\Big\}\\
&\le \mathbb{P}\Big\{\sup_{0\le t\le kT}\exp\big(\gamma_0M_t-\gamma_0^2\langle M\rangle_t/2\big)\ge e^{\gamma_0\rho}\Big\}. \qquad (1.6)
\end{aligned}
$$

We now note that $\exp(\gamma_0M_t-\gamma_0^2\langle M\rangle_t/2)$ is a supermartingale whose mean value does not exceed 1. Therefore, by a classical supermantingale inequality (e.g., see Theorem VI.T1 in [Mey66] or Theorem III.6.11 in [Kry95]), the expression on the right-hand side of (1.6) can be estimated by $e^{-\gamma_0\rho}$. The proof of (1.3) is complete. ☐

An obvious reformulation of Lemma 1.3 holds if $u(s)$ is a weak solution of (1.1) for $s\ge l$, $l\in\mathbb{R}$.

1.3 Estimates for pairs of solutions

Let $u_1(t,x)$ and $u_2(t,x)$ be two solutions of (1.1) that correspond to random initial functions $u_1^0(x)$ and $u_2^0(x)$, respectively. We set

$$U(t)=\big(u_1(t),u_2(t)\big),\quad U_0=(u_1^0,u_2^0),\quad R(t)=|u_1(t)|^2+|u_2(t)|^2,\quad R_0=R(0),$$

and assume that $\mathbb{E}R_0<\infty$.

Lemma 1.4. *For any $t\ge0$ we have*

$$\mathbb{E}\,R(t)\le e^{-2\alpha_1t}\mathbb{E}R_0+C_0(1-e^{-2\alpha_1t}),\quad C_0=\tfrac{B_0}{\alpha_1}. \qquad (1.7)$$

Proof. Applying Itô's formula to $R(t)$, taking the mean value, and using the inequality $\|u\|^2\ge\alpha_1|u|^2$, we find that

$$\mathbb{E}\,R(t)+2\alpha_1\int_0^t\mathbb{E}\,R(s)ds\le\mathbb{E}R_0+2B_0t.$$

Application of the Gronwall inequality results in (1.7). ☐

Now let us assume that U_0 is a non-random vector such that

$$R_0\le\rho_0,\quad \rho_0\ge C_0. \qquad (1.8)$$

Lemma 1.5. *Let $\theta_1\ge T_1:=\frac{1}{2\alpha_1}\ln\big(\frac{\rho_0}{C_0}\big)$. Then $\mathbb{P}\big\{R(\theta_1)\le4C_0\big\}\ge\frac12$.*

Proof. Due to (1.8) and (1.7), we have $\mathbb{E}\,R(t)\le C_0+\rho_0e^{-2\alpha_1t}$. If $t\ge T_1$, then the right-hand side of this inequality is no greater than $2C_0$. Therefore, applying the Chebyshev inequality, we obtain the required inequality. ☐

We now assume that

$$B_1 = \sum_{j=1}^{\infty} \alpha_j b_j^2 < \infty. \tag{1.9}$$

Lemma 1.6. *Suppose that conditions* (1.8) *and* (1.9) *are satisfied. Then for any* $\theta > 0$ *there is a* $\pi = \pi(\theta) > 0$ *not depending on* ρ_0 *such that*

$$\mathbb{P}\bigl\{|u_1(\theta_2)| \vee |u_2(\theta_2)| \le \theta\bigr\} \ge \pi(\theta),$$

where

$$\theta_2 \ge T_2 := \frac{1}{2\alpha_1} \ln \rho_0 + \frac{2}{\alpha_1} \ln \theta^{-1} + \frac{1}{2\alpha_1} \ln(64C_0). \tag{1.10}$$

Proof. 1) Without loss of generality, we can assume that $\zeta(0) = 0$. For any $T > 0$ and $\delta > 0$, we set

$$\Omega_{T,\delta} = \bigl\{\omega \in \Omega : \|\zeta(t)\| \le \delta \text{ for } 0 \le t \le T\bigr\}.$$

We claim that there is $\pi_0 = \pi_0(T, \delta) > 0$ such that $\mathbb{P}\bigl(\Omega_{T,\delta}\bigr) \ge \pi_0$. Indeed, for any integer $M \ge 1$, let us set $\zeta_M = \mathsf{P}_M\zeta$ and $\zeta_M^\perp = \mathsf{Q}_M\zeta$. It is clear that $\omega \in \Omega_{T,\delta}$ if the following two inequalities hold:

$$\sup_{0\le t\le T} \|\zeta_M(t)\| \le \delta/2, \qquad \sup_{0\le t\le T} \|\zeta_M^\perp(t)\| \le \delta/2. \tag{1.11}$$

The probability of the first event in (1.11) is no less than some $\pi_1(T, \delta, M) > 0$ due to the classical properties of a finite-dimensional Wiener process. In view of the Doob–Kolmogorov inequality (see [Mey66, Kry95]), the probability of the second event is bounded from below by the expression

$$\pi_2(T, \delta, M) = 1 - 4\delta^{-2}\mathbb{E}\,\|\zeta_M^\perp(T)\|^2 = 1 - 4T\delta^{-2} \sum_{j=M+1}^{\infty} \alpha_j b_j^2.$$

Using (1.9), we can find an integer $M = M(T, \delta)$ such that $\pi_2 \ge 1/2$. Since the events in (1.11) are independent, we conclude that $\pi_0 \ge \pi_1\pi_2 \ge \pi_1/2 > 0$.

2) We now fix $T > 0$ and $\delta > 0$ and consider a solution $u(t, x)$ of (1.1) that corresponds to some $\omega \in \Omega_{T,\delta}$. Let us write $u = \zeta + v$. Then $v(t, x)$ satisfies the equation

$$\dot v + Lv + B(v + \zeta, v + \zeta) = -L\zeta(t). \tag{1.12}$$

Since $\|\zeta(t)\| \le \delta$ for $0 \le t \le T$, then taking the scalar product of (1.12) and $2v$ and using the standard estimates for the cubic term $(B(v + \zeta, v + \zeta), v)$ (e.g., see [CF88]), we get

$$\frac{d}{dt}|v|^2 + 2\|v\|^2 \le C_1\delta|v|\,\|v\| + C_1\delta^2\|v\|^2 + 2\delta\|v\|, \quad 0 \le t \le T. \tag{1.13}$$

Here $C_1 > 0$ is a constant not depending on T, δ, and u. Assuming that $4C_1\delta^2 \le 1$ and $4C_1^2\delta^2 \le \alpha_1$, we see that the right-hand side of (1.13) does not exceed $\frac{3}{4}\|v\|^2 + \frac{\alpha_1}{4}|v|^2 + 4\delta^2$. Using the inequality $\|v\|^2 \ge \alpha_1|v|^2$, we arrive at

$$\frac{d}{dt}|v|^2 + \alpha_1|v|^2 \le 4\delta^2.$$

The Gronwall inequality now gives $|u(T)|^2 \le e^{-\alpha_1 T}|u(0)|^2 + 4\alpha_1^{-1}\delta^2$. Applying this inequality to two solutions u_1 and u_2 whose initial conditions are such that $R_0 \le 4C_0$, we see that, with probability no less than π_0, the following estimate holds:

$$\big(|u_1(T)| \vee |u_2(T)|\big)^2 \le 4\alpha_1^{-1}\big(B_0 e^{-\alpha_1 T} + \delta^2\big). \tag{1.14}$$

Let us take any $\theta > 0$. Choosing $T = T_2' := \frac{2}{\alpha_1}\ln\theta^{-1} + \frac{1}{\alpha_1}\ln(8C_0)$ and $\delta \le \theta\sqrt{\alpha_1/8}$, we see that the expression on the right-hand side of (1.14) does not exceed θ^2 with probability no less than $\pi_0 = \pi_0(T_2', \delta)$. Combining this with Lemma 1.5 and setting $T_2 = T_1 + T_2'$ and $\pi = \pi_0/2$, we obtain the required assertion. □

Lemma 1.6 states that with a positive probability any two solutions of the NS system (1.1) can be simultaneously pulled through a tiny neighbourhood of the origin. Moreover, the probability can be chosen to be independent from the initial conditions (cf. (5.16) in [KS00] and Lemma 3.1 in [KS01a]).

2 Proof of the main theorem

In this section we show that the main theorem follows from the existence of a specific coupling for solutions of the NS system. [2] Namely, we use the coupling to establish exponential decay of a Kantorovich type functional and then prove that this fact implies the exponential convergence to a unique stationary measure.

2.1 Coupling of solutions for the Navier–Stokes system

In this subsection, we use parameters $T \ge 1$, $\rho_0 \ge 1$, and $N \in \mathbb{N}$ which will be specified later. Let us fix an integer $k \ge 1$. For any integer l, $0 \le l \le k$, we define $Q_0(l,k)$ as the set of all quadruples of functions $(u_1(t), \zeta_1(t), u_2(t), \zeta_2(t))$, $t \in I_k := [0, kT]$, such that [3]

$$u_i \in \mathcal{H}(I_k), \quad \zeta_i \in D^T(I_k, V) \cap C([lT, kT]; V) \quad i = 1, 2, \tag{2.1}$$

$$|u_1(lT)| \vee |u_2(lT)| \le d, \tag{2.2}$$

$$\mathsf{P}_N u_1(t) = \mathsf{P}_N u_2(t), \quad \mathsf{Q}_N \zeta_1(t) = \mathsf{Q}_N \zeta_2(t), \quad lT \le t \le kT, \tag{2.3}$$

$$\mathcal{E}_i(t, lT) \le \rho + (B_0 + 1)(t - lT), \quad lT \le t \le kT, \quad i = 1, 2. \tag{2.4}$$

Here $\mathcal{H}(I_k) := C(I_k, H) \cap L^2(I_k, V)$, $d \in (0, 1]$ and $\rho > 0$ are parameters that will be defined in Theorem 2.1, and

$$\mathcal{E}_i(t, s) = \mathcal{E}(t, s)(u_i) := |u_i(t)|^2 + \int_s^t \|u_i(r)\|^2 dr. \tag{2.5}$$

[2] That is, a coupling for their distributions in the space of trajectories. See [Lin92] and Appendix in [KS01a] for some basic results on the coupling.

[3] In the case $l = k = 0$, the second relation in (2.3) should be ignored.

To shorten notations, we shall often write $\Theta_i = (u_i, \zeta_i)$. Let $Q(k)$ be the union of the sets $Q_0(l,k)$, $0 \le l \le k$, and let

$$Q(l,k) = Q_0(l,k) \setminus Q_0(l-1,k), \quad 0 \le l \le k,$$

where $Q(-1,k) = \varnothing$. We set

$$S(k) = \big(\mathcal{H}(I_k) \times D^T(I_k, V)\big)^2 \setminus Q(k),$$

where for a Banach space X we write $X^2 = X \times X$, and define

$$S_+(k) = \big\{(u_1, \zeta_1, u_2, \zeta_2) \in S(k) \,:\, R(kT) \le \rho_0\big\}, \quad S_-(k) = S(k) \setminus S_+(k),$$

where $R(t) = |u_1(t)|^2 + |u_2(t)|^2$.

The sets $Q(l,k)$ play crucial role in our construction of a coupling for solutions of the NS system. Besides, the events defined by relations (2.4) are used to construct cut-offs for (1.1) which we exploit to analyse the system. We note that similar cut-offs were used earlier in [EMS01].

Let $u_1(t)$ and $u_2(t)$, $t \in [lT, kT]$, be two weak solutions of (1.1) which satisfy (2.2), (2.3), where $u_1(lT)$ and $u_2(lT)$ are non-random vectors. Then, due to Lemma 1.3, we have

$$\mathbb{P}\Big\{\mathcal{E}_i(t, lT) \ge \rho + (B_0+1)(t - lT) \text{ for some } t \in [(r-1)T, rT]\Big\} \le \\ \le e^{-\gamma_0(\rho - d^2 + T(r-l-1))}, \quad (2.6)$$

since $\mathcal{E}_i(t, lT) \ge \rho + (B_0+1)(t-lT)$ implies that

$$\mathcal{E}_i(t, lT) - B_0(t - lT) \ge |u_i(lT)|^2 + (\rho - |u_i(lT)|^2) + T(r-l-1),$$

and $|u_i(lT)|^2 \le d^2$.

In the theorem below, $\rho' \ge 1$ is a constant which depends only on $\{b_j\}$ and $\{\alpha_j\}$; for weak solutions u_i and $\tilde u_i$ of the NS system (1.1), we denote the corresponding right-hand side by $\eta = \partial_t \zeta_i$ and $\tilde\eta = \partial_t \tilde\zeta_i$, respectively. For $i = 1, 2$ we abbreviate $\tilde\Theta_i(t) = (\tilde u_i(t), \tilde\zeta_i(t))$, $\Theta_i(t) = (u_i(t), \zeta_i(t))$, and recall that the processes ζ_i and $\tilde\zeta_i$ may be discontinuous at the points of the lattice $T\mathbb{Z}$; see discussion at the end of Subsection 2.1. Finally, we set $\mathbf{\Theta}_i^k = (\Theta_i(t), 0 \le t \le kT)$ and $\tilde{\mathbf{\Theta}}_i^{k-1} = (\tilde\Theta_i(t), 0 \le t \le (k-1)T)$.

