DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **10**, Number **1&2**, January & March **2004**

Website: http://AIMsciences.org

pp. **517–542**

QUALITATIVE PROPERTIES OF SOLUTIONS FOR LINEAR AND NONLINEAR HYPERBOLIC PDE'S

Armen Shirikyan

Laboratoire de Mathématiques
Université de Paris-Sud XI, Bâtiment 425
91405 Orsay Cedex, France

Leonid Volevich

Keldysh Institute of Applied Mathematics
Russian Academy of Sciences, 4 Miusskaya Square
125047 Moscow, Russia

Dedicated to Mark Iosifovich Vishik on the occasion of his eightieth birthday

Abstract. We present a number of results concerning large-time qualitative behavior of solutions for high-order hyperbolic equations and first-order hyperbolic systems. We discuss the properties of exponential stability and exponential dichotomy, construction of stable, unstable, and center manifolds, Grobman–Hartman type theorems on linearization of the phase portrait, and existence and uniqueness of time-bounded and almost periodic (AP) solutions.

0. **Introduction.** The aim of this article is to present, without proof, a number of results of the authors on qualitative behavior of solutions for linear and nonlinear hyperbolic PDE's. To be precise, let us consider the scalar equation

$$P(t,x,D_t,D_x)u(t,x)=f(t,x),\quad (t,x)\in\mathbb{R}^{n+1}, \tag{0.1}$$

where $D_t=-i\partial/\partial t$, $D_x=-i\partial/\partial x$, and $P(t,x,D_t,D_x)$ is a strictly hyperbolic operator of order m with smooth coefficients. As is known (see [15, 11, 7, 4]), the Cauchy problem for Eq. (0.1) is well-posed: for any continuous function $f(t,\cdot):[0,\infty)\to L^2(\mathbb{R}_x^n)$ there is a unique solution $u(t,x)$ for (0.1) that satisfies the initial conditions

$$D_t^j u(0,x)=u_j(x),\quad j=0,\dots,m-1, \tag{0.2}$$

where $u_j(x)$ are given functions belonging to appropriate functional classes. A natural question arises: what is the rate of growth of solutions as $t\to+\infty$?

In the case of equations with constant coefficients, an answer to the above question is given in terms of the characteristic roots of the full symbol $P(\tau,\xi)$ and the rate of growth of the right-hand side of the equation. For instance, if the L^2-norm of $f(t,\cdot)$ can be estimated by $e^{-\mu t}$ as $t\to+\infty$ and the roots of the polynomial $P(\tau,\xi)$ in τ lie above the line $\operatorname{Im}\tau=\mu$ and are separated from it, then any solution of the Cauchy problem grows at $+\infty$ no faster than $e^{-\mu t}$. This result admits a

1991 *Mathematics Subject Classification.* 35L25, 35L40, 35L60, 35L75, 35B15, 35B40.

Key words and phrases. hyperbolic equations and systems; bounded and almost periodic solutions; exponential dichotomy; stable, unstable and center manifolds; linearization.

The first author was supported by EPSRC (grant GR/N63055/01) and RFBR (grant 00-01-00387). The second author was supported by RFBR (grant 99-01-01157) and INTAS (project 899).

generalization to case in which some of the characteristic roots lie in the half-plane $\operatorname{Im}\tau < \mu$. More exactly, to single out a unique solution growing at $+\infty$ no faster than $e^{-\mu t}$, we need to impose as many initial conditions as the number of characteristic roots lying above the line $\operatorname{Im}\tau = \mu$ (see Section 4). These two results are simple consequences of a representation of solutions of equations with constant coefficients as contour integrals. Roughly speaking, one of the main results of this paper is that similar assertions are true for linear and nonlinear perturbations of hyperbolic equations (and systems) with constant coefficients. We also discuss the behavior of solutions in the neighborhood of a stationary point and construct time bounded and almost periodic solutions of nonhomogeneous problems.

We note that there is a substantial difference between the Cauchy problem for hyperbolic PDE's and the problem of their solvability in spaces of functions with given exponential rate of growth as time goes to infinity. Namely, the former contains a natural large parameter—the rate of growth of solutions at infinity—and is solvable for sufficiently large values of this parameter, whereas the latter has no "natural large parameter," and this is the main reason why we have to confine ourselves to the case of small perturbations of equations with constant coefficients. Examples show that in the general case qualitative properties of solutions described in this paper cannot be expressed only in terms of the characteristic roots of the symbol.

The properties of solutions discussed here in the case of hyperbolic PDE's were studied earlier for different classes of ordinary and partial differential equations. In particular, for the case of ODE's, many results on exponential stability and dichotomy, time-bounded and almost periodic solutions, stable, unstable, and center manifolds, and the phase portrait in the neighborhood of a stationary point can be found in [1, 5, 9, 8]. Similar questions for abstract evolution equations with infinite-dimensional phase space are investigated in [2, 13, 12] (see also references there). Application of these results to hyperbolic PDE's leads, as a rule, to the rather restrictive condition that the principal symbol has constant coefficients. We refer the reader to our papers [17]–[23] for further references concerning qualitative theory for hyperbolic equations and systems.

