Weak universality and singular SPDEs

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- The analysis of scaling limits of stochastic non–linear diffusion problems generates irregular random fields which should be described by *universal* non–linear SPDEs (i.e. independent of specific details of the microscopic model).
- The combination of irregularity and non–linearity is problematic and can generate unexpected phenomena which escape a purely analytical control: the statistical structure of the noise has to be carefully taken into account.
- Hairer's *regularity structures* provide a general tool do to so: they allows to describe the local features of the random fields in term of simpler objects. The effect of non–linear operations is then more easily understood given the improved description.
- In a parallel work, G.–Imkeller–Perkowski exploited tools from harmonic analysis to perform similar kind of analysis (Fourier–space counterpart of the regularity structure businness)
- *Paracontrolled distributions* are not as general as regularity structures but they provide an alternative approach in many relevant cases: KPZ, $\Phi^4_{2,3}$, Parabolic/Hamiltonian Anderson model in $2d$.

Talk based on joint work with: R. Catellier, K. Chouk, P. Imkeller, N. Perkowski, M. Furlan.

- 1d generalised Stochastic Burgers equation (gSBE)

$$\mathscr{L}u(t,x) = G(u(t,x))\partial_x u(t,x) + \xi(t,x), \qquad t \geqslant 0, x \in \mathbb{T},$$

where G is a smooth function, $\mathscr{L} = \partial_t - \Delta$.

- 1d Kardar–Parisi–Zhang equation (KPZ).

$$\mathscr{L}h(t,x) = (\partial_x h(t,x))^2 - C + \xi(t,x), \qquad t \geqslant 0, x \in \mathbb{T}$$

- Dynamic Φ^4_d model or stochastic quantisation equation ($d = 2, 3$) (SQE)

$$\mathscr{L}\varphi(t,x) = -\lambda\varphi(t,x)^3 - C\varphi(t,x) + \xi(t,x), \qquad t \geqslant 0, x \in \mathbb{T}^d,$$

- Generalised 2d parabolic Anderson model (gPAM)

$$\mathscr{L}u(t,x) = G(u(t,x))\xi(x) - CG'(u(t,x))G(u(t,x)), \qquad t \geqslant 0, x \in \mathbb{T}^2,$$

▷ G.-Imkeller-Perkowski, *Paracontrolled distributions and singular PDEs* (2012)

▷ G.-Perkowski, *Lectures on singular stochastic PDEs* (2015), *KPZ reloaded* (2016), *An introduction to singular SPDEs* (2017)

▷ Catellier-Chouk, *Paracontrolled Distributions and the 3-dimensional Stochastic Quantization Equation* (2013)

▷ Weber and Mourrat (2014–2016): SQE weak universality in 2d, SQE space–time global solution in 2d, time global solutions in 3d.

▷ G., Koch, Oh (2017): weak universality for 2d stochastic non-linear wave equation.

▷ Hairer–Xu (2016). weak universality for Φ^4_3 with regularity structures.

Other relevant literature:

▷ Hairer's 2014 Inventiones paper.

▷ Hairer–Quastel (2015). Weak universality for KPZ.

▷ Gonçalves–Jara (2014), G.–Perkowski (2016), G.–Perkowski–Diehl (2016). Weak universality for KPZ from particle systems via *energy solutions*.

▷ Cannizzaro-Chouk (2015): singular paracontrolled martingale problems. Allez-Chouk (2016): paracontrolled analysis of random unbounded operators. Bailleul and Bernicot (2016): higher order paracontrolled calculus.

▹ Scalar diffusion equation with slow reaction term

$$\mathscr{L}\psi(t,x) = \varepsilon^{\gamma} F(\psi(t,x)) + \eta(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{T}_\varepsilon^d$$

with $\mathbb{T}_\varepsilon = \mathbb{T}/\varepsilon$, F odd (typically $F = -\partial U$ for some potential U) and $\mathscr{L} = \partial_t - \Delta$.

▹ Parabolic rescaling $\psi_\varepsilon(t,x) = \varepsilon^{-\alpha}\psi(t/\varepsilon^2, x/\varepsilon)$ and $\xi_\varepsilon(t,x) = \varepsilon^{-d/2-1}\eta(t/\varepsilon^2, x/\varepsilon)$

$$\mathscr{L}\psi_\varepsilon(t,x) = \varepsilon^{\gamma-\alpha-2} F(\varepsilon^{\alpha}\psi_\varepsilon(t,x)) + \varepsilon^{(d-2)/2-\alpha}\xi_\varepsilon(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{T}_\varepsilon^d$$

Note that $\xi_\varepsilon \to \xi$ the space–time white noise.