Theorem 2.1. *For any $\rho_0 \ge 1$ and $\rho \ge \rho'$ there are $T(\rho, \rho_0) \ge 1$ and $d(\rho) \in (0,1]$ such that for any $T \ge T(\rho, \rho_0)$ and d, $0 < d \le d(\rho)$, and some appropriate constant $p_0 = p_0(d) > 0$ the following assertion holds for any integer $k \ge 1$. Let $\tilde u_1(t)$ and $\tilde u_2(t)$ be two weak solutions of the NS system defined for $t \in I_{k-1}$ on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. Then there is a probability space $(\Omega^k, \mathcal{F}^k, \mathbb{P}^k)$ and weak solutions $u_1(t)$ and $u_2(t)$ for the NS system defined on $(\Omega' \times \Omega^k, \mathcal{F}' \times \mathcal{F}^k, \mathbb{P}' \times \mathbb{P}^k)$ for $t \in I_k$ such that*

$$u_i(t; \omega', \omega^k) = \tilde u_i(t; \omega'), \quad t \in I_{k-1}\,; \quad \zeta_i(t; \omega', \omega^k) = \tilde\zeta_i(t; \omega'), \quad t \in [0, (k-1)T), \quad (2.7)$$

for $i = 1, 2$ and all ω' and ω^k. Moreover, the assertions below are satisfied:

(i) *For any l, $0 \le l \le k-1$, we have*

$$\int_{\Omega'} I_{\overline{Q}}(\omega')\,\mathbb{P}^k\big\{(\mathbf{\Theta}_1^k,\mathbf{\Theta}_2^k)\notin Q(l,k)\big\}\mathbb{P}'(d\omega') \le c\,e^{-\gamma_0\rho}e^{-\gamma_1 T(k-l-1)}\mathbb{P}'(\overline{Q}), \tag{2.8}$$

where $\overline{Q}$ is the event $\{(\tilde{\mathbf{\Theta}}_1^{k-1},\tilde{\mathbf{\Theta}}_2^{k-1})\in Q(l,k-1)\}$, and $c = 1+8e^{\gamma_0}$, $\gamma_1=\gamma_0\wedge 1$.

(ii) *If $(\tilde{\mathbf{\Theta}}_1^{k-1},\tilde{\mathbf{\Theta}}_2^{k-1})\in S_+(k-1)$, then*

$$\mathbb{P}^k\big\{(\mathbf{\Theta}_1^k,\mathbf{\Theta}_2^k)\in Q(k,k)\big\}\ge p_0. \tag{2.9}$$

(iii) *The constant $T(\rho,\rho_0)$ can be represented in the form*

$$T(\rho,\rho_0)=C_{(1)}\ln\rho_0 + C_{(2)}\rho + C_{(3)}, \tag{2.10}$$

where the constants $C_{(1)}$, $C_{(2)}$, and $C_{(3)}$ depend only on $\{b_j\}$ and $\{\alpha_j\}$.

Theorem 2.1 is proved below, in Section 3. To define the solutions u_1 and u_2, we construct there an operator which assigns to each pair of continuous curves $(\tilde{\boldsymbol{u}}_1,\tilde{\boldsymbol{u}}_2)$, $\tilde{\boldsymbol{u}}_i\in C(I_{k-1};H)$, a pair of processes $\big(U_1(t;\omega^k),U_2(t;\omega^k)\big)$, $(k-1)T\le t\le kT$, formed by weak solutions of (1.1) and equal to $\big(\tilde u_1,\tilde u_2\big)$ for $t=(k-1)T$. Next, if $\tilde u_1(t;\omega)$ and $\tilde u_2(t;\omega)$ are weak solutions as in Theorem 2.1, then we define the solutions u_1 and u_2 by relations (2.7) for $t\in I_{k-1}$ and set $u_i=U_i\big(t;\omega^k,\tilde{\boldsymbol{u}}_1(\cdot,\omega'),\tilde{\boldsymbol{u}}_2(\cdot,\omega')\big)$ for $(k-1)T\le t\le kT$. Denoting by μ_i the distribution in $C((k-1)T,kT;H)$ of a strong solution for (1.1) that is equal to $\tilde u_i((k-1)T)$ for $t=(k-1)T$, we clearly have

$$\mathcal{D}\big(U_i(\,\cdot\,;\tilde{\boldsymbol{u}}_1,\tilde{\boldsymbol{u}}_2)\big)=\mu_i,\quad i=1,2.$$

Hence, the pair (U_1,U_2) is a coupling for the measures (μ_1,μ_2). Thus, Theorem 2.1 is an analogue of Lemma 3.2 from [KS01a], which is the main lemma of that work, as well as of [Kuk02].

2.2 Exponential decay of a Kantorovich type functional

We now show that the above coupling theorem implies exponential convergence to zero of a Kantorovich type functional, similar to that used in [Kuk02]. Our arguments in this subsection and in the next one are related to those used in the theory of Markov chains for proving convergence to a stationary measure in the Kantorovich distance, cf. Section 14 in [Dob96].

For any two curves $\mathbf{\Theta}_i=(u_i(t),\zeta_i(t),t\in I_k)\in\mathcal{H}(I_k)\times D^T(I_k;V)$, $i=1,2$, satisfying (2.1), we set

$$f_k(\mathbf{\Theta})=\begin{cases}\left(\frac12\right)^{k-l} & \text{for}\quad \mathbf{\Theta}\in Q(l,k),\\ \mathcal{R}_k:=\varepsilon R(kT)+2 & \text{for}\quad \mathbf{\Theta}\in S(k),\end{cases} \tag{2.11}$$

where $\mathbf{\Theta}=(\mathbf{\Theta}_1,\mathbf{\Theta}_2)$, and $\varepsilon\in(0,1]$ will be chosen later.

We wish to study evolution of the mean value for $f_k(\boldsymbol{\Theta}^k)$ in the case when $\boldsymbol{\Theta}^k = (\boldsymbol{\Theta}_1^k, \boldsymbol{\Theta}_2^k)$ is the pair of trajectories $(\Theta_1(\cdot), \Theta_2(\cdot))$, where $\Theta_i = (u_i, \zeta_i)$, and u_1, u_2 are weak solutions for the NS system that are constructed by iterated application of Theorem 2.1.

More precisely, let u_1^0 and u_2^0 be two random variables with values in H such that $\mathbb{E}\,|u_i^0|^2 < \infty$, $i = 1, 2$. Using Theorem 2.1 with $k = 1$ and $\tilde{u}_i = u_i^0$, we construct a pair of weak solutions (u_1, u_2) defined for $0 \le t \le T$ and satisfying (2.8) – (2.9). Applying Theorem 2.1 again, we "extend" these solutions to the interval $[0, 2T]$, preserving the above-mentioned properties. Continuing this process, we obtain a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\Omega = \Omega^1 \times \cdots \times \Omega^k, \quad \mathcal{F} = \mathcal{F}^1 \times \cdots \times \mathcal{F}^k, \quad \mathbb{P} = \mathbb{P}^1 \times \cdots \times \mathbb{P}^k, \tag{2.12}$$

and a pair of weak solutions on $(\Omega, \mathcal{F}, \mathbb{P})$ that are defined for $0 \le t \le kT$ and satisfy (2.8) – (2.10).

We shall show that the mean value of $f_m(\boldsymbol{\Theta}^m)$ decays exponentially, provided that ρ_0 and T are large enough. Namely, let us introduce the functional $F_m(\boldsymbol{\Theta}^m) = \mathbb{E} f_m(\boldsymbol{\Theta}^m)$. We have the following result.

Theorem 2.2. *Suppose that $\rho_0 > 0$ and $T \ge 1$ are sufficiently large and that weak solutions $u_1(t)$ and $u_2(t)$, $0 \le t \le kT$, are constructed according to the above scheme. Then there are $\varepsilon > 0$, $\varkappa \in (0,1)$, and $\rho > 1$, not depending on the initial functions u_1^0 and u_2^0, such that*

$$F_m(\boldsymbol{\Theta}^m) \le \varkappa F_{m-1}(\boldsymbol{\Theta}^{m-1}), \quad 1 \le m \le k. \tag{2.13}$$

In particular, for any initial random variable u_1^0 and u_2^0 with finite second moment we have

$$F_0(\boldsymbol{\Theta}^0) \le \mathbb{E}\mathcal{R}_0 \le 2 + \mathbb{E}\,|u_1^0|^2 + \mathbb{E}\,|u_2^0|^2,$$

and therefore iterated application of inequality (2.13) implies that

$$F_m(\boldsymbol{\Theta}^m) \le \varkappa^m \bigl(2 + \mathbb{E}\,|u_1^0|^2 + \mathbb{E}\,|u_2^0|^2\bigr), \quad 1 \le m \le k. \tag{2.14}$$

We shall show in fact that, if $\rho_0 > 0$, $\rho > 1$ and $T \ge T(\rho, \rho_0)$ (see (2.10)) satisfy conditions (2.29) below, then inequality (2.13) holds for some appropriate constants $\varepsilon > 0$ and $\varkappa \in (0,1)$, depending on ρ_0, ρ, and T.

Proof of Theorem 2.2. In what follows, we denote by $T_{(1)}, T_{(2)}, \dots, \varepsilon_{(1)}, \varepsilon_{(2)}$, etc. various positive constants depending only on $\{b_j\}$ and $\{\alpha_j\}$. Let us introduce the events $\overline{S}(m)$, $\overline{S}_+(m)$, $\overline{S}_-(m)$, $\overline{Q}(l,m)$, and $\overline{Q}(m)$, where $\overline{S}(m) = \bigl\{\boldsymbol{\Theta}^m \in S(m)\bigr\}$, and the other sets are defined in a similar way. We note that these events depend only on $\boldsymbol{\omega}^m = (\omega^1, \dots, \omega^m)$, so they can be viewed as subsets of $\Omega^1 \times \cdots \times \Omega^m$.

We have

$$F_m(\boldsymbol{\Theta}^m) = F_m'(\boldsymbol{\Theta}^m) + \sum_{l=0}^{m-1} F_m^l(\boldsymbol{\Theta}^m),$$

where we set

$$F'_m(\mathbf{\Theta}^m) = \mathbb{E}\big\{I_{\overline{S}(m-1)} f_m(\mathbf{\Theta}^m)\big\}, \quad F^l_m(\mathbf{\Theta}^m) = \mathbb{E}\big\{I_{\overline{Q}(l,m-1)} f_m(\mathbf{\Theta}^m)\big\}.$$

In view of the definition of f_k (see (2.11)), the required inequality (2.13) will be established if we show that

$$F'_m(\mathbf{\Theta}^m) \le \varkappa\, \mathbb{E}\big\{I_{\overline{S}(m-1)} \mathcal{R}_{m-1}\big\}, \tag{2.15}$$

$$F^l_m(\mathbf{\Theta}^m) \le \varkappa\, 2^{-(m-l-1)} \mathbb{P}\big\{\overline{Q}(l, m-1)\big\}, \quad 0 \le l \le m-1. \tag{2.16}$$

Moreover, recalling relation (2.7) and the structure of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (see (2.12)), we see that, to prove (2.15), it suffices to verify that

$$\mathbb{E}^m f_m(\mathbf{\Theta}^m) \le \varkappa\, f_{m-1}(\mathbf{\Theta}^{m-1}) = \varkappa\, \mathcal{R}_{m-1}. \tag{2.17}$$

Here $\mathbf{\Theta}^{m-1}$ is any non-random trajectory in $S(m-1) = S_+(m-1) \cup S_-(m-1)$, $\mathbf{\Theta}^m|_{I_{m-1}} = \mathbf{\Theta}^{m-1}$, and for $t \in [(m-1)T, mT]$, $\Theta^m(t) = (u_1, \zeta_1, u_2, \zeta_2)$, where u_1 and u_2 are weak solutions for (1.1) depending on the random parameter $\omega \in \Omega^m$, while ζ_1 and ζ_2 are the corresponding right-hand sides.

1) We first prove (2.17) in the case $\mathbf{\Theta}^{m-1} \in S_+(m-1)$. Since now $\mathbf{\Theta}^m \in S(m) \cup Q(m,m)$ for each $\omega^m \in \Omega^m$, then we have

$$\begin{aligned}
\mathbb{E}^m f_m(\mathbf{\Theta}^m) &= \mathbb{E}^m\big\{I_{\overline{S}(m)} f_m(\mathbf{\Theta}^m)\big\} + \mathbb{E}^m\big\{I_{\overline{Q}(m,m)} f_m(\mathbf{\Theta}^m)\big\} \\
&\le \mathbb{E}^m\big\{I_{\overline{S}(m)} \mathcal{R}_m\big\} + \mathbb{P}^m\big\{\overline{Q}(m,m)\big\} \\
&\le \mathbb{E}^m\big\{\mathcal{R}_m\big\} - \mathbb{P}^m\big\{\overline{Q}(m,m)\big\},
\end{aligned} \tag{2.18}$$

because $I_{\overline{S}(m)} \mathcal{R}_m = \big(1 - I_{\overline{Q}(m,m)}\big)\mathcal{R}_m \le \mathcal{R}_m - 2 I_{\overline{Q}(m,m)}$.

