

A coupling approach to randomly forced nonlinear PDE's. II

Sergei Kuksin Andrey Piatnitski Armen Shirikyan

Abstract

We consider a class of discrete time random dynamical systems and establish the exponential convergence of its trajectories to a unique stationary measure. The result obtained applies, in particular, to the 2D Navier–Stokes system and multidimensional complex Ginzburg–Landau equation with random kick-force.

1 Main result

The present paper is an immediate continuation of [KS2] and is devoted to studying the following random dynamical system (RDS) in a Hilbert space H :

$$u^k = S(u^{k-1}) + \eta_k, \quad k \geq 1. \quad (1.1)$$

Here $S: H \rightarrow H$ is a locally Lipschitz operator such that $S(0) = 0$ and $\{\eta_k\}$ is a sequence of i.i.d. bounded random variables of the form

$$\eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j, \quad (1.2)$$

where $\{e_j\}$ is an orthonormal basis in H , $b_j \geq 0$ are some constants such that $\sum b_j^2 < \infty$, and ξ_{jk} are scalar random variables. The exact conditions imposed on S can be found in [KS2, Section 2] (see conditions (A) – (C)). Roughly speaking, they mean that S is compact and $S^n(u) \rightarrow 0$ as $n \rightarrow \infty$ uniformly on bounded subsets of H . Concerning the random variables ξ_{jk} , we assume that they satisfy the following condition:

- (D) *For any j , the random variables ξ_{jk} , $k \geq 1$, have the same distribution $\pi_j(dr) = p_j(r) dr$, where the densities $p_j(r)$ are functions of bounded variation such that $\text{supp } p_j \subset [-1, 1]$ and $\int_{|r| \leq \varepsilon} p_j(r) dr > 0$ for all $j \geq 1$ and $\varepsilon > 0$. We normalise the functions p_j to be continuous from the right.*

Let us denote by $\mathfrak{P} = \mathfrak{P}(k, v, \cdot)$ the Markov transition function for (1.1) and by \mathfrak{P}_k the associated Markov semigroup acting on the space of bounded

continuous functions on H . It was proved in [KS1, KS2] that, under the above conditions, the RDS (1.1) has a unique stationary measure μ , provided that

$$b_j \neq 0 \quad \text{for } 1 \leq j \leq N, \quad (1.3)$$

where $N \geq 1$ is sufficiently large. Moreover, it is shown in [KS2] that any trajectory u^k of the RDS (1.1) converges to μ (in an appropriate sense) with the rate $e^{-c\sqrt{k}}$. The aim of this paper is to prove that this convergence is exponential:

Theorem 1.1. *There is a constant $c > 0$ and an integer $N \geq 1$ such that if (1.3) holds, then*

$$|\mathfrak{P}_k f(u) - (\mu, f)| \leq C_R e^{-ck} (\sup_H |f| + \text{Lip}(f)) \quad \text{for } k \geq 0, \quad u \in B_H(R), \quad (1.4)$$

where $B_H(R)$ is the ball in H of radius R centred at zero, f is an arbitrary bounded Lipschitz function on H , and $C_R > 0$ is a constant depending on R solely.

As it is shown in [KS1], the conditions (A) – (D) (under which Theorem 1.1 is proved) are satisfied for the 2D Navier–Stokes system and multidimensional complex Ginzburg–Landau equation perturbed by a kick-force of the form

$$\eta(t, x) = \sum_{k=1}^{\infty} \eta_k(x) \delta(t - k),$$

where the kicks η_k are i.i.d. random variables which can be written in the form (1.2) in an appropriate functional space H .

We note that the exponential convergence to the stationary measure was established earlier for the Navier–Stokes system perturbed by a finite-dimensional white noise force. Namely, Bricmont, Kupiainen, Lefevre [BKL] showed that for μ -almost all¹ initial functions the corresponding solution converges to the stationary measure exponentially fast. See [KS2] for a discussion of these works and of the techniques they use. Our proof of the exponential convergence is much shorter. It exploits the coupling approach from [KS2].

For the reader’s convenience, we recall some notations used in [KS2].