▹ Let $\alpha = (d-2)/2$ to keep a noisy evolution:

$$\mathscr{L}\psi_\varepsilon = \varepsilon^{\gamma-d/2-1} F(\varepsilon^{(d-2)/2}\psi_\varepsilon) + \xi_\varepsilon$$

In the following we will concentrate on the $d = 3$ case.

▹ **Linear approximation**: assume $\psi_\varepsilon \simeq X_\varepsilon$ with $\mathscr{L} X_\varepsilon = \xi_\varepsilon$

$$\varepsilon^{\gamma - d/2-1} F(\varepsilon^{(d-2)/2} \psi_\varepsilon) \simeq \varepsilon^{\gamma - d/2-1} F(\varepsilon^{(d-2)/2} X_\varepsilon)$$

▹ Explicit Gaussian computations shows that

$$\varepsilon^{-(d-2)/2} F(\varepsilon^{(d-2)/2} X_\varepsilon) \to \mu X \qquad \text{(as space–time distributions)}$$

where $\mu = \mathbb{E}[G F(G)]$ and $G \sim \mathscr{N}(0, c)$ and $\mathscr{L} X = \xi$.

▹ Then ,if $\mu \neq 0$ we need to take $\gamma = 2$:

$$\varepsilon^{\gamma - d/2-1} F(\varepsilon^{(d-2)/2} \psi_\varepsilon) \simeq \varepsilon^{\gamma-2} \mu X \simeq \varepsilon^{\gamma-2} \mu \psi$$

The limiting equation is

$$\mathscr{L} \psi = \mu \psi + \xi$$

Result: we *expect* Gaussian fluctuations.

Rigorous analysis. Let $\psi_\varepsilon = X_\varepsilon + \theta_\varepsilon$ then (recall $d=3$, $\gamma=2$)

$$\mathscr{L}\theta_\varepsilon = \varepsilon^{-1/2} F(\varepsilon^{1/2} X_\varepsilon + \varepsilon^{1/2}\theta_\varepsilon)$$

▷ Taylor expansion gives the approximate equation

$$\mathscr{L}\theta_\varepsilon = \mu\tilde{X}_\varepsilon + \tilde{\mu}_\varepsilon \theta_\varepsilon + \varepsilon^{1/2} R_\varepsilon(\theta_\varepsilon)$$

$$\mu\tilde{X}_\varepsilon := \varepsilon^{-1/2} F(\varepsilon^{1/2} X_\varepsilon), \quad \tilde{\mu}_\varepsilon := F'(\varepsilon^{1/2} X_\varepsilon), \quad R_\varepsilon(\theta_\varepsilon) := \theta_\varepsilon^2 \int_0^1 F''(\varepsilon^{1/2} X_\varepsilon + \tau\varepsilon^{1/2}\theta_\varepsilon)(1-\tau)\mathrm{d}\tau$$

▷ Since $\|\varepsilon^{1/2} X_\varepsilon\|_{L^\infty} \lesssim \varepsilon^{-0-}$ we estabilish easily that

$$\varepsilon^{1/2} R_\varepsilon(\theta_\varepsilon) \xrightarrow[L^\infty]{} 0$$

▷ Direct Gaussian computations show ($\mathscr{C}^\alpha = C([0,T]; B^\alpha_{\infty,\infty}(\mathbb{T}^d))$)

$$\tilde{\mu}_\varepsilon \xrightarrow[\mathscr{C}^{0-}]{} \mu \in \mathbb{R} \qquad X_\varepsilon, \tilde{X}_\varepsilon \xrightarrow[\mathscr{C}^{-1/2-}]{} X \qquad \varepsilon^{1/2} R_\varepsilon(\theta_\varepsilon) \xrightarrow[L^\infty]{} 0$$

$$\mathscr{L}\theta_\varepsilon = \Theta_\varepsilon(\tilde{X}_\varepsilon, \tilde{\mu}_\varepsilon, \theta_\varepsilon) = \mu\tilde{X}_\varepsilon + \tilde{\mu}_\varepsilon\theta_\varepsilon + \varepsilon^{1/2}R_\varepsilon(\theta_\varepsilon) \in \mathscr{C}^{-1/2-}$$