Let us estimate each term on the right-hand side of (2.18). Using Lemma 1.4 and the fact that $R((m-1)T) \le \rho_0$ for $\mathbf{\Theta}^{m-1} \in S_+(m-1)$, we derive

$$\mathbb{E}^m\big\{\mathcal{R}_m\big\} \le \varepsilon\, e^{-2\alpha_1 T} \rho_0 + \varepsilon C_0 + 2. \tag{2.19}$$

Furthermore, in view of (2.9), we have

$$\mathbb{P}^m\big\{\overline{Q}(m,m)\big\} \ge p_0. \tag{2.20}$$

We now note that $f_{m-1}(\mathbf{\Theta}^{m-1}) \ge 2$ for $\mathbf{\Theta}^{m-1} \in S_+(m-1)$. Combining this with (2.19) and (2.20), we see that inequality (2.17) holds if

$$\varepsilon\,(e^{-2\alpha_1 T} \rho_0 + C_0) + 2 - p_0 \le 2\varkappa.$$

The latter is satisfied if we choose

$$\varkappa \ge \varkappa_1 := 1 - p_0/4, \quad \varepsilon \le \varepsilon_1 := \frac{p_0}{2(\rho_0 e^{-2\alpha_1 T} + C_0)}. \tag{2.21}$$

2) Let us prove (2.17) for $\mathbf{\Theta}^{m-1} \in S_-(m-1)$. Lemma 1.4 implies that

$$\mathbb{E}^m f_m(\mathbf{\Theta}^m) \le \mathbb{E}^m\big\{\mathcal{R}_m\big\} \le \varepsilon\, e^{-2\alpha_1 T} R((m-1)T) + \varepsilon C_0 + 2.$$

Taking into account the fact that $R((m-1)T) \geq \rho_0$ for $\mathbf{\Theta}^{m-1} \in S_-(m-1)$, we conclude that inequality (2.17) with $\varkappa = 3/4$ holds if

$$\varepsilon \geq \varepsilon_{(2)} := \frac{4}{3\rho_0 - 8C_0}, \tag{2.22}$$

provided that $e^{-2\alpha_1 T} \leq 3/8$, i.e., $T \geq T_{(2)}$, and $\rho_0 > 8C_0/3$.

3) It remains to establish (2.16). Abbreviating $\overline{Q}(l, m-1)$ to $\overline{Q}$, we note that

$$\begin{aligned} F_m^l(\mathbf{\Theta}^m) &\leq 2^{-(m-l)}\mathbb{E}\big\{I_{\overline{Q}}\, I_{\overline{Q}(l,m)}\big\} + \mathbb{E}\big\{I_{\overline{Q}}\, I_{\overline{S}(m)}\mathcal{R}_m\big\} \\ &\leq 2^{-(m-l)}\mathbb{P}(\overline{Q}) + \mathbb{E}\big\{I_{\overline{Q}\cap\overline{S}(m)}\mathcal{R}_m\big\}. \end{aligned} \tag{2.23}$$

Let us denote the second term on the right-hand side of (2.23) by E. Then to prove (2.16) with $\varkappa = 3/4$, we have to check that

$$E \leq 2^{-(m-l+1)}\mathbb{P}(\overline{Q}). \tag{2.24}$$

If $\mathbb{P}(\overline{Q}) = 0$, then the inequality holds trivially. Assuming that $\mathbb{P}(\overline{Q}) \neq 0$, we denote by $\overline{\mathbb{P}}$ the conditional probability on $\overline{Q}$, $\overline{\mathbb{P}}(A) = \mathbb{P}(\overline{Q} \cap A)/\mathbb{P}(\overline{Q})$, and by $\overline{\mathcal{F}}$ the σ-algebra of measurable subsets of $\overline{Q}$. For $t \in J_m = [(m-1)T, mT]$ the processes $u_1(t)$ and $u_2(t)$ (which are two out of the four components of Θ^m) depend on $(\overline{\omega}, \omega^m) \in \overline{Q}\times\Omega^m$, while increments of the processes ζ_1 and ζ_2 depend on ω^m. For $i = 1, 2$ and $t \in J_m$, let us denote by $\mathcal{F}_t^i$ the σ-algebra in $\overline{Q} \times \Omega^m$ generated by $\overline{\mathcal{F}}$ and the random variables $\zeta_i(s) - \zeta_i((m-1)T)$, $(m-1)T \leq s \leq t$. Then $u_i(t)$, $t \in J_m$, is a Markov process with respect to the filtration $\{\mathcal{F}_t^i\}$. To estimate E, we introduce a Markov time σ^i with respect to $\mathcal{F}_t^i$, $i = 1, 2$, by the formula

$$\sigma^i = \min\big\{t \in J_m : \mathcal{E}_i(t, lT) \geq \rho + (B_0 + 1)(t - lT)\big\},$$

where $\sigma^i = mT$ if the set $\{\cdots\}$ is empty, and $\mathcal{E}_i(t,s)$ is defined by (2.5). For $i = 1, 2$, we have $\overline{Q} \cap \overline{S}(m) = S_1^i \cup S_2^i$, where

$$S_1^i := \overline{Q} \cap \{(m-1)T \leq \sigma^i < mT\}, \quad S_2^i := \overline{Q} \cap \{\sigma^i = mT\} \cap \overline{S}(m).$$

The sets S_1^i and S_2^i do not intersect, and therefore $I_{\overline{Q}\cap\overline{S}(m)} = I_{S_1^i} + I_{S_2^i}$. If $\omega \in S_2^i$, then $|u_i(mT)|^2 \leq K'_{lm} = \rho + (B_0 + 1)(m - l)T$. Hence, denoting by $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{E}}$ the probability and the expectation corresponding to the probability space $\overline{Q} \times \Omega^m$, we have

$$\widehat{\mathbb{E}}\big\{I_{S_2^i}|u_i(mT)|^2\big\} \leq K'_{lm}\widehat{\mathbb{P}}\{S_2^i\}. \tag{2.25}$$

Furthermore, since S_1^i belongs to $\mathcal{F}_{\sigma^i}$, then using the strong Markov property and Lemma 1.4 with $u_1 = u_2$, we derive

$$\widehat{\mathbb{E}}\big\{I_{S_1^i}|u_i(mT)|^2\big\} = \widehat{\mathbb{E}}\big\{I_{S_1^i}\widehat{\mathbb{E}}\big(|u_i(mT)|^2 \,\big|\, \mathcal{F}_{\sigma^i}\big)\big\} \leq K_{lm}\widehat{\mathbb{P}}\{S_1^i\}, \tag{2.26}$$

where $K_{lm} = K'_{lm} + \frac{C_0}{2}$. Due to (2.25) and (2.26), we have

$$E \le (\varepsilon K_{lm} + 1)\big(\mathbb{P}(S_1^1) + \mathbb{P}(S_2^1) + \mathbb{P}(S_1^2) + \mathbb{P}(S_2^2)\big)$$
$$= 2(\varepsilon K_{lm} + 1)\mathbb{P}\big(\overline{Q} \cap \overline{S}(m)\big) = (2\varepsilon K_{lm} + 2)\mathbb{P}\big(\overline{Q} \cap \overline{Q}(l,m)^c\big).$$

Therefore (2.24) holds if

$$(2\varepsilon K_{lm} + 2)\mathbb{P}\big(\overline{Q} \cap \overline{Q}(l,m)^c\big) \le 2^{-(m-l+1)}\mathbb{P}\big(\overline{Q}\big).$$

Since $\mathbb{P}\big(\overline{Q} \cap \overline{Q}(l,m)^c\big)$ is equal to the left-hand side of (2.8), this relation is fulfilled if

$$c\, e^{-\gamma_0 \rho} e^{-\gamma_1 T(m-l-1)}\big((2\rho + 2T(B_0+1)(m-l) + C_0)\varepsilon + 2\big) \le 2^{-(m-l+1)}.$$

Denoting $m - l - 1 = r$, we rewrite this inequality as

$$c\, e^{-\gamma_0 \rho} e^{-r(\gamma_1 T - \ln 2)}\big(2\varepsilon(\rho + T(B_0+1)(r+1)) + C_0 + 2\big) \le \tfrac{1}{4}. \tag{2.27}$$

Considering separately the cases $r = 0$ and $r \ge 1$, we see that (2.27) holds for all r and any $T \ge \frac{\ln 2 + 1}{\gamma_1} =: T_{(3)}$ if

$$\rho \ge \rho_{(3)} \ln T, \quad \varepsilon \le \varepsilon_{(3)}. \tag{2.28}$$

We have thus shown that the required inequalities (2.15) and (2.16) hold under the conditions (2.21), (2.22), and (2.28). These conditions are compatible for any $T \ge T_{(2)} \vee T_{(3)}$, provided that ρ_0 is large enough. Indeed, since $T \ge T_{(2)}$, we have $e^{-2\alpha_1 T} \le 3/8$. Therefore $\varepsilon_{(2)} < \varepsilon_1 < \varepsilon_{(3)}$ if $\rho_0 \ge \rho_0(p_0)$. Choosing $\varepsilon = \varepsilon_1$, we see that the conditions above hold if $\varkappa = \varkappa_1$ and

$$\rho_0 \ge \rho_0(p_0), \quad T \ge T_{(2)} \vee T_{(3)}, \quad \rho \ge \rho_{(3)} \ln T. \tag{2.29}$$

It remains to note that these restrictions are consistent with the assumption $T \ge T(\rho, \rho_0)$, where $T(\rho, \rho_0)$ is given in (2.10), if ρ and ρ_0 are large enough. The proof of Theorem 2.2 is complete. $\square$

2.3 Exponential convergence of the transition function

Let $L^\alpha(H)$, $\alpha \in (0, 1]$, be the space of real-valued bounded Hölder continuous functions on H. We endow $L^\alpha(H)$ with the natural norm

$$\|g\|_{L^\alpha} := \sup_{u \in H} |g(u)| + \sup_{u \ne v} \frac{|g(u) - g(v)|}{|u - v|^\alpha}.$$

Let $\|\cdot\|^*_{L^\alpha}$ be the dual norm on the space of signed measures on $(H, \mathcal{B}(H))$:

$$\|\mu\|^*_{L^\alpha} = \sup\big|(\mu, g)\big|,$$

where the supremum is taken over all functions $g \in L^\alpha(H)$ such that $\|g\|_{L^\alpha} \le 1$. In the case $\alpha = 1$ we shall omit the corresponding superscript.

The space $\mathcal{P}(H)$ of probability Borel measures on H is complete with respect to the distance defined by $\|\cdot\|_{L^\alpha}^*$. Indeed, in the case $\alpha = 1$ this assertion is proved in [Dud02]. In view of the inclusion $L(H) \subset L^\alpha(H) \subset C_b(H)$ and the equivalence of the weak* convergence and the topology defined by $\|\cdot\|_L^*$ (see [Dud02]), the topologies for all metrics $\|\cdot\|_{L^\alpha}^*$, $\alpha \in (0,1]$, coincide. This implies the required assertion.

We recall that Markov semigroups $\mathfrak{P}_t : C_b(H) \to C_b(H)$ and $\mathfrak{P}_t^* : \mathcal{P}(H) \to \mathcal{P}(H)$ corresponding to the transition function $P_t(u,\Gamma)$ are given by the formulas

$$\mathfrak{P}_t f(u) = \int_H P_t(u,dv) f(v), \quad \mathfrak{P}_t^* \mu(\Gamma) = \int_H P_t(v,\Gamma)\mu(dv).$$

Let $\mathcal{P}_2(H)$ be the set of measures $\mu \in \mathcal{P}(H)$ with finite second moment $\mathfrak{m}_2(\mu) := \int_H |u|^2 \mu(du)$. We now use Theorems 2.1 and 2.2 to establish the following result:

Theorem 2.3. *There are positive constants C and σ such that for any $\alpha \in (0,1]$ and any initial measures $\lambda_i \in \mathcal{P}_2(H)$, $i = 1,2$, we have*

$$\|\mathfrak{P}_t^*\lambda_1 - \mathfrak{P}_t^*\lambda_2\|_{L^\alpha}^* \le C\left(1 + \mathfrak{m}_2(\lambda_1) + \mathfrak{m}_2(\lambda_2)\right)e^{-\alpha\sigma t}. \quad t \ge 0, \tag{2.30}$$

Moreover, there is a stationary measure $\mu \in \mathcal{P}_2(H)$ such that

$$\|\mathfrak{P}_t^*\lambda - \mu\|_{L^\alpha}^* \le C\left(1 + \mathfrak{m}_2(\lambda)\right)e^{-\alpha\sigma t}, \quad t \ge 0, \quad \lambda \in \mathcal{P}_2(H). \tag{2.31}$$

Corollary 2.4. *For any $u \in H$, $\alpha \in (0,1]$ and $t \ge 0$ we have the inequality $\|P_t(u,\cdot) - \mu\|_{L^\alpha}^* \le C\big(1 + |u|^2\big)e^{-\alpha\sigma t}$, where the constants C and σ are defined in Theorem 2.3.*

This assertion follows immediately from inequality (2.31) in which λ is the δ-measure concentrated at the point u.

Corollary 2.5. *The NS system has a unique stationary measure $\mu \in \mathcal{P}(H)$.*

Indeed, the existence is established in Theorem 2.3. Furthermore, as is shown in [EMS01], any stationary measure has a finite second moment. Passing to the limit in (2.31) as $t \to \infty$, we see that, if λ is a stationary measure, then it must coincide with μ.