Notations

We abbreviate a pair of random variables ξ_1, ξ_2 or points u_1, u_2 to $\xi_{1,2}$ and $u_{1,2}$, respectively. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an integer $k \geq 1$ (the case $k = \infty$ is not excluded), we denote by Ω^k the space $\Omega \times \cdots \times \Omega$ (k times) endowed with the σ -algebra $\mathcal{F} \times \cdots \times \mathcal{F}$ and the measure $\mathbb{P} \times \cdots \times \mathbb{P}$. The points of Ω^k will be denoted by $\omega^k = (\omega_1, \dots, \omega_k)$, where $\omega_j \in \Omega$.

$C_b(H)$ is the space of bounded continuous functions on H with the supremum norm $\|\cdot\|_{\infty}$.

¹We denote by μ the unique stationary measure.

$L(H)$ is the space of bounded Lipschitz functions on H endowed with the norm $\|f\|_L = \sup_H |f| + \text{Lip}(f)$, where $\text{Lip}(f)$ is the Lipschitz constant of f . $\mu_v(k)$ denotes the measure $\mathfrak{P}(k, v, \cdot)$. $B_H(R)$ is the closed ball of radius $R > 0$ centred at zero.

Acknowledgements. The first and third authors were supported by grant GR/N63055/01 from EPSRC.

2 Proof of the theorem

Step 1. For any two probability Borel measures μ_1 and μ_2 on H we set

$$\|\mu_1 - \mu_2\|_L^* = \sup_{\|f\|_L \leq 1} |(\mu_1 - \mu_2, f)|.$$

In view of Lemma 1.2 in [KS2], to prove the theorem it suffices to show that for any $R > 0$ there is $C_R > 0$ such that

$$\|\mu_{u_1}(k) - \mu_{u_2}(k)\|_L^* \leq C_R e^{-ck} \quad \text{for } u_1, u_2 \in B_H(R), \quad k \geq 1,$$

where $c > 0$ is a constant not depending on R . As in [KS2], we can restrict our consideration to the compact invariant set \mathcal{A} (see formula (2.5) in [KS2]). Moreover, by Lemma 1.3 in [KS2], the required inequality (1.4) will be proved if we show that for any $u_1, u_2 \in \mathcal{A}$ and any integer $k \geq 1$ there is a coupling $y_{1,2}(k) = y_{1,2}(k, u_1, u_2)$ for the measures $\mu_{u_{1,2}}(k)$ such that

$$\mathbb{P}\{\|y_1(k) - y_2(k)\| \geq C e^{-ck}\} \leq C e^{-ck} \quad \text{for } k \geq 1, \quad (2.1)$$

where $\|\cdot\|$ is the norm in H and $C > 0$ is a constant not depending on $u_1, u_2 \in \mathcal{A}$ and k . Finally, repeating the argument in Step 2 of the proof of Theorem 2.1 in [KS2, Section 3.2], we see that it suffices to find an integer $l \geq 1$ and to construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of couplings $y_{1,2}^n(u_1, u_2, \omega^n)$, $\omega^n \in \Omega^n$, for the measures $\mu_{u_{1,2}}(nl)$, $n \geq 1$, such that the maps $y_{1,2}^n$ are measurable with respect to (u_1, u_2, ω^n) and satisfy the inequality

$$\mathbb{P}\{\|y_1^n - y_2^n\| \geq e^{-c'n}\} \leq e^{-c'n} \quad \text{for } n \geq 1. \quad (2.2)$$

If (2.2) is established, then (2.1) holds with $c = c'/l$ and some constant $C > 1$.

Step 2. To prove (2.2), we shall need the following result, which is a particular case of Lemma 3.3 in [KS2].