This paper deals with high-order scalar equations and first-order systems. As far as the Cauchy problem is concerned, the results in these two cases are almost identical, although the methods of proofs differ essentially. On the other hand, the qualitative behavior of solutions for scalar equations admits more complete description, and as a rule, we first discuss the case of systems and then indicate generalizations for scalar equations.

The paper is organized as follows. In Section 1, we consider two simple examples and use them to reveal the main difficulties arising in the general theory. In the second section, we recall well-known results on solvability of the Cauchy problem for linear and nonlinear hyperbolic PDE's. Section 3 is devoted to energy estimates for solutions of linear equations. In Section 4, we discuss qualitative properties of solutions for linear equations and systems. Finally, Section 5 deals with nonlinear problems.

We note once again that this paper is a survey of the theory developed by the authors over the last five years, and the emphasis here is on the main ideas only. We refer the reader to [17, 18, 19, 20, 21, 23] for rigorous (and rather involved) proofs of the theorems presented in this survey.

Notation. We denote by $t \in \mathbb{R}$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ the time and space variables, respectively, and by $\tau \in \mathbb{C}$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ their dual variables. To shorten the notation, we set $y = (t, x)$ and $\eta = (\tau, \xi)$. Let $\partial = (\partial_t, \partial_x)$ and $D = (D_t, D_x) = (-i\partial_t, -i\partial_x)$.

Let $J \subset \mathbb{R}$ be a closed interval, let X be a Banach space, let $\mathfrak{M} \subset \mathbb{R}$ be a countable module, and let $\Omega \subset \mathbb{R}^d$ be an open set. We will use the following function spaces.

$H^{(s)} = H^{(s)}(\mathbb{R}^n)$ is the Sobolev space of order s with scalar product

$$(u, v)_s = \int_{\mathbb{R}^n} \hat{u}(\xi)\overline{\hat{v}(\xi)}(1 + |\xi|^2)^s d\xi,$$

where $\hat{w}(\xi)$ is the Fourier transform of the function $w(x)$. We denote by $\|\cdot\|_{(s)}$ the corresponding norm.

$C_b^\infty(\overline{\Omega})$ is the space of infinitely smooth functions on Ω that are bounded together with all their derivatives.

$C(J, X)$ is the space of continuous functions on J with range in X.

$C_b(J, X)$ is the space of bounded functions $f \in C(J, X)$.

$C_b^\infty(\mathbb{R}, H^{(\infty)})$ is the space of functions $u(t, x)$ such that $\partial_t^j u \in C_b(\mathbb{R}, H^{(s)})$ for any nonnegative integers j and s.

$L^1(J, X)$ is the space of Bochner-measurable functions $f(t)\colon J \to X$ such that

$$\|f\|_{L^1(J,X)} := \int_J \|f(t)\|_X dt < \infty.$$

$L^1_{\text{loc}}(J, X)$ is the space of Bochner-measurable functions $f(t)\colon J \to X$ such that $\|f\|_{L^1(I,X)} < \infty$ for any finite interval $I \subset J$.

$AP(X, \mathfrak{M})$ (accordingly, $LAP(X, \mathfrak{M})$) is the space of Bohr (Levitan) almost periodic functions on $\mathbb{R}$ with range in X whose module of Fourier exponents is contained in $\mathfrak{M}$.

We will use the same notation for spaces of scalar and vector functions.

1. **Two examples.**

The aim of this section is to consider two simple examples modeling hyperbolic equations and systems and to reveal the main difficulties arising in the general theory. We begin with a high-order ODE and study some qualitative properties of its solutions. We next turn to a fully nonlinear perturbation of the damped Klein–Gordon equation and show that the qualitative behavior of its solutions is similar to the case of an ODE with stable characteristic roots. The results presented in this section are illustrations of those discussed in §§ 3–5 for hyperbolic equations and systems.

1.1. **Ordinary differential equations.**

Let us consider the linear ODE

$$P(D_t) + \varepsilon Q(t, D_t)u = 0, \quad t \in \mathbb{R}, \tag{1.1}$$

where $D_t = -id/dt$, $\varepsilon \in \mathbb{C}$ is a small parameter, and $P(\tau)$ and $Q(t, \tau)$ are polynomials in τ of degree $m \geq 1$ with constant and bounded continuous coefficients, respectively. Let us assume that $P(\tau) \neq 0$ for $\operatorname{Im} \tau = 0$ and denote by m_+ and m_- the number of roots for $P(\tau)$ that belong to the half-planes $\operatorname{Im} \tau > 0$ and $\operatorname{Im} \tau < 0$, respectively.