Corollaries 2.4 and 2.5 imply the Main Theorem stated in the Introduction.

Proof of Theorem 2.3. The existence of a limiting measure and inequality (2.31) follow easily from estimate (2.30) and the completeness of $\mathcal{P}(H)$ (cf. [KS01a, Lemma 1.2]). Therefore, we confine ourselves to the proof of (2.30).

Step 1. We fix arbitrary $t > 0$ and $\alpha \in (0,1]$. Let $k = k(t)$ be the smallest integer such that $t \le kT$, where T is the constant in Theorem 2.2, and let u_i^0, $i = 1,2$, be random variables in H with distribution λ_i. We denote by $u_1(t)$ and $u_2(t)$, $0 \le t \le kT$, the weak solutions of the NS system as in Theorem 2.2. Inequality (2.30) will be proved if we show that (cf. [KS01a, Lemma 1.3])

$$p(t) := \mathbb{P}\big\{|u_1(t) - u_2(t)| > C_1 e^{-\sigma t}\big\} \le C_1 e^{-\sigma t}\big(1 + \mathfrak{m}_2(\lambda_1) + \mathfrak{m}_2(\lambda_2)\big), \tag{2.32}$$

where $C_1 > 0$ is a constant not depending on the initial functions.

Step 2. Let $c \in (0,1)$ be such that $\ln \varkappa^{-1} \geq (1-c)\ln 4$, where $\varkappa$ is the constant in (2.30). We define the event

$$\overline{G}(k) = \left\{ \mathbf{\Theta}^k = (\mathbf{\Theta}_1^k, \mathbf{\Theta}_2^k) \in G(k) \right\}, \quad G(k) = \bigcup_{l=0}^{[ck]} Q(l,k),$$

where $[s]$ denotes the integer part of s and $\mathbf{\Theta}_i^k = \big((u_i(t), \zeta_i(t)), 0 \leq t \leq kT\big)$. Clearly,

$$p(t) \leq \mathbb{P}\big(\overline{G}(k)^c\big) + \mathbb{P}\Big(\overline{G}(k) \cap \big\{|u_1(t) - u_2(t)| > C_1 e^{-\sigma t}\big\}\Big).$$

We shall show that

$$\mathbb{P}\big(\overline{G}(k)^c\big) \leq e^{-\sigma t}\big(2 + \mathfrak{m}_2(\lambda_1) + \mathfrak{m}_2(\lambda_2)\big), \tag{2.33}$$

$$\mathbb{P}\Big(\overline{G}(k) \cap \big\{|u_1(t) - u_2(t)| > C_1 e^{-\sigma t}\big\}\Big) = 0, \tag{2.34}$$

where $C_1 > 0$ is sufficiently large. Then (2.32) would follow.

Step 3. We first prove (2.33). In view of (2.14) and the definition of the functional F_k, we have

$$\sum_{l=0}^{k} 2^{l-k}\mathbb{P}\big\{\mathbf{\Theta}^k \in Q(l,k)\big\} + 2\mathbb{P}\big\{\mathbf{\Theta}^k \in S(k)\big\} \leq \varkappa^k\big(2 + \mathfrak{m}_2(\lambda_1) + \mathfrak{m}_2(\lambda_2)\big). \tag{2.35}$$

Since $\overline{G}(k)^c$ is contained in the event $\big\{\mathbf{\Theta}^k \in \big(\bigcup_{l=[ck]+1}^{k} Q(l,k)\big) \cup S(k)\big\}$, it follows that

$$\begin{aligned}\mathbb{P}\big(\overline{G}(k)^c\big) &\leq e^{(\ln \varkappa + (1-c)\ln 2)k}\big(2 + \mathfrak{m}_2(\lambda_1) + \mathfrak{m}_2(\lambda_2)\big) \\ &\leq e^{-\sigma k T}\big(2 + \mathfrak{m}_2(\lambda_1) + \mathfrak{m}_2(\lambda_2)\big),\end{aligned} \tag{2.36}$$

where $\sigma = (1-c)T^{-1}\ln 2$. Recalling that $k \geq t/T$, we see that (2.36) implies (2.33).

Step 4. It remains to establish (2.34). We claim that, if $\omega \in \overline{G}(k)$ and C_1 is sufficiently large, then

$$|u_1(t) - u_2(t)| \leq C_1 e^{-\sigma t}. \tag{2.37}$$

Indeed, by the definition of the set $G(k)$, if $\mathbf{\Theta}^k \in G(k)$, then there is an integer l, $0 \leq l \leq [ck]$, such that the relations (2.1)–(2.3) are satisfied and

$$\int_{lT}^{s} \|u_1(r)\|^2 dr \leq \rho + (B_0 + 1)(s - lT), \quad lT \leq s \leq kT,$$

where ζ_1 and ζ_2 are the right-hand sides corresponding to u_1 and u_2, respectively. Therefore, in view of Proposition 4.1 with $M = \sigma$, for $u = u_1 - u_2$ we have the estimate

$$|u(t)| = |w(t)| \leq 2\, d \exp\big(C\rho - \sigma(t - lT)\big), \tag{2.38}$$

where $w = \mathsf{Q}_N u$. We now note that $lT \le ckT \le c(t+T)$ and therefore $t - lT \ge (1-c)t - cT$. Hence, $|u(t)| \le 2\,d\,e^{cT+C\rho}e^{-\sigma t}$. This coincides with inequality (2.37), where $C_1 = 2\,d\,e^{cT+C\rho}$. The proof of Theorem 2.3 is complete. $\square$

When proving Theorem 2.3, we established the following assertion: there are positive constants C_1 and σ such that, for any $t \ge 0$,

$$\mathbb{P}\bigl\{|u_1(t) - u_2(t)| \le C_1 e^{-\sigma t}\bigr\} \ge 1 - C_1\bigl(1 + \mathfrak{m}_2(\lambda_1) + \mathfrak{m}_2(\lambda_2)\bigr)e^{-\sigma t}. \qquad (2.39)$$

In particular, the processes $u_1(t)$ and $u_2(t)$ converge exponentially fast (as $t \to \infty$) in probability. In fact, they converge almost surely as well. This result is important for some applications, and we prove it now.

Iterating infinitely the construction described at the beginning of Subsection 2.2, we get the process $U(t) = (u_1(t), u_2(t))$, $t \ge 0$. Its components u_1 and u_2 are weak solutions of (1.1) defined on the probability space $\Omega = \Omega^1 \times \Omega^2 \times \cdots$. For $m \ge 1$ we denote by $\Pi_m : \Omega \to \Omega^1 \times \Omega^2 \times \cdots \times \Omega^m$ the natural projection and for $0 \le l \le m$ we set

$$\widehat{Q}(l,m) = \Pi_m^{-1}\overline{Q}(l,m), \quad \widehat{Q}(m) = \bigcup_{l \le m} \widehat{Q}(l,m) \subset \Omega.$$

Then $\widehat{Q}(0) \subset \widehat{Q}(1) \subset \cdots$ and $\widehat{Q}(m)^c = \Pi_m^{-1}\overline{S}(m)$. Due to (2.35),

$$\mathbb{P}\bigl(\widehat{Q}(m)^c\bigr) \le \varkappa^k\bigl(2 + \mathfrak{m}_2(\lambda_1) + \mathfrak{m}_2(\lambda_2)\bigr).$$

Hence, $\widehat{Q} = \cup_m \widehat{Q}(m)$ is an event of full measure. For $\omega \in \widehat{Q}$ let $m(\omega)$ be the smallest integer such that $\omega \in \widehat{Q}(m)$. Due to (2.38), for $t \ge T' = m(\omega)T$ we have

$$|u_1(t) - u_2(t)| \le 2\,d\,e^{C\rho}e^{-\sigma(t-T')}. \qquad (2.40)$$

We have proved the following result:

Proposition 2.6. *Let λ_1 and λ_2 be any two measures from $\mathcal{P}_2(\mathcal{H})$. Then there exists a random variable $T' \ge 0$ which is finite almost surely and weak solutions $u_1(t)$ and $u_2(t)$, $t \ge 0$, of Eq.* (1.1) *such that $\mathcal{D}(u_i(0)) = \lambda_i$, $i = 1,2$, and inequality* (2.40) *holds for $t \ge T'$.*

3 Proof of Theorem 2.1

3.1 Theorem on isomorphism

In this subsection, we show that the NS system is isomorphic (in an appropriate sense) to an auxiliary problem with trivial dynamics in high Fourier modes. A similar result is used in [KS00, KS01b] in the case of a kick force.

Let us set

$$v = \mathsf{P}_N u, \quad w = \mathsf{Q}_N u, \quad \varphi = \mathsf{P}_N \zeta, \quad \psi = \mathsf{Q}_N \zeta. \qquad (3.1)$$

Applying the projections P_N and Q_N to the NS system (1.1), we write it in the following equivalent form:

$$\dot{v} + Lv + \mathsf{P}_N B(v+w) = \dot{\varphi}(t), \tag{3.2}$$

$$\dot{w} + Lw + \mathsf{Q}_N B(v+w) = \dot{\psi}(t), \tag{3.3}$$

where $B(u) = B(u,u)$. Let us supplement Eqs. (3.2), (3.3) with the initial conditions

$$v(0) = v^0, \tag{3.4}$$

$$w(0) = w^0, \tag{3.5}$$

and fix an arbitrary $T > 0$. The theory of deterministic NS equations implies that for any $v^0 \in H_N$, $w^0 \in H_N^\perp$, $\varphi \in C(0,T;H_N)$, and $\psi \in C(0,T;V \cap H_N^\perp)$ the problem (3.2) – (3.5) has a unique solution (v,w), $v \in C(0,T;H_N)$, $w \in \mathcal{H}_N^\perp(0,T) := C(0,T;H_N^\perp) \cap L^2(0,T;V \cap H_N^\perp)$ (e.g., see [Lio69]).

Let us now assume that $v \in C(0,T;H_N)$ and $\psi \in C(0,T;V \cap H_N^\perp)$ are given deterministic functions. In this case, we can regard (3.3) as an equation for w.

Lemma 3.1. *For any v, ψ as above and any $w^0 \in H_N^\perp$, the problem* (3.3), (3.5) *has a unique solution $w \in \mathcal{H}_N^\perp(0,T)$, and the associated resolving operator $\mathcal{W}\colon (v,\psi,w^0) \mapsto w$ regarded as a map from $C(0,T;H_N) \times C(0,T;V \cap H_N^\perp) \times H_N^\perp$ to $\mathcal{H}_N^\perp(0,T)$ is continuous. Furthermore, the function $w(t)$ does not depend on $v(s)$ and $\psi(s)$, $s > t$.*

Proof. The proof is based on standard arguments, and therefore we only outline it. We seek the solution in the form $w = \psi + w'$. Substitution of this expression into (3.3) and (3.5) results in the following problem for the function w':

$$\dot{w}' + Lw' + \mathsf{Q}_N B(v + \psi + w') = 0, \quad w'(0) = w^0 - \psi(0).$$

The unique solvability of this problem and the continuity of the associated resolving operator can be proved using well-known methods of the theory for deterministic NS equations (e.g, see [Lio69, Chapter I]). This implies the required assertion on unique solvability of the original problem. The last statement of the lemma is obvious. $\square$

In what follows, we shall use the notations $\boldsymbol{v}_t = (v(s), 0 \le s \le t)$, $\boldsymbol{\psi}_t = (\psi(s), 0 \le s \le t)$, and $\mathcal{W}_t(\boldsymbol{v}_t, \boldsymbol{\psi}_t, w^0) = w(t)$, where $w = \mathcal{W}(v,\psi,w^0)$.

Along with (3.2), (3.3), let us consider the system

$$\dot{v} + Lv + \mathsf{P}_N B(v + \mathcal{W}_t(\boldsymbol{v}_t, \boldsymbol{a}_t, w^0)) = \dot{\varphi}(t), \tag{3.6}$$

$$\dot{a} = \dot{\psi}(t). \tag{3.7}$$

We claim that for any $v_0 \in H_N$ and $w_0 \in H_N^\perp$ the problem (3.6), (3.7), (3.4) has a unique solution (v,a), $v \in C(0,T;H_N)$, $a \in C(0,T;V \cap H_N^\perp)$, such that

$$a(0) = \psi(0). \tag{3.8}$$

Indeed, let us fix an arbitrary pair $(v^0, w^0) \in H_N \times H_N^\perp$ and denote by (v, w) the unique solution of (3.2) – (3.5). It follows from the definition of the operator $\mathcal{W}_t$ that (v, ψ) is a solution for (3.6) – (3.8), (3.4). This implies the existence of a solution. To prove the uniqueness, assume that (v, a) is a solution of (3.6) – (3.8), (3.4). It follows from (3.7), (3.8) that $a(t) = \psi(t)$, and therefore the pair $(v, w = \mathcal{W}_t(\boldsymbol{v}_t, \boldsymbol{\psi}_t, w^0))$ satisfies (3.2) – (3.5). So , by virtue of the uniqueness for the problem (3.2) – (3.5), the function $v(t)$ is uniquely defined.