Lemma 2.1. *Under the conditions of Theorem 1.1, there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, positive constants $d_0 < 1/2$ and θ , and an integer $l \geq 1$ such that for any $u_1, u_2 \in \mathcal{A}$ the measures $\mu_{u_{1,2}}(l)$ admit a coupling $U_{1,2} = U_{1,2}(u_1, u_2; \omega)$ such that the following assertions hold:*

- (i) *The maps $U_{1,2}(u_1, u_2, \omega)$ are measurable with respect to $(u_1, u_2, \omega) \in \mathcal{A} \times \mathcal{A} \times \Omega$.*

(ii) If $\|u_1 - u_2\| > d_0$, then

$$\mathbb{P}\{\|U_1 - U_2\| \leq d_0\} \geq \theta. \quad (2.3)$$

(iii) If $d = \|u_1 - u_2\| \leq 2^{-r}d_0$ for some integer $r \geq 0$, then

$$\mathbb{P}\{\|U_1 - U_2\| \leq d/2\} \geq 1 - 2^{-r-3}. \quad (2.4)$$

Remark 2.2. In [KS2], it is proved that the probability on the left-hand side of (2.4) can be estimated from below by $1 - 2^{-r-1}$. However, it is not difficult to see that the term 2^{-r-1} can be replaced by 2^{-r-3} if the constant d_0 is sufficiently small.

Let us fix arbitrary $u_1, u_2 \in \mathcal{A}$ and define a sequence of random variables $y_{1,2}^n = y_{1,2}^n(u_1, u_2, \boldsymbol{\omega}^n)$, $\boldsymbol{\omega}^n = (\boldsymbol{\omega}^{n-1}, \omega_n) \in \Omega^n$, by the rule $y_{1,2}^0 = u_{1,2}$ and

$$y_{1,2}^n(u_1, u_2, \boldsymbol{\omega}^n) = U_{1,2}(y_1(u_1, u_2, \boldsymbol{\omega}^{n-1}), y_2(u_1, u_2, \boldsymbol{\omega}^{n-1}), \omega_n), \quad n \geq 1.$$

We shall show that $y_{1,2}^n$ satisfy (2.2) for all $n \geq 0$.

Step 3. Let us introduce a probability space $(\Omega^\infty, \mathcal{F}^\infty, \mathbb{P}^\infty)$ as the countable product of $(\Omega, \mathcal{F}, \mathbb{P})$ and note that the random variables $y_{1,2}^n$, $n \geq 0$, can be extended to Ω^∞ by the natural formula

$$\tilde{y}_{1,2}^n(u_1, u_2, \boldsymbol{\omega}^\infty) = y_{1,2}^n(u_1, u_2, \boldsymbol{\omega}^n), \quad \boldsymbol{\omega}^\infty = (\boldsymbol{\omega}^n, \omega_{n+1}, \omega_{n+2}, \dots).$$

Thus, without loss of generality, we can assume that they are defined on the same probability space Ω^∞ . To simplify notation, we denote the extended random variables by the same symbol $y_{1,2}^n$ and write $(\Omega, \mathcal{F}, \mathbb{P})$ instead of $(\Omega^\infty, \mathcal{F}^\infty, \mathbb{P}^\infty)$.

For any non-negative integers r and n , we define the events

$$Q_{n,r} = \{\omega \in \Omega : d_r \leq \|y_1^n(\omega) - y_2^n(\omega)\| < d_{r-1}\},$$

where $d_r = 2^{-r}d_0$ for $r \geq 1$ and $d_{-1} = +\infty$. Let us denote $p_{n,r} = \mathbb{P}(Q_{n,r})$ and set

$$\zeta_n = \sum_{r=0}^{\infty} 2^{-r} p_{n,r}.$$

We claim that

$$\zeta_n \leq \gamma^n, \quad n \geq 0, \quad (2.5)$$

where $\gamma < 1$ is a positive constant not depending on $u_1, u_2 \in \mathcal{A}$ and n .

Taking inequality (2.5) for granted, let us complete the proof of (2.2).

For any real number $s \geq 0$, we denote by $[s]$ its integer part. Let us choose $\alpha > 0$ so small that $\beta := 2^\alpha \gamma < 1$ and consider the event

$$R_n := \{\|y_1^n(\omega) - y_2^n(\omega)\| \geq d_{[\alpha n]}\} = \bigcup_{r=0}^{[\alpha n]} Q_{n,r}.$$

In view of (2.5), we have

$$\mathbb{P}(R_n) = \sum_{r=0}^{[\alpha n]} p_{n,r} \leq 2^{[\alpha n]} \sum_{r=0}^{[\alpha n]} 2^{-r} p_{n,r} \leq 2^{\alpha n} \zeta_n \leq (2^\alpha \gamma)^n = \beta^n.$$

Since $d_0 \leq 1/2$, we see that $d_{[\alpha n]} = 2^{-[\alpha n]} d_0 \leq 2^{-\alpha n}$. We have thus proved that

$$\mathbb{P}\{\|y_1^n(\omega) - y_2^n(\omega)\| \geq 2^{-\alpha n}\} \leq \beta^n.$$

This inequality implies (2.2) with $c' = \min\{\alpha \log 2, \log \beta^{-1}\}$.