▷ **Parabolic estimates** give $\theta_\varepsilon \in \mathscr{C}^{3/2-}$

▷ **Continuity of the product** in $\mathscr{C}^\alpha \times \mathscr{C}^\beta \to \mathscr{C}^{\alpha\wedge\beta}$ when $\alpha + \beta > 0$ results in continuity for

$$\tilde{\mu}_\varepsilon \times \theta_\varepsilon \in \mathscr{C}^{-0-} \times \mathscr{C}^{3/2-} \to \tilde{\mu}_\varepsilon\theta_\varepsilon$$

Hence, the family of maps Θ_ε

$$(\tilde{X}_\varepsilon, \tilde{\mu}_\varepsilon, \theta_\varepsilon) \in \mathscr{C}^{-1/2-} \times \mathscr{C}^{-0-} \times \mathscr{C}^{3/2-} \mapsto \Theta_\varepsilon(\tilde{X}_\varepsilon, \tilde{\mu}_\varepsilon, \theta_\varepsilon) \in \mathscr{C}^{-1/2-}$$

is continuous (and even locally Lipshitz). From this easy to deduce that

$$\theta_\varepsilon \xrightarrow[\mathscr{C}^{3/2-}]{} \theta \qquad \mathscr{L}\theta = \mu X + \mu\theta$$

$$\psi_\varepsilon = X_\varepsilon + \theta_\varepsilon \xrightarrow[\mathscr{C}^{-1/2-}]{} \psi = X + \theta \qquad \mathscr{L}\psi = \mu\psi + \xi$$

Suppose now that we choose F such that $\mu = 0$. In this case by symmetry

$$\mathbb{E}[H_{c,2}(G)F(G)] = 0$$

but if

$$\lambda = \mathbb{E}[H_{c,3}(G)F(G)] \neq 0$$

we have

$$\varepsilon^{-3(d-2)/2}F(\varepsilon^{(d-2)/2}X_\varepsilon) \xrightarrow[\mathscr{C}^{-3/2-}]{} \lambda X^{\diamond 3} \qquad \text{(as space–time distributions)}$$

where $X^{\diamond 3} = \lim_{\varepsilon\to 0} H_{\gamma_\varepsilon,3}(X_\varepsilon)$.

Now we guess that

$$\varepsilon^{\gamma - d/2 - 1}F(\varepsilon^{(d-2)/2}\psi_\varepsilon) \simeq \varepsilon^{d-4+\gamma}\lambda X^{\diamond 3}$$

so something non–trivial can be obtained if $\gamma = 4 - d = 1$.

$$\mathscr{L}\varphi_\varepsilon = \varepsilon^{-3/2}F(\varepsilon^{1/2}\varphi_\varepsilon) + \xi_\varepsilon$$

▷ Taylor expansion with $\varphi_\varepsilon = X_\varepsilon + \theta_\varepsilon$

$$\mathscr{L}\theta_\varepsilon = \varepsilon^{-3/2}F(\varepsilon^{1/2}X_\varepsilon) + \varepsilon^{-1}F'(\varepsilon^{1/2}X_\varepsilon)\theta_\varepsilon + \varepsilon^{-1/2}F''(\varepsilon^{1/2}X_\varepsilon)\theta_\varepsilon^2 + F'''(\varepsilon^{1/2}X_\varepsilon)\theta_\varepsilon^3 + \varepsilon^{1/2}R_\varepsilon$$

$$\varepsilon^{-3/2}F(\varepsilon^{1/2}X_\varepsilon) \xrightarrow[\mathscr{C}^{-3/2-}]{} \lambda X^{\diamond 3}, \qquad \varepsilon^{-1}F'(\varepsilon^{1/2}X_\varepsilon) \xrightarrow[\mathscr{C}^{-1-}]{} \lambda X^{\diamond 2}, \qquad \varepsilon^{-1/2}F''(\varepsilon^{1/2}X_\varepsilon) \xrightarrow[\mathscr{C}^{-1/2-}]{} \lambda X$$

Problem: the previous strategy will not work since

$$\mathscr{L}\theta_\varepsilon \in \mathscr{C}^{-3/2-} \quad \Rightarrow \quad \theta_\varepsilon \in \mathscr{C}^{1/2-}$$

and the products

$$\varepsilon^{-1}F'(\varepsilon^{1/2}X_\varepsilon)\theta_\varepsilon, \qquad \varepsilon^{-1/2}F''(\varepsilon^{1/2}X_\varepsilon)\theta_\varepsilon$$

are then **not** under control.