The above arguments show that the systems (3.2), (3.3) and (3.6), (3.7) are equivalent. Namely, let us fix $w^0 \in H_N^\perp$ and introduce the operators

$$\Phi(w^0;\,\cdot\,) : (v, a) \mapsto \big(v, \mathcal{W}_t(\boldsymbol{v}_t, \boldsymbol{a}_t, w^0)\big), \tag{3.9}$$

$$\Psi(w^0;\,\cdot\,) : (v, w) \mapsto \Big(v, \psi(0) + w(t) - w^0 - \int_0^t \big(Lw + \mathsf{Q}_N B(v + w)\big)\, ds\Big). \tag{3.10}$$

It is easy to see that the map $\Phi(w^0)$ is continuous from the space $C(0, T; H_N) \times C(0, T; V \cap H_N^\perp)$ to $C(0, T; H_N) \times \mathcal{H}_N^\perp(0, T)$, and $\Psi(w^0)$ is continuous from $C(0, T; H_N) \times L^2(0, T; V \cap H_N^\perp)$ to $C(0, T; H_N) \times L^2(0, T; V^*)$, where V^* is the adjoint space for V. What has been said implies that (v, a) is a solution of (3.6) – (3.8), (3.4) if and only if $\Phi(w^0; v, a)$ satisfies (3.2) – (3.5) and that (v, w) is a solution of (3.2) – (3.5) if and only if $\Psi(w^0; v, w)$ satisfies (3.6) – (3.8), (3.4).

The following theorem establishes the equivalence of the systems (3.2), (3.3) and (3.6), (3.7) in the stochastic case. Its proof is an obvious consequence of the above-mentioned properties of the operators $\Phi(w^0)$ and $\Psi(w^0)$.

Theorem 3.2. *Suppose that $\varphi(t)$ and $\psi(t)$ are the projections of the process $\zeta(t, x)$ to the subspaces H_N and $H_N^\perp$, respectively* (*see* (3.1))*. Then a pair of processes (v, a) is a weak solution of the problem* (3.6) – (3.8), (3.4) *if and only if $\Phi(w^0; v, a)$ is a weak solution for* (3.2) – (3.5)*. Similarly, the pair (v, w) is a weak solution of* (3.2) – (3.5) *if and only if $\Psi(w^0; v, w)$ is a weak solution for* (3.6) – (3.8), (3.4).

3.2 General scheme for constructing a coupling

To explain the scheme, let us assume that, for $i = 1, 2$ $\tilde{u}_i(t)$ is a weak solution for (1.1), defined for $-\tilde{t} \le t \le 0$ with some $-\tilde{t} < 0$, and that $\partial_t \tilde{\zeta}_i(t)$ is the corresponding right-hand side. For a fixed value of the random parameter, we denote $u_i^0 = \tilde{u}_i(0)$ and $\zeta_i^0 = \tilde{\zeta}_i(0)$, $i = 1, 2$. Below we construct a special pair of weak solutions for (1.1) with initial conditions u_1^0, u_2^0. They form a coupling for the pair of strong solutions with the same initial data.

Our construction depends on parameters $\theta \in (0, 1]$ and $\theta_2 \ge T_2(\theta)$, where θ is chosen in Subsection 3.4 and the function $T_2(\theta)$ is defined in Lemma 1.6. We set $T = \theta_2 + \theta$ and denote by μ_1 and μ_2 the measures generated on $C(0, T; H)$ by solutions of (1.1) starting from u_1^0 and u_2^0, respectively. Below we define a coupling $U_{1,2}(\omega, u_1^0, u_2^0)$ for the measures $\mu_{1,2}$, given by measurable functions of its arguments and valued in $C(0, T; H)$ (i.e., $U_i = U_i(t; \omega, u_1^0, u_2^0)$). In fact, the operators U_1, U_2 also depend on $Q_N\zeta_1^0$, but since the dependence on the last argument is rather irrelevant, we omit it from our notations.

We start with defining three coupling operators in the following three cases (which have non-empty intersection):

(a) (u_1^0, u_2^0) is an arbitrary pair of functions in H;

(b) the projections of u_1^0 and u_2^0 to H_N coincide: $\mathsf{P}_N u_1^0 = \mathsf{P}_N u_2^0$;

(c) $|u_1^0| \vee |u_2^0| \le \rho_0$, where $\rho_0 > 0$ is defined in Theorem 2.2.

The equation (1.1) will not change if we add a constant to the process ζ. Using this observation we renormilize ζ as follows:

$$\zeta(t) := \zeta(t) - \zeta(0) + \tilde{\zeta}_1(0). \tag{3.11}$$

Now $\zeta(0) = \tilde{\zeta}_1(0)$.

In the case (a), we choose the trivial coupling. Namely, let $u_i(t,x)$, $t \in [0,T]$, $i = 1,2$, be the solution of Eq. (1.1) starting from u_i^0. We set $U_i^a(t;\omega,u_i^0) = u_i(t)$ It is clear that U_i^a is a measurable function of (ω, u_i^0), and (U_1^a, U_2^a) is a coupling for (μ_1, μ_2).

We now consider the case (b). For $i = 1,2$, let us set

$$\lambda_i = \mathcal{D}\big(\mathsf{P}_N u_i(t), \mathsf{Q}_N\zeta(t), 0 \le t \le T\big), \tag{3.12}$$

where u_i is the solution of the problem (1.1), (1.2) with $u^0 = u_i^0$ (so λ_i is a measure on $C(0,T;H_N) \times C(0,T;V \cap H_N^{\perp})$). In other words, λ_i is the image of the measure μ_i under the mapping $\Psi(w_i^0)$, where $w_i^0 = \mathsf{Q}_N u_i^0$ and the operator $\Psi(w^0)$ is defined by (3.10). Let $(\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_2)$, $\boldsymbol{\Upsilon}_i = (\boldsymbol{v}_i, \boldsymbol{a}_i) = (v_i(t), a_i(t), 0 \le t \le T)$, be a maximal coupling for (λ_1, λ_2). The coupling $(\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_2)$ depends on the functional parameter $(u_1^0, u_2^0, Q_N\tilde{\zeta}_1(0))$. We can assume that it is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and is a measurable function of $(\omega, u_1^0, u_2^0, Q_N\tilde{\zeta}_1(0))$ $\in \Omega \times H^2 \times H_N^{\perp}$ (see Appendix in [KS01a] and references therein).

Let us set

$$U_i^b = \Phi(w_i^0; \boldsymbol{\Upsilon}_i) = v_i + \mathcal{W}(\boldsymbol{v}_i, \boldsymbol{a}_i, w_i^0).$$

It follows from Theorem 3.2 and the measurability of $(\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_2)$ that the processes $U_i^b = v_i + w_i$, $i = 1,2$, are weak solutions of (1.1), and the pair (U_1^b, U_2^b) is a measurable coupling for (μ_1, μ_2).

Finally, let us consider the case (c). We first define some auxiliary operators. We fix arbitrary initial functions u_i^0, $i = 1,2$, and a sufficiently small constant $\theta > 0$ and denote by $u_i(t)$, $0 \le t \le \theta$, a solution of (1.1), (1.2) starting from u_i^0. Let λ_1 and λ_2' be distributions [4] of the random variables $(\mathsf{P}_N u_1(t), \mathsf{Q}_N\zeta(t), 0 \le t \le \theta)$ and $(\mathsf{P}_N u_2'(t), \mathsf{Q}_N\zeta(t), 0 \le t \le \theta)$, respectively,

[4] Note that the measure λ_1 formally does not coincide with the one introduced for the case (b) since they are defined on different spaces. However, we use the same notation because their meaning is the same.

where $u_2'(t) := u_2(t) + \frac{\theta - t}{\theta}\mathsf{P}_N(u_1^0 - u_2^0)$. We note that $u_2'(\theta) = u_2(\theta)$ and $\mathsf{P}_N u_2'(0) = \mathsf{P}_N u_1^0$.

We now repeat the construction of the case (b). Namely, let $(\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_2')$, where $\boldsymbol{\Upsilon}_1 = (v_1(t), a_1(t), 0 \le t \le \theta)$ and $\boldsymbol{\Upsilon}_2' = (v_2'(t), a_2(t), 0 \le t \le \theta)$, be a maximal coupling for (λ_1, λ_2') that is defined on a probability space $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$ and depends on $(\omega^1, u_1^0, u_2^0) \in \Omega^1 \times H^2$ in a measurable manner. We define $v_2 := v_2' - \frac{\theta - t}{\theta}\mathsf{P}_N(u_1^0 - u_2^0)$, $\boldsymbol{v}_2 = (v_2(t), 0 \le t \le \theta)$, $\boldsymbol{\Upsilon}_2 = (v_2(t), a_2(t), 0 \le t \le \theta)$ and set

$$U_i = \Phi(w_i^0; \boldsymbol{\Upsilon}_i) = v_i + \mathcal{W}(\boldsymbol{v}_i, \boldsymbol{a}_i, w_i^0), \quad i = 1, 2,$$

where $w_i^0 = \mathsf{Q}_N u_i^0$. It is clear that (U_1, U_2) is a coupling for (μ_1, μ_2). Moreover, the construction implies that $\mathsf{P}_N U_1|_{t=\theta} = \mathsf{P}_N U_2|_{t=\theta}$ as soon as $\boldsymbol{\Upsilon}_1 = \boldsymbol{\Upsilon}_2'$.

We are now ready to define the coupling operators in the case (c). Assuming that the right-hand side in (1.1) is defined on a probability space Ω^0 independent of Ω^1, we set

$$U_i^c(t; \omega, u_1^0, u_2^0) = \begin{cases} u_i(t, x; \omega_0) & \text{for} \quad 0 \le t \le \theta_2, \\ U_i(t; \omega_1, u_1(\theta_2), u_2(\theta_2)) & \text{for} \quad \theta_2 \le t \le T, \end{cases} \tag{3.13}$$

where $T = \theta_2 + \theta$, $\omega = (\omega^0, \omega^1)$, and $u_i(t, x; \omega^0)$ is the solution of Eq. (1.1) starting from u_i^0. The Markov property implies that (U_1^c, U_2^c) is a coupling for the measures μ_1 and μ_2 (defined on the space $C(0, T; H)$). We note that for $t \in [0, \theta_2]$ we have $\zeta_1 = \zeta_2 = \zeta$, so the renormalization (3.11) of the process ζ for $t \ge \theta_2$ is trivial, the operators U_1, U_2 do not depend on $\zeta_1(\theta)$ and the processes ζ_1, ζ_2 are continuous for $t \in [0, T]$.

Let us use the above coupling operators to construct the weak solutions mentioned in Theorem 2.1. Let us denote

$$T_m = mT, \quad 0 \le m \le k.$$

We can assume that the operators U_i^a, U_i^b, and U_i^c are defined on the same probability space $(\Omega^k, \mathcal{F}^k, \mathbb{P}^k)$. Let us set $u_i(t; \omega', \omega^k) = \tilde{u}_i(t; \omega')$ for $0 \le t \le T_{k-1}$ and define $u_i = u_i(t, x)$ for $T_{k-1} \le t \le T_k$ by the formula

$$u_i = \begin{cases} U_i^b\big(t; \omega^k, u_1(T_{k-1}), u_2(T_{k-1})\big), & (\tilde{\boldsymbol{\Theta}}_1^{k-1}, \tilde{\boldsymbol{\Theta}}_2^{k-1}) \in Q(k-1), \\ U_i^c\big(t; \omega^k, u_1(T_{k-1}), u_2(T_{k-1})\big), & (\tilde{\boldsymbol{\Theta}}_1^{k-1}, \tilde{\boldsymbol{\Theta}}_2^{k-1}) \in S_+(k-1), \\ U_i^a\big(t; \omega^k, u_i(T_{k-1})\big), & \text{otherwise.} \end{cases} \tag{3.14}$$

The relation (2.7) holds trivially, so we only need to prove inequalities (2.8) – (2.10). To this end, we first establish some auxiliary assertion and then, in Subsection 3.4, we derive the required estimates.

3.3 Auxiliary lemmas

In this subsection, we establish some properties of distributions of solutions for the problem (1.1), (1.2). For the kick-forced NS system (0.1), (0.2), the results we need follow from explicit formulas in terms of iterated integrals (see Section 5.2 in [KS00]). For the white-forced case we are concerned with now, the explicit formulas which imply the desired results are given by an infinite-dimensional version of Girsanov's theorem. In the particular case when there is no noise in high Fourier modes (i.e., $b_j = 0$ for $j > N$), our arguments are related to those in [EMS01], and Lemmas 3.3 and 3.5 can be viewed as revised versions of the corresponding statements in [EMS01].

We begin with an estimate for the variational distance between the measures λ_1 and λ_2 defined in (3.12).

