Global controllability and mixing for the Burgers equation with localised finite-dimensional external force

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The aim of this short note to give a summary of some recent results on global controllability of the viscous Burgers equation and mixing properties of the associated stochastic problem. We also outline the main ideas of the proofs. The details are given in [1,2].

Keywords: Burgers equation, exact and approximate controllability, mixing, localised external force

1. Introduction

We consider the problem

$$\partial_t u - \nu \partial_x^2 u + u \partial_x u = f(t,x), \tag{1}$$

$$u(t,0) = u(t,1) = 0, \tag{2}$$

$$u(0,x) = u_0(x), \tag{3}$$

where $x \in I = (0,1)$ and $t \geq 0$, $\nu > 0$ is a fixed parameter, $u = u(t,x)$ is an unknown function with range in $\mathbb{R}$, u_0 is a given initial condition, and f is an external force. Let us fix a closed interval $[a,b] \subset I$ and denote by E the two-dimensional vector space spanned by the functions

$$e_1(x) = \sin\Bigl(\pi \frac{x-a}{b-a}\Bigr), \quad e_2(x) = \sin\Bigl(2\pi \frac{x-a}{b-a}\Bigr), \quad x \in [a,b], \tag{4}$$

extended to $I \setminus [a,b]$ by zero. We deal with following two situations:

Control problem. *The external force has the form*

$$f(t,x) = h(x) + \zeta(t,x), \tag{5}$$

where[a] $h \in H^1(I)$ *is a given function and* $\zeta(t,x)$ *is a control with range in* E.

Stochastic problem. *The external force is stochastic and is given by the relation*

$$f(t,x) = h(x) + \eta(t,x), \tag{6}$$

in which $h \in H^1(I)$ *is a deterministic function and* η *is an* E*-valued white noise of the form*

$$\eta(t,x) = \frac{\partial}{\partial t}\sum_{j=1}^{2} b_j \beta(t) e_j(x), \tag{7}$$

where $b_j > 0$ *are some numbers and* $\{\beta_j\}$ *are independent standard Brownian motions.*

In the first case, we are interested in global approximate controllability to trajectories, while the second problem concerns the uniqueness of a stationary distribution for the Markov process associated with (1), (6), (7) and the large-time behaviour of its trajectories. These two questions were intensively studied in the literature, and we now describe some results concerning the Burgers equation.

The problem of asymptotic behaviour of trajectories for stochastic Burgers equation was first studied by Sinai[3] who established the convergence of the laws of solutions to a measure independent of the initial condition. This result was refined and developed in a number of works, and it is now well known that the Markov process corresponding to problem (1), (6) with $h \equiv 0$ has a unique stationary distribution, and all other solutions converge to it weakly as $t \to +\infty$; see the paper[4] and the references therein for the exact result and discussion. The control problem was investigated by Fursikov and Imanuvilov[5,6], who proved the local exact controllability to trajectories and the absence of global approximate controllability. Agrachev and Sarychev[7,8] (see also[9]) developed a general approach, applicable to the Burgers equation, that enables one to study the approximate controllability of nonlinear PDE's by a low-dimensional control. Chapouly[10] applied Coron's return method to study the exact controllability of the Burgers equation by two boundary controls combined with one distributed control. The reader is referred to the books[11–15] for more details on these two subjects and further results for other equations.

[a]We refer the reader to the Notation below for the definition of functional spaces.

The aim of this short note is to give a summary of the results established in [1,2]. We first discuss the problem of global exponential stabilisation of non-stationary solutions for (1), (5) by a localised control. This property combined with the local exact controllability implies that the Burgers equation is exactly controllable to trajectories by a control localised in the physical space. Moreover, invoking a result of Phan and Rodrigues [16], one can easily deduce that the Burgers equation is approximately controllable to trajectories by a two-dimensional localised control. We next turn to the stochastic counterpart of these results. Namely, the contraction property for the resolving operator of the Burgers equation combined with the global approximate controllability to trajectories implies that problem (1), (6), (7) has a unique stationary distribution, which is asymptotically stable (in the weak topology of measures) as $t \to +\infty$. We confine ourselves to giving some ideas of the proofs of these results, referring the reader to [1,2] for details.