▷ Need better understanding of the structure of the solution...

Consider the case $F(x) = \lambda(x^3 - c_\varepsilon x)$:

$$\mathscr{L}\varphi_\varepsilon = \lambda(\varphi_\varepsilon^3 - \varepsilon^{-1}c_\varepsilon\varphi_\varepsilon) + \xi_\varepsilon$$

and let $\varphi_\varepsilon = X_\varepsilon + \lambda Y_\varepsilon + \lambda\varphi_\varepsilon^Q$, $(c_\varepsilon = 3c_{1,\varepsilon} + 9\lambda c_{2,\varepsilon})$

$$\mathscr{L}Y_\varepsilon + \mathscr{L}\varphi_\varepsilon^Q = X_\varepsilon^{\diamond 3} + 3\lambda X_\varepsilon^{\diamond 2}(Y_\varepsilon + \varphi_\varepsilon^Q) + 3\lambda^2 X_\varepsilon(Y_\varepsilon + \varphi_\varepsilon^Q)^2 + \lambda^3(Y_\varepsilon + \varphi_\varepsilon^Q)^3 - 9\lambda c_{2,\varepsilon}\varphi_\varepsilon$$

$$X_\varepsilon^{\diamond 3} = X_\varepsilon^3 - 3c_{1,\varepsilon}X_\varepsilon \in \mathscr{C}^{-3/2-} \qquad X_\varepsilon^{\diamond 2} = X_\varepsilon^2 - c_{1,\varepsilon} \in \mathscr{C}^{-1-}$$

Choosing $\mathscr{L}Y_\varepsilon = X_\varepsilon^{\diamond 3}$ we get rid of the first term.

▷ $\mathscr{L}\varphi_\varepsilon^Q \in \mathscr{C}^{-1-} \Rightarrow \varphi_\varepsilon^Q \in \mathscr{C}^{1-}$ and $\mathscr{L}Y_\varepsilon \in \mathscr{C}^{-3/2-} \Rightarrow Y_\varepsilon \in \mathscr{C}^{1/2-}$.

▷ **Problem:** slightly better situation but still not ok:

$$\underbrace{X_\varepsilon^{\diamond 2}}_{\mathscr{C}^{-1-}}(\underbrace{Y_\varepsilon + \varphi_\varepsilon^Q}_{\mathscr{C}^{1/2-} + \mathscr{C}^{1-}}) \qquad \underbrace{X_\varepsilon}_{\mathscr{C}^{-1/2-}}(\underbrace{Y_\varepsilon + \varphi_\varepsilon^Q}_{\mathscr{C}^{1/2-} + \mathscr{C}^{1-}})^2$$

Decomposition of a product into *paraproducts* and *resonant term*

$$fg = f \prec g + f \circ g + f \succ g$$

Theorem **(Bony, Meyer)**

$$(f,g) \in \mathscr{C}^\alpha \times \mathscr{C}^\beta \to f \prec g = g \succ f \in \mathscr{C}^{\beta + \alpha \wedge 0}, \qquad \alpha, \beta \in \mathbb{R} \setminus \mathbb{N}$$

$$(f,g) \in \mathscr{C}^\alpha \times \mathscr{C}^\beta \to f \circ g \in \mathscr{C}^{\alpha+\beta}, \qquad \alpha + \beta > 0$$

Paralinearization:

$$f \in \mathscr{C}^\alpha \to R(f) = G(f) - G'(f) \prec f \in \mathscr{C}^{2\alpha}, \qquad \alpha > 0$$

A single new *key* ingredient:

Lemma **(G-Imkeller-Perkowski)**

$$(f,g,h) \in \mathscr{C}^\alpha \times \mathscr{C}^\beta \times \mathscr{C}^\gamma \to C(f,g,h) = (f \prec g) \circ h - f(g \circ h) \in \mathscr{C}^{\alpha+\beta+\gamma}, \qquad \alpha + \beta + \gamma > 0$$