Lemma 3.3. *Let $T > 0$ and $\rho \geq 1$ be arbitrary constants and let $d \leq d_\rho := (3K_N)^{-1/2} e^{-\rho(C+\gamma_0)} \leq 1$, where C is the constant in (4.3) and $K_N \geq 1$ is a suitable constant depending only on N. Then for any initial functions u_1^0 and u_2^0 such that $\mathsf{P}_N u_1^0 = \mathsf{P}_N u_2^0$ and $|u_1^0| \vee |u_2^0| \leq d$ we have*

$$\|\lambda_1 - \lambda_2\|_{\mathrm{var}} \leq C_1 e^{-\gamma_0 \rho}\,, \qquad C_1 = 1 + 2e^{\gamma_0}\,. \tag{3.15}$$

Proof. Step 1. The random process $(\mathsf{P}_N u_i(t), \mathsf{Q}_N \zeta(t))$ is a solution of the system (3.6), (3.7) supplemented with the initial conditions (3.4), (3.8), where $v^0 = \mathsf{P}_N u_i^0$, $w^0 = w_i^0 = \mathsf{Q}_N u_i^0$. Along with (3.6), (3.7), let us consider the truncated systems

$$\dot v + Lv + \chi_i(t, \boldsymbol{v}_t, \boldsymbol{a}_t, w_i^0)\, B_N\big(v + \mathcal{W}_t(\boldsymbol{v}_t, \boldsymbol{a}_t, w_i^0)\big) = \dot\varphi(t), \tag{3.16}$$

$$\dot a = \dot\psi(t), \tag{3.17}$$

where $0 \leq t \leq T$, $B_N(u) = \mathsf{P}_N B(u,u)$, and the function χ_i is defined by the following rule: $\chi_i(t, \boldsymbol{v}_t, \boldsymbol{a}_t, w_i^0) = 1$ if (cf. (2.4))

$$|u_i'(t)|^2 + \int_0^s \|u_i'(r)\|^2 dr \leq \rho + (B_0+1)s \quad \text{for} \quad 0 \leq s \leq t, \tag{3.18}$$

where $u_i'(t) = v(t) + \mathcal{W}_t(\boldsymbol{v}_t, \boldsymbol{a}_t, w_i^0)$, and $\chi_i(t, \boldsymbol{v}_t, \boldsymbol{a}_t, w_i^0) = 0$ otherwise. We denote by $(z_i(t), a_i(t))$, $0 \leq t \leq T$, a solution of (3.16), (3.17) such that

$$z_i(0) = v^0, \quad a_i(0) = \psi(0). \tag{3.19}$$

The random process (z_i, a_i) is uniquely defined. Indeed, it follows from (3.17), (3.19) that $a_i(t) = \psi(t)$. Substituting this formula into (3.16), we obtain the finite-dimensional stochastic equation with a constant diffusion and a Lipschitz drift. Therefore, by Theorem 4.6 in [LS77], it has a unique strong solution satisfying the initial condition (3.19). We also note that, since the noise in Eqs. (3.16), (3.17) is additive, its solutions can be treated pathwise.

We set $\boldsymbol{z}_i(t) = (z_i(s), 0 \leq s \leq t)$ and define $\boldsymbol{u}_i(t)$ and $\boldsymbol{a}_i(t) \equiv \boldsymbol{a}_t$ in a similar way. If $\chi(t, \boldsymbol{z}_i(t), \boldsymbol{a}_t, w_i^0) = 1$ for $t \leq t'$, then $z_i(t) = \mathsf{P}_N u_i(t)$ for $t \leq t'$. Therefore, denoting by N_i the event $\{\boldsymbol{z}_i(T) \neq \mathsf{P}_N \boldsymbol{u}_i(T)\}$, we have

$$N_i = \big\{\mathcal{E}(t,0)(u_i) \geq (B_0+1)t + \rho \text{ for some } t \in [0,T]\big\} \tag{3.20}$$

(see (2.5)). Hence, in view of inequality (2.6) with $l = 0$ and $r = 1$, we have

$$\mathbb{P}(N_i) \le \Big\{ \sup_{t\in[0,T]} \big(\mathcal{E}(t,0)(u_i) - (B_0+1)t\big) \ge \rho \Big\} \le e^{-\gamma_0(\rho-1)}, \tag{3.21}$$

where $i = 1, 2$, and we used that $d \le 1$.

Let us denote by ν_i distribution of $(z_i(t), \mathsf{Q}_N\zeta(t), t \in [0,T])$. Then, due to (3.21), we have

$$\|\lambda_1 - \lambda_2\|_{\mathrm{var}} \le 2e^{-\gamma_0(\rho-1)} + \|\nu_1 - \nu_2\|_{\mathrm{var}}, \tag{3.22}$$

since $\|\lambda_i - \nu_i\|_{\mathrm{var}} \le \mathbb{P}\{\boldsymbol{z}_i(T) \ne \mathsf{P}_N \boldsymbol{u}_i(T)\}$. Thus, to bound $\|\lambda_1 - \lambda_2\|_{\mathrm{var}}$, we have to estimate the variational distance between the measures ν_1 and ν_2.

Step 2. We claim that the measures ν_1 and ν_2 are absolutely continuous with respect to each other and, moreover, we have the estimate

$$\int_{X_0} \left(\frac{d\nu_2}{d\nu_1}\right)^2 d\nu_1 \le \sqrt{M}, \tag{3.23}$$

where $X_0 = C(0,T;H)$ and $M = \exp(6K_N d^2 e^{2C\rho})$. Taking inequality (3.23) for granted, let us complete the proof of (3.15). We have

$$\|\nu_1 - \nu_2\|_{\mathrm{var}} = \frac{1}{2}\int_{X_0} \left|1 - \frac{d\nu_2}{d\nu_1}\right| d\nu_1 \le \frac{1}{2}\left(\int_{X_0} \left|1 - \frac{d\nu_2}{d\nu_1}\right|^2 d\nu_1\right)^{1/2} \le \tfrac{1}{2}\big(\sqrt{M} - 1\big)^{1/2}.$$

Hence, for $d \le d_\rho$ we obtain $\|\nu_1 - \nu_2\|_{\mathrm{var}} \le \frac{1}{2}\big(\exp(e^{-2\gamma_0\rho}) - 1\big)^{1/2} \le e^{-\gamma_0\rho}$. This estimate and (3.22) imply (3.15). Thus, it remains to establish the absolute continuity of the measures ν_1 and ν_2 and inequality (3.23). To this end, we use an infinite-dimensional variant of Girsanov's theorem.

Step 3. Let us set

$$\alpha(t,\omega) = \boldsymbol{b}^{-1}\Big(\chi_1\big(t, \boldsymbol{z}_1(t), \boldsymbol{a}_t, w_1^0\big)\big\{B_N\big(z_1 + \mathcal{W}_t(\boldsymbol{z}_1(t), \boldsymbol{a}_t, w_1^0)\big) - \\ - B_N\big(z_1 + \mathcal{W}_t(\boldsymbol{z}_1(t), \boldsymbol{a}_t, w_2^0)\big)\big\}\Big), \tag{3.24}$$

where $\boldsymbol{b}$ is the diagonal $N \times N$ matrix with elements b_j, $j = 1, \dots, N$, and $\boldsymbol{b}^{-1}$ is its inverse. As we show below, the function α is uniformly bounded:

$$\big|\alpha(t,\omega)\big|^2 \le K_N d^2 e^{2C\rho - 2t}, \tag{3.25}$$

where $K_N \ge 1$ is a constant depending only on N. It follows that

$$\mathbb{E}\, e^{6\int_0^T |\alpha(t,\omega)|^2 dt} \le M = e^{6K_N d^2 e^{2C\rho}} < \infty. \tag{3.26}$$

We claim that

$$\nu_2(d\boldsymbol{\Upsilon}) = e^{G(\boldsymbol{\Upsilon})}\nu_1(d\boldsymbol{\Upsilon}), \tag{3.27}$$

where $\boldsymbol{\Upsilon} = (\boldsymbol{v}, \boldsymbol{a}) \in C(0,T; H_N \times H_N^\perp)$ and

$$G(\boldsymbol{\Upsilon}) = -\int_0^T \big(\alpha, \boldsymbol{b}^{-1} d\varphi\big) - \frac{1}{2}\int_0^T |\alpha|^2 dt. \tag{3.28}$$

If the system (3.16), (3.17) was finite-dimensional, the above assertion would follow from Theorem 7.18 in [LS77]. [5] In our situation, formulas (3.27) and (3.28) are obtained by a reduction to the finite-dimensional case (see Subsection 4.2 in Appendix).

We can now complete the proof of (3.23). In view of (3.28), the left-hand side of (3.23) is equal to

$$\begin{aligned}\mathbb{E}\exp\Big(-2\int_0^T \big(\alpha, \boldsymbol{b}^{-1} d\varphi\big) - \int_0^T |\alpha|^2 dt\Big)& \\ \le \bigg(\mathbb{E}\exp\Big(-4\int_0^T \big(\alpha, \boldsymbol{b}^{-1} d\varphi\big) - 8\int_0^T |\alpha|^2 dt\Big)\bigg)^{1/2} &\times \\ \times \bigg(\mathbb{E}\exp\Big(6\int_0^T |\alpha|^2 dt\Big)\bigg)^{1/2} &\le \sqrt{M},\end{aligned}$$

where we used (3.26) and the fact that the process $\exp\big(-4\int_0^T(\alpha, \boldsymbol{b}^{-1} d\varphi) - 8\int_0^T |\alpha|^2 dt\big)$ is a supermartingale.

To prove (3.25), we use the Foiaş–Prodi estimate (see Proposition 4.2). By construction, the function $w_i = \mathcal{W}_t(\boldsymbol{z}_1(t), \boldsymbol{a}_t, w_i^0)$ satisfies Eq. (4.7) with $v = z_1$ and $h = \dot{a}$, as well as the initial condition $w_i(0) = w_i^0$. Therefore, if N is sufficiently large, then, by (4.3), we have

$$|w_1(t) - w_2(t)| \le 2d\, e^{-t + C\rho} \tag{3.29}$$

for $0 \le t \le T$, whence follows (3.25). □

We now establish some estimates for the variational distance between distributions of the processes $\Upsilon_i(t) = (\mathsf{P}_N u_i(t), \mathsf{Q}_N \zeta_i(t))$, $i = 1, 2$, on the interval $J_k = [T_{k-1}, T_k]$, where $u_i = u_i(t; \omega', \omega_k)$ are the weak solutions for (1.1) defined by (3.14) and $\zeta_i(t)$ are the corresponding right-hand sides. Namely, let us fix $\omega' \in \Omega'$ and denote by $\lambda_i(\omega')$ the measure generated by $(\Upsilon_i(t; \omega', \omega_k), t \in J_k)$ on the space $C(J_k; H_N) \times C(J_k; V \cap H_N^\perp)$. Recall that the event $\overline{Q} = \overline{Q}(l, k-1)$ depends on the parameters $\rho \ge 1$ and $d \in (0, 1]$.

Lemma 3.4. *Let d_ρ be the constant defined in Lemma 3.3. Then, for sufficiently large $\rho > 0$ and $T \ge 1$ and for any $d \in (0, d_\rho]$ and $0 \le l \le k-1$, we have*

$$\int_{\overline{Q}} \big\|\lambda_1(\omega') - \lambda_2(\omega')\big\|_{\mathrm{var}} \mathbb{P}'(d\omega') \le C_2 e^{-\gamma_0 \rho} e^{-\gamma_1 T(k-l-1)} \mathbb{P}'(\overline{Q}), \tag{3.30}$$

where $C_2 = 1 + 4e^{\gamma_0}$ and $\gamma_1 = \gamma_0 \wedge 1$.

[5] Girsanov's theorems presented in the less technical book [Øk98] are "almost sufficient" for our purposes.

Proof. For any $\omega' \in \overline{Q}$, let $y_i(t;\omega')$, $t \in J_k$, be a strong solution for (1.1) that is equal to $u_i(T_{k-1};\omega')$ for $t = T_{k-1}$. This solution depends on the random parameter $\omega \in \Omega$, independent of ω'. Distribution of $y_i(t;\omega')$ on the interval J_k coincides with that of $u_i(t;\omega')$. For $t < T_{k-1}$ we define $y_i(t;\omega') = u_i(t;\omega')$. We also set $x_i(t;\omega',\omega) = \mathsf{P}_N y_i(t;\omega',\omega)$. Let us note that due to the definition of the set $\overline{Q}$ and the renormalization (3.11) we have

$$\mathsf{Q}_N\zeta_1(T_{k-1}-0) = \mathsf{Q}_N\zeta_2(T_{k-1}-0) = \mathsf{Q}_N\zeta(T_{k-1}) \quad \text{if} \quad k \geq 2. \tag{3.31}$$

Step 1. We first consider the case $l = 0$. The proof of (3.30) is by induction on k. Abbreviating $\overline{Q}(0,k)$ to $\overline{Q}_k$, we shall show that inequality (3.30) holds together with the estimate

$$\mathbb{P}(\overline{Q}_k) \geq 1 - \tfrac{1}{2}(1 - 3^{-k}), \quad k \geq 0, \tag{3.32}$$

provided that the initial functions u_1^0 and u_2^0 satisfy

$$\mathsf{P}_N u_1^0 = \mathsf{P}_N u_2^0\,, \qquad |u_1^0| \vee |u_2^0| \leq d. \tag{3.33}$$

For $k = 1$ inequality (3.30) coincides with (3.15), and for $k = 0$ (3.32) follows from (3.33).