Notation

We write $I = (0,1)$ and $\mathbb{R}_+ = [0,+\infty)$ and denote by $L^p(I)$ and $H^s(I)$ the usual Lebesgue and Sobolev spaces on I, endowed with the standard norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_s$, respectively, and by $H_0^s(I)$ the closure in $H^s(I)$ of the space of infinitely smooth functions on I with compact support. Very often, we shall omit the interval I from the notation and write L^p, H^s, and H_0^s. Given a closed interval $J \subset \mathbb{R}$ and a separable Banach space X, let $C(J,X)$ be the space of continuous functions from J to X and $L^p(J,X)$ be the space of Borel-measurable functions $f : J \to X$ such that

$$\|f\|_{L^p(J,X)}^p := \int_J \|f(t)\|_X^p \mathrm{d}t < \infty,$$

with obvious modification in the case $p = \infty$.

2. Controllability to trajectories

In this section, we consider problem (1)–(3), in which $u_0 \in L^2$, and the external force f has the form (5). We shall always assume that $h \in H^1$ is a fixed function, and ζ is a control taking values in the closed subspace

$$F = \{v \in H_0^1(I) : \operatorname{supp} v \subset [a,b]\},$$

where $[a,b] \subset I$ is an interval. Let us recall that the initial-boundary value problem (1)–(3) is well posed in the space $H := L^2$. Namely, for any $u_0 \in H$

and $f \in L^1_{\text{loc}}(\mathbb{R}_+, H)$, there is a unique function

$$u \in L^2_{\text{loc}}(\mathbb{R}_+, H^1_0) \cap W^2_{\text{loc}}(\mathbb{R}_+, H^{-1})$$

satisfying (1) and (3). We denote by $\mathcal{R}_t(u_0, f)$ the mapping that takes the pair (u_0, f) to $u(t)$. The following theorem is the main result of[2].

Theorem 2.1. *Let $\nu > 0$, $[a,b] \subset I$, and $h \in H^1$ be given. Then there are positive numbers C and γ such that the following property holds: for arbitrary $u_0, \hat{u}_0 \in H$ there is a piecewise continuous function $\zeta : \mathbb{R}_+ \to F$ such that*

$$\|\mathcal{R}_t(u_0, h+\zeta) - \mathcal{R}_t(\hat{u}_0, h)\|_{H^1} + \|\zeta(t)\|_{H^1} \le Ce^{-\gamma t} \min\big(\|u_0 - \hat{u}_0\|_{L^1}^{2/5}, 1\big), \quad (8)$$

where $t \ge 1$.

In other words, any solution of (1), (2) corresponding to the zero control $\zeta \equiv 0$ can be exponentially stabilized by an H^1-smooth control localized in a given interval $[a,b]$. Before giving some ideas of the proof of this result, we state two corollaries. The first one concerns the exact controllability to trajectories for problem (1), (2).

Corollary 2.1. *Under the hypotheses of Theorem 2.1, there is $T > 0$ such that, given initial conditions $u_0, \hat{u}_0 \in H$, one can find a control $\zeta \in L^2([0,T], F)$ for which*

$$\mathcal{R}_T(u_0, h+\zeta) = \mathcal{R}_T(\hat{u}_0, h).$$

To prove this property, it suffices to use the control constructed in Theorem 2.1 to steer a solution of (1), (2) sufficiently close to the reference trajectory $\hat{u}(t) = \mathcal{R}_t(\hat{u}_0, h)$ and then to apply the Fursikov–Imanuvilov result about local exact controllability to trajectories; see Section I.6 in[6].

The second corollary concerns the global approximate controllability to trajectories for the Burgers equation by a localised control of low dimension. Recall that we denote by E the vector span of the functions $e_1, e_2 \in H^1_0$ defined by (4) on $[a,b]$ and extended by zero outside $[a,b]$.