▷ Paralinearization:

$$(Y_\varepsilon+\varphi_\varepsilon^Q)^2 = 2(Y_\varepsilon+\varphi_\varepsilon^Q)\prec(Y_\varepsilon+\varphi_\varepsilon^Q)+\underbrace{R(Y_\varepsilon+\varphi_\varepsilon^Q)}_{\mathscr{C}^{1-}}$$

▷ Decomposition

$$X_\varepsilon(Y_\varepsilon+\varphi_\varepsilon^Q)^2 = X_\varepsilon\prec(Y_\varepsilon+\varphi_\varepsilon^Q)^2+X_\varepsilon\circ(Y_\varepsilon+\varphi_\varepsilon^Q)^2+X_\varepsilon\succ(Y_\varepsilon+\varphi_\varepsilon^Q)^2$$

$$X_\varepsilon\circ(Y_\varepsilon+\varphi_\varepsilon^Q)^2 = 2\,X_\varepsilon\circ[(Y_\varepsilon+\varphi_\varepsilon^Q)\prec(Y_\varepsilon+\varphi_\varepsilon^Q)]+X_\varepsilon\circ R(Y_\varepsilon+\varphi_\varepsilon^Q)$$

▷ Commutator lemma

$$=2(Y_\varepsilon+\varphi_\varepsilon^Q)\,X_\varepsilon\circ(Y_\varepsilon+\varphi_\varepsilon^Q)+2\underbrace{C(Y_\varepsilon+\varphi^Q,Y_\varepsilon,X_\varepsilon)}_{\mathscr{C}^{1/2-}}+X_\varepsilon\circ R(Y_\varepsilon+\varphi_\varepsilon^Q)$$

$$=2(Y_\varepsilon+\varphi_\varepsilon^Q)\,X_\varepsilon\circ Y_\varepsilon+2(Y_\varepsilon+\varphi_\varepsilon^Q)(X_\varepsilon\circ\varphi_\varepsilon^Q)+2C(Y_\varepsilon+\varphi^Q,Y_\varepsilon,X_\varepsilon)+X_\varepsilon\circ R(Y_\varepsilon+\varphi_\varepsilon^Q)$$

▷ Assume that we can control $X_\varepsilon\circ Y_\varepsilon$ in $\mathscr{C}^{-0-}$.

$$3\lambda X_\varepsilon^{\diamond 2}(Y_\varepsilon+\varphi_\varepsilon^Q)-9\lambda c_{2,\varepsilon}\varphi_\varepsilon=3\lambda X_\varepsilon^{\diamond 2}\succ(Y_\varepsilon+\varphi_\varepsilon^Q)+3\lambda X_\varepsilon^{\diamond 2}\prec(Y_\varepsilon+\varphi_\varepsilon^Q)$$

$$+3\lambda X_\varepsilon^{\diamond 2}\circ Y_\varepsilon-3c_{2,\varepsilon}X_\varepsilon+3\lambda X_\varepsilon^{\diamond 2}\circ\varphi_\varepsilon^Q-9\lambda^2 c_{2,\varepsilon}(Y_\varepsilon+\varphi_\varepsilon^Q)$$

▷ Need of a further renormalization: the quantity $X_\varepsilon^{\diamond 2}\diamond Y_\varepsilon=X_\varepsilon^{\diamond 2}\circ Y_\varepsilon-3c_{2,\varepsilon}X_\varepsilon$ is under control in $\mathscr{C}^{-1/2-}$.

▷ **Paracontrolled ansatz**:

$$\varphi_\varepsilon^Q=\varphi_\varepsilon^\star\prec\!\!\prec Q_\varepsilon+\varphi_\varepsilon^\#$$

with $Q_\varepsilon\in\mathscr{C}^{1-}$, $\varphi_\varepsilon^\star\in\mathscr{C}^{1/2-}$ and $\varphi_\varepsilon^\#\in\mathscr{C}^{3/2-}$.

$$3\lambda X_\varepsilon^{\diamond 2}\circ\varphi_\varepsilon^Q-9\lambda^2 c_{2,\varepsilon}(Y_\varepsilon+\varphi_\varepsilon^Q)=3\lambda X_\varepsilon^{\diamond 2}\circ(\varphi_\varepsilon^\star\prec\!\!\prec Q_\varepsilon)-9\lambda^2 c_{2,\varepsilon}(Y_\varepsilon+\varphi_\varepsilon^Q)+3\lambda X_\varepsilon^{\diamond 2}\circ\varphi_\varepsilon^\#$$

$$=3\lambda\varphi_\varepsilon^\star X_\varepsilon^{\diamond 2}\circ Q_\varepsilon-9\lambda^2 c_{2,\varepsilon}(Y_\varepsilon+\varphi_\varepsilon^Q)+3\lambda C_{\prec\!\!\prec}(\varphi_\varepsilon^\star,Q_\varepsilon,X_\varepsilon^{\diamond 2})+3\lambda X_\varepsilon^{\diamond 2}\circ\varphi_\varepsilon^\#$$

How to choose φ_ε' and Q_ε?