Step 4. Thus, it remains to establish (2.5). Since $\zeta_0 \leq 1$, it is sufficient to show that

$$\zeta_n \leq \gamma \zeta_{n-1}, \quad n \geq 1. \quad (2.6)$$

We have

$$\begin{aligned} \zeta_n &= \sum_{r=0}^{\infty} 2^{-r} \mathbb{P}(Q_{n,r}) = \sum_{r=0}^{\infty} 2^{-r} \sum_{m=0}^{\infty} p_{n-1,m} \mathbb{P}\{Q_{n,r} | Q_{n-1,m}\} \\ &\leq \sum_{m=0}^{\infty} p_{n-1,m} \left\{ \sum_{r=0}^m \mathbb{P}\{Q_{n,r} | Q_{n-1,m}\} + 2^{-(m+1)} \sum_{r=m+1}^{\infty} \mathbb{P}\{Q_{n,r} | Q_{n-1,m}\} \right\}. \end{aligned} \quad (2.7)$$

Let us estimate the two sums in r in the right-hand side of (2.7). In view of inequality (2.4) with $d \in [d_m, d_{m-1})$, for $m \geq 1$ we have

$$\sum_{r=0}^m \mathbb{P}\{Q_{n,r} | Q_{n-1,m}\} = \mathbb{P}\{\|y_1^n - y_2^n\| \geq d_m | Q_{n-1,m}\} \leq 2^{-m-2}, \quad (2.8)$$

$$\sum_{r=m+1}^{\infty} \mathbb{P}\{Q_{n,r} | Q_{n-1,m}\} = \mathbb{P}\{\|y_1^n - y_2^n\| < d_m | Q_{n-1,m}\} \leq 1. \quad (2.9)$$

We now consider the case $m = 0$. Inequality (2.3) implies that

$$\sigma_n := \mathbb{P}\{Q_{n,0} | Q_{n-1,0}\} \leq 1 - \theta.$$

Hence, denoting by $Q_{n,0}^c$ the complement of $Q_{n,0}$, we derive

$$\begin{aligned} \mathbb{P}\{Q_{n,0} | Q_{n-1,0}\} + 2^{-1} \sum_{r=1}^{\infty} \mathbb{P}\{Q_{n,r} | Q_{n-1,0}\} &= \sigma_n + 2^{-1} \mathbb{P}\{Q_{n,0}^c | Q_{n-1,0}\} \\ &= \sigma_n + (1 - \sigma_n)/2 \leq 1 - \theta/2. \end{aligned} \quad (2.10)$$

Substitution of (2.8) – (2.10) into (2.7) results in

$$\zeta_n \leq (1 - \theta/2) p_{n-1,0} + \frac{3}{4} \sum_{m=1}^{\infty} 2^{-m} p_{n-1,m} \leq \gamma \zeta_{n-1},$$

if we choose $\gamma = \max\{1 - \theta/2, 3/4\} < 1$. The proof of Theorem 1.1 is complete.

References

- [BKL] J. BRICMONT, A. KUPIAINEN, R. LEFEVERE, *Exponential mixing for the 2D stochastic Navier–Stokes dynamics*, preprint (2000).
- [EMS] W. E, J. C. MATTINGLY, YA. G. SINAI, *Gibbsian dynamics and ergodicity for the stochastically forced Navier–Stokes equation*, preprint (2000).
- [KS1] S. KUKSIN, A. SHIRIKYAN, *Stochastic dissipative PDE’s and Gibbs measures*, *Comm. Math. Phys.* **213** (2000), 291–330.
- [KS2] S. KUKSIN, A. SHIRIKYAN, *A coupling approach to randomly forced nonlinear PDE’s. I*, *Comm. Math. Phys.*, see www.ma.hw.ac.uk/~kuksin.