Corollary 2.2. *Under the hypotheses of Theorem 2.1, there is $T > 0$ such that, given $u_0, \hat{u}_0 \in H$ and $\varepsilon > 0$, one can find a control $\zeta \in C^\infty([0,T], E)$ for which*

$$\|\mathcal{R}_T(u_0, h+\zeta) - \mathcal{R}_T(\hat{u}_0, h)\|_{L^2} \le \varepsilon.$$

Thus, the global approximate controllability to trajectories holds, provided that the time of control is sufficiently large. To establish this property, it suffices to combine Corollary 2.1 with the results of Section 5 in[16].

Scheme of the proof of Theorem 2.1. We begin with two remarks:

Regularisation. The Burgers equation is a globally well-posed nonlinear parabolic PDE and, hence, possesses a regularising property. Namely, if $f(t,x)$ is a bounded function of time with range in H^1, then for any initial condition $u_0 \in H$ the corresponding solution $u(t,x)$ of (1)–(3) is bounded in H^2 on the half-line $[1,+\infty)$ by universal constant R not depending on u_0. Thus, there is no loss of generality in assuming that both initial functions u_0 and $\hat{u}_0$ belong to the ball of radius R centred at zero in the space $H_0^1 \cap H^2$.

Interpolation. The following inequality is well known:

$$\|v\|_{H^1} \le C\|v\|_{L^1}^{2/5}\|v\|_{H^2}^{3/5} \quad \text{for any } v \in H^2.$$

Therefore, if we construct a bounded control $\zeta : \mathbb{R}_+ \to F$ such that

$$\|\mathcal{R}_t(u_0, h+\zeta) - \mathcal{R}_t(\hat{u}_0, h)\|_{L^1} \le C_1 e^{-\gamma_1 t}\|u_0 - \hat{u}_0\|_{L^1}, \quad t \ge 0, \tag{9}$$

then the required estimate for the difference between two solutions will follow from the above interpolation inequality and the boundedness of the H^2-norms.

A key for the proof of (9) is the following result about the linear PDE

$$\partial_t w - \nu\partial_x^2 w + \partial_x(a(t,x)w) = 0, \tag{10}$$

where $a(t,x)$ is a given sufficiently regular function.

Proposition 2.1. *For any closed interval $I' \subset I$, one can find positive numbers ε and $q < 1$ such that any solution $w(t,x)$ of (10) satisfies one of the inequalites*

$$\|w(1)\|_{L^1} \le q\|w(0)\|_{L^1} \quad \text{or} \quad \|w(1)\|_{L^1(I')} \ge \varepsilon\|w(0)\|_{L^1}. \tag{11}$$

In other words, either the L^1-norm w contracts by a factor strictly less than 1, or a nontrivial mass is concentrated on I'. Now note that if v and $\hat{u}$ are two solutions of (1), (2) with $\zeta \equiv 0$, then their difference $w = v - \hat{u}$ satisfies Eq. (10) and, hence, one of the inequalities in (11). In both cases, we can modify $v(t,x)$ on the domain $[0,1] \times [a,b]$ so that the resulting function $u(t,x)$ satisfies the inequality

$$\|u(1) - \hat{u}(1)\|_{L^1} \le \theta\,\|u(0) - \hat{u}(0)\|_{L^1}, \tag{12}$$

where $\theta < 1$ depends only on the size of initial conditions. Since v and u coincide on $[0,1]\times(I\setminus[a,b])$, we see that $u(t,x)$ is a solution of the controlled equation (1) in which $\operatorname{supp}\zeta(t,\cdot) \subset [a,b]$. Iteration of (12) gives the required inequality (9). □

3. Mixing of the stochastic flow

In this section, we consider problem (1)–(3), in which the right-hand side is a stochastic process of the form (6), (7). The existence and uniqueness of a solution is well known, and our goal is to study the large-time behaviour of trajectories. The family of all solutions for (1), (2) form a Markov process in the phase space $H = L^2(I)$, and we denote by $P_t(u, \Gamma)$ its transition function and by

$$\mathfrak{P}_t : C_b(H) \to C_b(H), \quad \mathfrak{P}_t^* : \mathcal{P}(H) \to \mathcal{P}(H)$$

the corresponding Markov operators. Here $C_b(H)$ stands for the space of bounded continuous functions $g : H \to \mathbb{R}$, endowed with the L^∞-norm, and $\mathcal{P}(H)$ denotes the set of all probability Borel measures on H. The latter is endowed with the topology of weak convergence, which is denotes by $\rightharpoonup$; see Chapter 1 in[15] for the definition of all these concepts.