Note that

$$\mathscr{L}\varphi_\varepsilon^Q = 3\lambda X_\varepsilon^{\diamond 2} \succ (Y_\varepsilon + \varphi_\varepsilon^Q) + \cdots$$

but also

$$\mathscr{L}\varphi_\varepsilon^Q = \mathscr{L}(\varphi_\varepsilon^\star \lll Q_\varepsilon + \varphi_\varepsilon^\#) = \varphi_\varepsilon' \lll \mathscr{L}Q_\varepsilon + \mathscr{L}\varphi_\varepsilon^\# + \cdots$$

so a natural choice is $\varphi_\varepsilon^\star = 3\lambda(Y_\varepsilon + \varphi_\varepsilon^Q)$ and $\mathscr{L}Q_\varepsilon = X_\varepsilon^{\diamond 2}$. In this case

$$3\lambda X_\varepsilon^{\diamond 2} \circ \varphi_\varepsilon^Q - 9\lambda^2 c_{2,\varepsilon}(Y_\varepsilon + \varphi_\varepsilon^Q) = 3\lambda\varphi_\varepsilon^\star (X_\varepsilon^{\diamond 2} \circ Q_\varepsilon - c_{2,\varepsilon}) + 3\lambda C_{\lll}(\varphi_\varepsilon^\star, Q_\varepsilon, X_\varepsilon^{\diamond 2}) + 3\lambda X_\varepsilon^{\diamond 2} \circ \varphi_\varepsilon^\#$$

and it can be shown that $X_\varepsilon^{\diamond 2} \diamond Q_\varepsilon = X_\varepsilon^{\diamond 2} \circ Q_\varepsilon - c_{2,\varepsilon}$ is under control in $\mathscr{C}^{0-}$.

Conclusion: starting from the equation

$$\mathscr{L}\varphi_\varepsilon^Q = 3\lambda X_\varepsilon^{\diamond 2}(Y_\varepsilon + \varphi_\varepsilon^Q) + 3\lambda^2 X_\varepsilon (Y_\varepsilon + \varphi_\varepsilon^Q)^2 + \lambda^3 (Y_\varepsilon + \varphi_\varepsilon^Q)^3 - 9\lambda c_{2,\varepsilon}\varphi_\varepsilon$$

and performing the change of variables

$$\varphi_\varepsilon^Q = \varphi_\varepsilon^\star \lll Q_\varepsilon + \varphi_\varepsilon^\#, \qquad \varphi_\varepsilon^\star = 3\lambda(Y_\varepsilon + \varphi_\varepsilon^Q)$$

we get an equation for $\varphi_\varepsilon^\#$ which reads

$$\mathscr{L}\varphi_\varepsilon^\# = \Phi_\varepsilon^\#(\mathbb{X}_\varepsilon, \varphi_\varepsilon^Q, \varphi_\varepsilon^\#)$$

where the r.h.s. depends continuosly on $\varphi_\varepsilon^Q, \varphi_\varepsilon^\#$ and on the datum of $\mathbb{X}_\varepsilon$:

$$\mathbb{X}_\varepsilon = (X_\varepsilon, X_\varepsilon^{\diamond 2}, X_\varepsilon^{\diamond 3}, X_\varepsilon \circ Y_\varepsilon, X_\varepsilon^{\diamond 2} \diamond Y_\varepsilon, X_\varepsilon^{\diamond 2} \diamond Q_\varepsilon)$$

The limit as $\varepsilon \to 0$ can be now estabilished via standard PDE estimates (for short times) once it is known that $\mathbb{X}_\varepsilon \to \mathbb{X}$ in a suitable topology.

Remark: Fundamental ideas coming from Lyons' Rough Paths theory.