Let us assume that for $k = m - 1 \geq 0$ the required assertions are established and prove them for $k = m$. If $m = 1$, then the Step *i)* should be omitted.

i) (proof of (3.30)). The arguments below almost literally repeat the derivation of (3.15), and therefore we only outline them. The random process $(x_i(t), \mathsf{Q}_N\zeta(t))$ is the solution of the system (3.6), (3.7) (with segment $[0,T]$ replaced by J_m), supplemented with the initial conditions (3.4), (3.8), where

$$v^0 = \mathsf{P}_N u_i(T_{m-1}), \quad w^0 = w_i^0 = \mathsf{Q}_N u_i(T_{m-1}). \tag{3.34}$$

Along with (3.6), (3.7), let us consider the truncated systems (3.16), (3.17) for $t \in J_m$, where we set $v(s) = \mathsf{P}_N u_i(s)$ for $0 \leq s \leq T_{m-1}$. Since $u_i \in Q(0, m-1)$, for $t \leq T_{k-1}$ we have $\chi_i(t, \boldsymbol{v}_t, \boldsymbol{a}_t, w_i^0) = 1$ and $u_i'(t) = u_i(t)$. We define $z_i(t) = \mathsf{P}_N u_i(t)$ for $0 \leq t \leq T_{m-1}$ and for $t \in J_m$ define $(z_i(t), a_i(t))$ as a solution of (3.16), (3.17) such that

$$z_i(T_{m-1}) = v^0, \quad a_i(T_{m-1}) = \psi(T_{m-1}) = Q_N\zeta(T_{m-1}).$$

Let us denote by N_i the event $\{z_i(s) \neq \mathsf{P}_N u_i(s)$ for some $0 \leq s \leq T_m\}$. Repeating the arguments in the proof of Lemma 3.3, where now in (3.21) the segment $[0,T]$ should be replaced by $J_m \subset [0, T_m]$, we show that

$$\mathbb{P}(N_i) \leq e^{-\gamma_0(\rho - 1 + T(m-1))} \leq 2e^{-\gamma_0(\rho-1+T(m-1))}\mathbb{P}'(\overline{Q}_{m-1}).$$

Here we used the fact that $\mathbb{P}'(\overline{Q}_{m-1}) \geq 1/2$ (see (3.32) with $k = m-1$). It follows that

$$\int_{\overline{Q}_{m-1}} \big\|\lambda_1(\omega') - \lambda_2(\omega')\big\|_{\text{var}} \mathbb{P}'(d\omega') \leq 4e^{-\gamma_0(\rho-1+T(k-1))}\mathbb{P}'(\overline{Q}_{m-1}) + \\ + \int_{\overline{Q}_{m-1}} \big\|\nu_1(\omega') - \nu_2(\omega')\big\|_{\text{var}} \mathbb{P}'(d\omega'), \tag{3.35}$$

where $\nu_i(\omega')$ is the distribution of $\big((z_i(t),a(t)),t\in J_m\big)$.

Thus, inequality (3.30) will be established if we prove the following estimate for the variational distance between $\nu_1(\omega')$ and $\nu_2(\omega')$ (and next integrate it with respect to $\omega'\in\overline{Q}_{m-1}$):

$$\big\|\nu_1(\omega')-\nu_2(\omega')\big\|_{\mathrm{var}}\le e^{-\gamma_0\rho-T(m-1)}. \tag{3.36}$$

This can be done with the help of the arguments used in the proof of Lemma 3.3. The only difference is that the integrals over the time interval $[0,T]$ should be replaced by integrals over J_m. Due to (3.31) and (3.34), for $\omega\in\overline{Q}_m$ the processes v, w, and $\psi=a$ are continuous on the segment $[0,T_m]$. Therefore the estimate (3.29) holds for $t\le T_m$, where the constant M (see (3.26)) is now replaced by $M(m)=\exp(6K_Nd^2e^{2C\rho-2T(m-1)})$.

ii) (proof of (3.32)). By the definition of $\overline{Q}_m$ we have

$$\mathbb{P}(\overline{Q}^c_m)\le\mathbb{P}(\overline{Q}^c_{m-1})+\int_{\overline{Q}_{m-1}}\mathbb{P}^m\big\{B_1(\omega')\big\}\mathbb{P}'(d\omega')+\int_{\overline{Q}_{m-1}}\mathbb{P}^m\big\{B_2(\omega')\big\}\mathbb{P}'(d\omega')\,, \tag{3.37}$$

where

$$\begin{aligned}B_1(\omega')&=\big\{\exists\, t\in J_m \text{ such that } \mathsf{P}_Nu_1(t)\ne\mathsf{P}_Nu_2(t) \text{ or } \mathsf{Q}_N\zeta_1(t)\ne\mathsf{Q}_N\zeta_2(t)\big\},\\ B_2(\omega')&=\bigcup_{i=1,2}\big\{\exists\, t\in J_m \text{ such that } \mathcal{E}_i(t,0)>\rho+(B_0+1)t\big\}.\end{aligned}$$

By construction, for $\omega'\in\overline{Q}_{m-1}$ the random processes $\big(\Upsilon_i(t),t\in J_m\big)$, $i=1,2$, form a maximal coupling for the measures $\lambda_1(\omega')$ and $\lambda_2(\omega')$. Therefore, $\mathbb{P}^m\big\{B_1(\omega')\big\}=\big\|\lambda_1(\omega')-\lambda_2(\omega')\big\|_{\mathrm{var}}$. Evoking (2.6) to majorize the second integral in (3.37) we see that the sum of the two integrals is bounded by

$$\int_{\overline{Q}_{m-1}}\big\|\lambda_1(\omega')-\lambda_2(\omega')\big\|_{\mathrm{var}}\mathbb{P}'(d\omega')+2\,e^{-\gamma_0(\rho-1+T(m-1))}\le\tfrac12\,3^{-m}+\tfrac12\,3^{-m}=3^{-m}, \tag{3.38}$$

where we used inequality (3.30) with $l=0$ and $k=m$ (which is already proved). By the induction hypothesis, $\mathbb{P}(\overline{Q}^c_{m-1})\le\frac12(1+3^{-m+1})$ (see (3.32) with $k=m-1$). Therefore (3.37) and (3.38) imply (3.32) with $k=m$.

This completes the induction step and the proof of (3.30) for $l=0$.

Step 2. We now consider the case $l\ge1$. The curves $u_i(t)$, $0\le t\le(k-1)T$, depend on the random parameter ω'. We can assume that it has the form $\omega'=(\widetilde\omega,\widehat\omega)\in\widetilde\Omega\times\widehat\Omega=\Omega'$, where $\widetilde\omega$ and $\widehat\omega$ correspond to the time intervals $[0,T_l]$ and $[T_l,T_{k-1}]$, respectively, and are independent.

Let us consider the set $\overline{Q}=\overline{Q}(l,k-1)\subset\Omega'$. It can be written as

$$\overline{Q}=\big\{(\widetilde\omega,\widehat\omega)\,|\,\widetilde\omega\in\widetilde\Omega_0,\ \widehat\omega\in\widehat Q(\widetilde\omega)\big\}, \tag{3.39}$$

where $\widetilde\Omega_0$ is formed by $\widetilde\omega\in\widetilde\Omega$ such that $(u_1(T_l),u_2(T_l))$ satisfies (3.33). Applying inequality (3.30) with $l=0$ and k replaced by $k-l$, for any fixed $\widetilde\omega\in\widetilde\Omega_0$ we obtain

$$\int_{\widehat{Q}(\widetilde{\omega})} \big\|\lambda_1(\widetilde{\omega},\widehat{\omega})-\lambda_2(\widetilde{\omega},\widehat{\omega})\big\|_{\mathrm{var}}\widehat{\mathbb{P}}(d\widehat{\omega}) \le C_2 e^{-\gamma_0\rho}e^{-\gamma_1 T(k-l-1)}\widehat{\mathbb{P}}\big(\widehat{Q}(\widetilde{\omega})\big).$$

Integration of this inequality with respect to $\widetilde{\omega}\in\widetilde{\Omega}_0$ results in (3.30). The proof of Lemma 3.4 is complete. □

Finally, we shall need an estimate for the variational distance between the measures λ_1 and λ_2', which were defined in Subsection 3.2 when constructing the coupling operators in the case (c).

Lemma 3.5. *There is $\theta_{(1)}>0$ such that if $\theta\le\theta_{(1)}$ and $|u_1^0|\vee|u_2^0|\le\theta$, then $\big\|\lambda_1-\lambda_2'\big\|_{\mathrm{var}}\le\frac14$.*

Proof. The proof is similar to and easier than that of Lemma 3.3, and we only outline it. We fix an arbitrary constant $\theta\in(0,1]$ and recall that λ_1 and λ_2' are the distributions of the processes $\Upsilon_1=(v_1,a_1)$ and $\Upsilon_2'=(v_2',a_2)$. The first process is a solution of (3.6), (3.7), defined for $0\le t\le\theta$, while

$$v_2'(t)=v_2(t)+\frac{\theta-t}{\theta}v^{\triangle}\qquad v^{\triangle}=\mathsf{P}_N(u_1^0-u_2^0)\,,\quad 0\le t\le\theta\,, \tag{3.40}$$

where (v_2,a_2) satisfies (3.6), (3.7) with $w^0=\mathsf{P}_N u_2$. Therefore (v_2',a_2) is a solution for the following equation:

$$\dot v'+Lv'+\Big(\frac{1}{\theta}\,v^{\triangle}-\frac{\theta-t}{\theta}\,Lv^{\triangle}+B_N(v_2+\mathcal{W}_t(\boldsymbol{v}_2,\boldsymbol{a}_t,w_2^0)\Big)=\dot\varphi(t), \tag{3.41}$$

$$\dot a=\dot\psi(t), \tag{3.42}$$

where $B_N=\mathsf{P}_N B$ and we view v_2 as a function of $v'=v_2'$, defined in (3.40). The processes satisfy the initial conditions

$$v_1(0)=v_2'(0)=\mathsf{P}_N u_1^0,\quad a_1(0)=a_2(0)=\psi(0). \tag{3.43}$$

Along with (3.6), (3.7) and (3.41), (3.42), let us consider the truncated system (3.16), (3.17) with $i=1$ and its analogue for Υ_2':

$$\dot v'+Lv'+\chi_2(t)\Big(\frac{1}{\theta}\,v^{\triangle}-\frac{\theta-t}{\theta}\,Lv^{\triangle}+B_N\Big)=\dot\varphi,\qquad \dot a=\dot\psi\,. \tag{3.44}$$

Here B_N is the same as in (3.41) and χ_2 is defined as in Step 1 of the proof of Lemma 3.3 with u_2' replaced by $u'=v'+\mathcal{W}_t(\boldsymbol{v}_2,\boldsymbol{a}_t,w_2^0)$ (v_2 is the function of v' as above). Arguing as in the proof of Lemma 3.3 and choosing ρ in (3.18) to be sufficiently large (this ρ can be different from the constant, used in Lemmas 3.3 and 3.4), we achieve that

$$\mathbb{P}\Big\{\text{a solution of (3.6), (3.7) (or of (3.41), (3.42)) differs from}\\ \text{the solution of (3.16), (3.17) (or of (3.44), respectively)}\Big\}\le\frac18\,. \tag{3.45}$$

Let ν_1 and ν_2' be the distributions of solutions for problems (3.16), (3.17) and (3.44), respectively, that are supplemented with the initial conditions (3.43). Due to (3.45), to prove the lemma it suffice to check that

$$\|\nu_1 - \nu_2'\|_{\text{var}} \le \tfrac{1}{8}. \tag{3.46}$$

By the definition of χ_2 (see (3.18), where u_2 is replaced by u'), $\chi_2 \neq 0$ implies that $|u'| \le C = \rho + B_0 + 1$. Therefore, $|u_2| \le C_1$ since $u_2 = u' - \frac{\theta - t}{\theta} v^\triangle$ and $|v^\triangle| \le 2\theta$ (the constants $C, C_1, \dots$ depend on ρ and $\{b_j\}$, $\{\alpha_j\}$). Due to basic properties of the nonlinearity B, [6] this implies that $|B_N(v_2 + \mathcal{W}_t)| = |B_N(u_2)| \le C_2$ if $\chi_2 \neq 0$. So the term $\chi_2(t)(\dots)$ in (3.44) is bounded by some constant C_3. The corresponding term $\chi_1 B_N$ in (3.16) is bounded for similar reasons. Therefore, now the function $\alpha(t,\omega)$, analogous to that defined in (3.24), is bounded by a constant C_4, and we get that

$$\mathbb{E} \exp\Bigl(6 \int_0^\theta |\alpha|^2 \, dt\Bigr) \le e^{C_5 \theta} =: M.$$

Now, as in the proof of Lemma 3.2, Girsanov's theorem implies that $\|\nu_1 - \nu_2'\|_{\text{var}} \le \frac{1}{2}(\sqrt{M} - 1)^{1/2} \le C_6 \theta^{1/2}$. So (3.46) holds if θ is sufficiently small, and the lemma is proved. □

3.4 Proof of inequalities (2.8) − (2.10)

1) We first prove (2.8). To this end, we repeat the argument used in the proof of Lemma 3.4 (see the derivation of (3.32)). Let us note that, for any $\omega' \in \overline{Q} = \overline{Q}(l, k-1) \subset \Omega'$, the curves ζ_1 and ζ_2 are continuous on $[lT, kT]$ due to (3.31) and to the definition of the set $Q(l, k-1)$. Therefore,

$$\bigl\{(\mathbf{\Theta}_1^k, \mathbf{\Theta}_2^k) \notin Q(l,k)\bigr\} \subset B_1(\omega') \cup B_2(\omega')$$

(cf. (3.37)), where

$$B_1(\omega') = \bigl\{\exists\, t \in J_k \text{ such that } \mathsf{P}_N u_1(t) \neq \mathsf{P}_N u_2(t) \text{ or } \mathsf{Q}_N \zeta_1(t) \neq \mathsf{Q}_N \zeta_2(t)\bigr\},$$
$$B_2(\omega') = \bigcup_{i=1,2} \bigl\{\exists\, t \in J_k \text{ such that } \mathcal{E}_i(t, T_l) > \rho + (B_0 + 1)(t - T_l)\bigr\},$$

where $J_k = [T_{k-1}, T_k]$. It follows that the left-hand side in (2.8) can be estimated by the sum $\beta_1 + \beta_2$, where

$$\beta_i = \int_{\Omega'} I_{\overline{Q}}(\omega') \mathbb{P}^k\bigl\{B_i(\omega')\bigr\} \mathbb{P}'(d\omega').$$

By construction, for $\omega' \in \overline{Q}$ the random processes $\bigl(\mathsf{P}_N u_i(t), \mathsf{Q}_N \zeta_i(t), t \in J_k\bigr)$, $i = 1, 2$, form a maximal coupling for the measures $\lambda_1(\omega')$ and $\lambda_2(\omega')$. Therefore, $\mathbb{P}^k\bigl\{B_1(\omega')\bigr\} = \bigl\|\lambda_1(\omega') - \lambda_2(\omega')\bigr\|_{\text{var}}$. Using (3.30), we find that

$$\beta_1 = \int_{\overline{Q}} \bigl\|\lambda_1(\omega') - \lambda_2(\omega')\bigr\|_{\text{var}} \mathbb{P}'(d\omega') \le C_2 e^{-\gamma_0 \rho} e^{-\gamma_1 T(k-l-1)} \mathbb{P}'(\overline{Q}).$$

[6] Namely, we use the estimate $\|B(u,u)\|_{-2} \le C|u|^2$.