Theorem 3.1. *Let $\nu > 0$, $[a, b] \subset I$, and $h \in H^1$ be given and let the coefficients b_j defining the random force (7) be positive. Then there is a unique measure $\mu \in \mathcal{P}(H)$ such that $\mathfrak{P}_t^* \mu = \mu$ for all $t \geq 0$. Moreover, μ is asymptotically stable in the sense that, for any $g \in C_b(H)$ and $\lambda \in \mathcal{P}(H)$, we have*

$$\mathfrak{P}_t g \to (g, \mu) \quad \text{in } C_b(H) \text{ as } t \to +\infty, \tag{13}$$

$$\mathfrak{P}_t^* \lambda \rightharpoonup \mu \quad \text{in } \mathcal{P}(H) \text{ as } t \to +\infty, \tag{14}$$

where (g, μ) stands for the integral of g over H against the measure μ.

Scheme of the proof. The existence of a stationary measure is a simple consequence of the Bogolyubov–Krylov argument (e.g., see Section 14 in[11]), and therefore we confine ourselves to outlining the proof of uniqueness and asymptotic stability. To this end, let us recall a sufficient condition for the validity of these properties. Namely, as is proved in Section 3.1 of[15], the uniqueness of a stationary measure μ and the convergence relations (13) and (14) are valid if the following two properties hold:

Stability. *There is a function $\gamma(r) \geq 0$ going to zero as $r \to 0$ such that, for any 1-Lipschitz function $g \in C_b(H)$ satisfying $\|g\|_{L^\infty} \leq 1$, we have*

$$\sup_{t \geq 1} \bigl| \bigl(g, P_t(u, \cdot)\bigr) - \bigl(g, P_t(v, \cdot)\bigr) \bigr| \leq \gamma(r) \quad \text{for } \|u - v\|_H \leq r. \tag{15}$$

Given an initial condition $u_0 \in H$ and a closed ball $B \subset H$, let $\tau_{u_0}(B)$ be the first instant of time when the trajectory $\mathcal{R}_t(u_0, h + \eta)$ hits B.

Recurrence. *There is a sequence of balls $B_m \subset H$, whose diameters go zero as $m \to \infty$, and functions $p_m(t)$, going to zero as $t \to +\infty$, such that*

$$\mathbb{P}\{\tau_v(B_m) > t\} \le p_m(t) \quad \textit{for any } v \in H,\ m \ge 1,\ t > 0. \tag{16}$$

The stability is a straightforward consequence of the regularisation for the viscous Burgers equation and the following *contraction property*:

$$\|\mathcal{R}_t(u_0, h+\eta) - \mathcal{R}_t(v_0, h+\eta)\|_{L^1} \le \|u_0 - v_0\|_{L^1} \quad \text{for } t \ge 0,\ u_0, v_0 \in H. \tag{17}$$

Let us consider the recurrence. A regularisation property for the stochastic Burgers equation enables one to find positive numbers $s \in (1,2)$, T_1, R, and p such that

$$\mathbb{P}\big\{\|\mathcal{R}_{T_1}(u_0, h+\eta)\|_{H^s} \le R\big\} \ge p \quad \text{for any } u_0 \in H.$$

Furthermore, the controllability property established in Corollary 2.2 implies that, if $\hat{u} \in H_0^1 \cap H^2$ is a time-independent solution of (1) with $f = h$, then there is $T_2 > 0$ such that, for any $u_0 \in H_0^1 \cap H^s$ satisfying $\|u_0\|_s \le R$, we have

$$\mathbb{P}\big\{\|\mathcal{R}_{T_2}(u_0, h+\eta) - \hat{u}\|_{L^2} \le 1/m\big\} \ge \varepsilon_m \quad \text{for } m \ge 1,$$

where $\varepsilon_m > 0$ are some numbers not depending on u_0. The above two inequalities imply the required estimate (16), in which $B_m \subset H$ is the ball of radius $1/m$ centred at $\hat{u}$; e.g., see the proof of Proposition 3.3.6 in [15]. □

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