Another way to state the previous analysis is to say that to each decomposition

$$\psi = X + Y + \psi^\star \lll Q + \psi^\#$$

of a function ψ we can associate a corresponding decomposition of the power φ^3

$$\psi^3 = G(\mathbb{X}, \psi^\star, \psi^\#) = X^3 + 3(Y + \psi^\star \lll Q + \psi^\#) \prec X^2 + \cdots$$

$$\mathbb{X} = (X, X^2, X^3, Y \circ X, Y \circ X^2, Q \circ X^2)$$

in such a way that, for any $a, b \in \mathbb{R}$

$$\psi^3 - 3a\psi - 3bX - 3b\psi^\star = G(R_{a,b}\mathbb{X}, \psi^\star, \psi^\#)$$

with $R_{a,b}\mathbb{X} = (X, X^2 - a, X^3 - 3a, Y \circ X, Y \circ (X^2 - a) - bX, Q \circ (X^2 - a) - b)$

Now if ψ is a distribution for which

$$\psi = X + Y + \psi^{\star} \prec\!\!\prec Q + \psi^{\#}$$

we can set

$$\psi_{\varepsilon} = \rho_{\varepsilon} * \psi = \rho_{\varepsilon} * X + \rho_{\varepsilon} * Y + \psi^{\star} \prec\!\!\prec \rho_{\varepsilon} * Q + \rho_{\varepsilon} * \psi^{\#} + [\psi^{\star} \prec\!\!\prec, \rho_{\varepsilon} *]Q$$

so that $X_{\varepsilon} = \rho_{\varepsilon} * X$

$$\psi_{\varepsilon}^{3} - 3a_{\varepsilon}\psi_{\varepsilon} - 3bX_{\varepsilon} - 3b\,\psi^{\star} = G(R_{a_{\varepsilon},b_{\varepsilon}}\mathbb{X}_{\varepsilon}, \psi^{\star}, \rho_{\varepsilon} * \psi^{\#} + [\psi^{\star} \prec\!\!\prec, \rho_{\varepsilon} *]Q)$$

$$\mathbb{X}_{\varepsilon} = (X_{\varepsilon}, X_{\varepsilon}^{2}, X_{\varepsilon}^{3}, Y_{\varepsilon} \circ X_{\varepsilon}, Y_{\varepsilon} \circ X_{\varepsilon}^{2}, Q_{\varepsilon} \circ X_{\varepsilon}^{2})$$

with $Z_{\varepsilon} = \rho_{\varepsilon} * Z$ for $Z = X, Y, Q$. And if $R_{a_{\varepsilon},b_{\varepsilon}}\mathbb{X}_{\varepsilon} \to \mathbb{X}_{\mathrm{ren}}$ as $\varepsilon \to 0$ then

$$\psi_{\varepsilon}^{3} - 3a_{\varepsilon}\psi_{\varepsilon} - 3bX_{\varepsilon} - 3b\,\psi^{\star} \to \psi_{\mathrm{ren}}^{3} = G(\mathbb{X}_{\mathrm{ren}}, \psi^{\star}, \psi^{\#})$$

At this point we are capable of defining solutions of the singular SPDE which we expect in the limit of the reaction diffusion equation.

1) Given the noise ξ we construct its associated "model"

$$R_{c_{1,\varepsilon},c_{2,\varepsilon}}\mathbb{X}_\varepsilon \to \mathbb{X}(\xi) = (X, X^{\diamond 2}, X^{\diamond 3}, Y \diamond X, Y \diamond X^{\diamond 2}, Q \diamond X^{\diamond 2})$$

by passing to the limit in the canonical model $R_{c_{1,\varepsilon},c_{2,\varepsilon}}\mathbb{X}_\varepsilon$ for ξ_ε with suitable renormalization constants $c_{1,\varepsilon}, c_{2,\varepsilon}$.

2) For fixed $\mathbb{X}$ we say that a distribution φ is a solution to

$$\mathscr{L}\varphi = \lambda\varphi^{\diamond 3} + \xi$$

if $\varphi = X + \lambda Y + \lambda\varphi^\star \ll Q + \lambda\varphi^\#$ and

$$\mathscr{L}\varphi = \lambda G(\mathbb{X}, \lambda\varphi^\star, \lambda\varphi^\#) + \xi$$

This implies in particular that $\varphi^\star$ can be chosen to be equal to $3\lambda(Y + \lambda\varphi^\star \ll Q + \lambda\varphi^\#)$

Thanks!