Due to (2.6) (see also (3.32)), we have $\beta_2 \le 4\, e^{-\gamma_0(\rho-1+T(k-l-1))}\mathbb{P}'(\overline{Q})$. Hence, $\beta_1+\beta_2 \le \mathbb{P}'(\overline{Q})(1+8e^{\gamma_0})e^{-\gamma_1 T(k-l-1)}$. This completes the proof of (2.8).

2) We now turn to (2.9). Let $d = d_\rho > 0$ be the constant from Lemma 3.4. We recall that $T = \theta_2 + \theta$, where $\theta \in (0,1]$ is chosen below, $\theta_2 = T_2(\theta)$, and T_2 is defined in (1.10). The parameter θ will be chosen so small that T satisfies the second inequality in (2.29). Let us denote $g_i = U_i(\theta; u_1^0, u_2^0)$, $i = 1,2$, where the coupling operators $U_{1,2}$ were defined in Subsection 3.2 (see (3.14)), and we omitted their dependence on the random parameters. By the definition of $Q(k,k)$, we have

$$\mathbb{P}^k\big\{(\boldsymbol{\Theta}_1^k, \boldsymbol{\Theta}_2^k) \in Q(k,k)\big\} \ge p_1 p_2, \tag{3.47}$$

where

$$\begin{aligned} p_1 &= \mathbb{P}^k\big\{|u_1(T_{k-1}+\theta_2)| \vee |u_2(T_{k-1}+\theta_2)| \le \theta\big\},\\ p_2 &= \inf_{(u_1^0,u_2^0)} \mathbb{P}^k\big\{|g_1| \vee |g_2| \le d,\ \mathsf{P}_N g_1 = \mathsf{P}_N g_2 \,\big|\, |u_1^0| \vee |u_2^0| \le \theta\big\}. \end{aligned}$$

In view of Lemma 1.6, we have $p_1 \ge \pi(\theta)$.

To estimate p_2, we apply Lemma 1.4, where $R_0 \le 2\theta^2$ is a constant. Then, due to (1.7) with $t = \theta \le 1$, we get:

$$\mathbb{E}R(\theta) \le 2\theta^2 + C_{(1)}\theta \le C_{(2)}\theta.$$

Choosing $\theta = d^2/4C_{(2)}$ and applying the Chebyshev inequality we find that $\mathbb{P}^k\{R(\theta) \ge d^2\} \le \frac{1}{4}$. That is,

$$\mathbb{P}^k\big\{|g_1| \vee |g_2| > d \,\big|\, |u_1^0| \vee |u_2^0| \le \theta\big\} \le \tfrac{1}{4}. \tag{3.48}$$

Furthermore, as was explained in Subsection 3.2, if $\boldsymbol{\Upsilon}_1 = \boldsymbol{\Upsilon}_2'$, then $\mathsf{P}_N g_1 = \mathsf{P}_N g_2$. Since $(\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_2')$ is a maximal coupling for (λ_1, λ_2'), then Lemma 3.5 implies that

$$\mathbb{P}^k\big\{\mathsf{P}_N g_1 \ne \mathsf{P}_N g_2 \,\big|\, |u_1^0| \vee |u_2^0| \le \theta\big\} \le \big\|\lambda_1 - \lambda_2'\big\|_{\mathrm{var}} \le \tfrac{1}{4}\,, \tag{3.49}$$

if $0 \le \theta \le \theta_{(1)}$. Combining (3.48) and (3.49), we see that $p_2 \ge \frac{1}{2}$. Now (3.47) implies (2.9) with $p_0 = \frac{1}{2}\pi(\theta)$, where $\theta = (d^2/4C_{(2)}) \wedge \theta_{(1)}$.

3) Due to (1.10) with $d = d_\rho$ as in Lemma 3.3, we have

$$\begin{aligned} T_2(\theta) &\le C_{(3)} \ln \rho_0 + C_{(4)} \ln\big((d^2/4C_{(3)}) \wedge \theta_{(1)}\big)^{-1} + C_{(5)}\\ &\le C_{(3)} \ln \rho_0 + C_{(6)}\rho + C_{(7)} =: T_2'. \end{aligned}$$

Since our arguments apply for any $T \ge T_2(\theta) + \theta$ and $\theta \le 1$, then we can choose $T(\rho, \rho_0) = T_2' + 1$. This proves (2.10) with some new constants $C_{(1)} - C_{(3)}$. The proof of Theorem 2.1 is complete.

4 Appendix

In the first subsection of this appendix, we present a well-known estimate for the difference between two solutions for deterministic NS equations (see [FP67]). Since the solutions of equations with additive noise can be treated pathwise, that estimate established in the deterministic case remains valid for problems discussed in this paper. The second subsection is devoted to the proof of an infinite-dimensional version of Girsanov's formula.

4.1 Foias–Prodi estimate

We shall assume that the right-hand side $\eta(t)$ in (1.1) is the time derivative of a deterministic function belonging to $C(\mathbb{R}_+, V)$. In this case, the Cauchy problem (1.1), (1.2) is uniquely solvable in the space $C(\mathbb{R}_+, H) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+, V)$.

Proposition 4.1. *Let u_1 and u_2 be two solutions of the NS system* (1.1) *with right-hand sides η_1 and η_2, respectively, such that*

$$\int_s^t \|u_1(r)\|^2 dt \le \rho + K(t-s), \quad s \le t \le s+T, \tag{4.1}$$

where s, ρ, K, and T are non-negative constants. For any $M > 0$ there is an integer $N = N(K, M) \ge 1$ such that, if

$$\mathsf{P}_N u_1(t) = \mathsf{P}_N u_2(t), \quad \mathsf{Q}_N \eta_1(t) = \mathsf{Q}_N \eta_2(t) \quad \textit{for} \quad s \le t \le s+T, \tag{4.2}$$

then

$$|u_1(t) - u_2(t)| \le e^{-M(t-s)+C\rho} |u_1(s) - u_2(s)|, \quad s \le t \le s+T, \tag{4.3}$$

where $C > 0$ does not depend on solutions and all other parameters.

Proof. We only sketch the well-known proof [FP67]. Without loss of generality, we shall assume that $s = 0$. Taking into account (4.2), we see that, on the interval $[0, T]$, the difference $w = \mathsf{Q}_N(u_1 - u_2)$ satisfies the equation

$$\dot{w} + Lw + \mathsf{Q}_N\big(B(w, u_1) + B(u_2, w)\big) = 0. \tag{4.4}$$

Taking the scalar product of (4.4) and $2w$ in the space H and using the relation $(B(u_2, w), w) = 0$ and the inequality $|(B(w, u_1), w)| \le C_1 |w|\,\|w\|\,\|u_1\|$, we derive

$$\partial_t |w|^2 + 2\|w\|^2 \le 2C_1 |w|\,\|w\|\,\|u_1\|. \tag{4.5}$$

Since $\|w\|^2 \ge \alpha_{N+1} |w|^2$, it follows from (4.5) that

$$\partial_t |w|^2 + \big(\alpha_{N+1} - C_1^2 \|u_1\|^2\big)|w|^2 \le 0. \tag{4.6}$$

Let us choose N so large that $\alpha_{N+1} \ge 2M + C_1^2 K$. In view of the Gronwall inequality, from (4.1) and (4.6) we obtain

$$|w(t)|^2 \le |w(0)|^2 \exp\Big(-\alpha_{N+1} t + C_1^2 \int_0^t \|u_1(r)\|^2 dr\Big) \le |w(0)|^2 \exp\big(C_1^2 \rho - 2Mt\big),$$

whence follows inequality (4.3) with $C = C_1^2/2$. $\square$

When proving Lemma 3.3 and 3.4, we used a more general assertion concerning solutions for the projection of the NS system onto high Fourier modes. Namely, let us fix an integer $N \geq 1$ and consider the equation

$$\dot{w} + Lw + \mathsf{Q}_N B(v + w, v + w) = h(t), \tag{4.7}$$

where $v \in C(\mathbb{R}_+, H_N)$ is a given function, and the right-hand side h is the derivative of a function belonging to $C(\mathbb{R}_+, V \cap H_N^{\perp})$.

Proposition 4.2. *Let $w_i \in C(\mathbb{R}_+, H_N^{\perp}) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+, V)$, $i = 1, 2$, be two solutions of Eq. (4.7) such that inequality (4.1) holds for $u_1 = v + w_1$ and some nonnegative constants s, ρ, K, and T. For any $M > 0$ there is an integer $N_0 = N_0(K, M) \geq 1$ such that, if $N \geq N_0$, then*

$$|w_1(t) - w_2(t)| \leq e^{-M(t-s)+C\rho} |w_1(s) - w_2(s)|, \quad s \leq t \leq s + T,$$

where $C > 0$ does not depend on solutions and all other parameters.

The proof of Proposition 4.2 literally repeats the arguments used in derivation of (4.3), and therefore we omit it.

4.2 An infinite-dimensional Girsanov formula

Here we prove (3.27) and (3.28). We have to verify that

$$\int_{X_0} f(\boldsymbol{\Upsilon})\, \nu_2(d\boldsymbol{\Upsilon}) = \int_{X_0} f(\boldsymbol{\Upsilon})\, e^{G(\boldsymbol{\Upsilon})} \nu_1(d\boldsymbol{\Upsilon}) \tag{4.8}$$

for any continuous function $f(\boldsymbol{v}, \boldsymbol{a})$ such that $0 \leq f \leq 1$. To this end, we first note that, for any nonnegative continuous function $g(\boldsymbol{\Upsilon}) = g(\boldsymbol{v}, \boldsymbol{a})$,

$$\int_{X_0} g(\boldsymbol{\Upsilon})\, \nu_i(d\boldsymbol{\Upsilon}) = \int_{X_{0N}^{\perp}} \hat{\nu}(d\boldsymbol{a}) \int_{X_{0N}} g(\boldsymbol{v}, \boldsymbol{a}) \nu_{iN}(\boldsymbol{a}, d\boldsymbol{v}),$$

where $X_{0N} = C(0, T; H_N)$, $X_{0N}^{\perp} = C(0, T; H_N^{\perp})$, $\hat{\nu}(d\boldsymbol{a})$ is the distribution of the random variable $(\mathsf{Q}_N \zeta(t), 0 \leq t \leq T)$, and $\nu_{iN}(\boldsymbol{a}, d\boldsymbol{v})$ is the distribution of the solution for Eq. (3.16) with fixed $\boldsymbol{a} \in C(0, T; H_N^{\perp})$ and the initial condition $v(0) = \mathsf{P}_N u_i^0$. Therefore, relation (4.8) will be established if we show that

$$\int_{X_{0N}} f(\boldsymbol{v}, \boldsymbol{a}) \nu_{2N}(\boldsymbol{a}, d\boldsymbol{v}) = \int_{X_{0N}} f(\boldsymbol{v}, \boldsymbol{a})\, e^{G(\boldsymbol{v}, \boldsymbol{a})} \nu_{1N}(\boldsymbol{a}, d\boldsymbol{v}), \tag{4.9}$$

where $\boldsymbol{a} \in C(0, T; H_N^{\perp})$ is an arbitrary deterministic function. It remains to note that (4.8) follows from the usual finite-dimensional Girsanov theorem applied to the system (3.16) with fixed $\boldsymbol{a}$; e.g., see Theorem 7.18 in [LS77]. Applicability of the theorem (i.e., the fact that $\int e^{G(\boldsymbol{v}, \boldsymbol{a})} d\nu_{1N}(\boldsymbol{a}, d\boldsymbol{v}) = 1$) follows from (3.26) due to Novikov's theorem [Kry95, Theorem IV.3.5].

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