Consider the Cauchy problem for (1.1):

$$D_t^j u(0) = u_j \in \mathbb{C}, \quad j = 0, \dots, m-1. \tag{1.2}$$

Theorem 1.1. *Under the above conditions, there are positive constants C, ε_0 and μ and subspaces $\mathbb{E}^+, \mathbb{E}^- \subset \mathbb{C}^m$, depending on ε, such that the following assertions hold for $|\varepsilon| \le \varepsilon_0$.*

(i) *A vector $[u_0, \dots, u_{m-1}] \in \mathbb{C}^m$ belongs to $\mathbb{E}^\pm$ if and only if the solution $u(t)$ of the problem* (1.1), (1.2) *satisfies the inequality*[1]

$$\sum_{j=0}^{m} |D_t^j u(t)| \le C\, e^{-\mu|t|} \sum_{j=0}^{m-1} |u_j|, \quad \pm t \ge 0. \tag{1.3}$$

(ii) *The phase space $\mathbb{C}^m$ can be represented as the direct sum*

$$\mathbb{C}^m = \mathbb{E}^+ \dotplus \mathbb{E}^-.$$

(iii) *There are constants $r_{jk}^\pm(\varepsilon) \in \mathbb{C}^m$, $j = m_\pm, \dots, m-1$, $k = 0, \dots, m_\pm - 1$, such that*

$$\mathbb{E}^\pm = \left\{ [u_0, \dots, u_{m-1}] \in \mathbb{C}^m : u_j = \sum_{k=0}^{m_\pm - 1} r_{jk}^\pm(\varepsilon) u_k,\ j = m_\pm, \dots, m-1 \right\}. \tag{1.4}$$

If properties (i) and (ii) hold, then one says that Eq. (1.1) possesses the property of exponential dichotomy. In the particular case when $m_- = 0$, the solutions of (1.1) are exponentially asymptotically stable as $t \to +\infty$. Similarly, if $m_+ = 0$, then the solutions are exponentially stable as $t \to -\infty$.

Proof. We will only give the main ideas of the proof. For simplicity, we assume that the polynomial $P(\tau)$ has simple roots.

Step 1. We first consider the case when $\varepsilon = 0$ and $P(\tau) \ne 0$ for $\operatorname{Im} \tau \le 0$, i. e., all the roots of $P(\tau)$ are stable and lie above a line $\operatorname{Im} \tau = \sigma > 0$. In this situation, an explicit formula for solutions of (1.1), (1.2) implies that

$$\sum_{j=0}^{k} |D_t^j u(t)| \le C_k\, e^{-\sigma t} \sum_{j=0}^{m-1} |u_j|, \quad t \ge 0, \tag{1.5}$$

where $k \ge 0$ is an arbitrary integer. It follows that the operator $P(D_t)$ possesses Green's function that is zero for $t < 0$ and decays as $e^{-\sigma t}$ for $t > 0$ together with all its derivatives.

For $\mu \ge 0$ and an integer $k \ge 0$, we denote by $C_\mu^k(\mathbb{R}_+)$ the space of k times continuously differentiable functions $h \colon \mathbb{R}_+ \to \mathbb{C}$ such that

$$\sum_{j=0}^{k} |D_t^j h(t)| \le \text{const}\ e^{-\mu t}, \quad t \ge 0.$$

In the case $k = 0$, we drop the corresponding superscript from the notation. What has been said implies that for any $f \in C_\mu^k(\mathbb{R}_+)$, where $0 \le \mu < \sigma$, the problem

$$P(D_t) u = f(t), \quad D_t^j u(0) = 0, \quad j = 0, \dots, m-1, \tag{1.6}$$

has a unique solution $u \in C_\mu^{k+m}(\mathbb{R}_+)$.

Furthermore, using an explicit formula (in terms of Green's function) for solutions of the inhomogeneous equation

$$P(D_t) u = f(t), \tag{1.7}$$

[1] Here and henceforth, a formula involving the indices $\pm$ and $\mp$ is a brief notation for the two formulas corresponding to the upper and lower signs. When referring to the formula with upper (lower) sign, we will use a number with subscript $+$ $(-)$.

it is not difficult to show that if $P(\tau) \neq 0$ for $\operatorname{Im}\tau \geq 0$, then Eq. (1.7) with right-hand side $f \in C_\mu^k(\mathbb{R}_+)$, $\mu \geq 0$, has a unique solution in the space $C_\mu^{k+m}(\mathbb{R}_+)$.

Step 2. We now assume that $P(\tau)$ has roots in both upper and lower half-planes. Denoting by $\tau_j^\pm$, $j = 1, \ldots, m_\pm$, the roots lying in the half-plane $\pm \operatorname{Im}\tau > 0$, we can represent the solution of (1.1) in the form (recall that $\varepsilon = 0$)

$$u(t) = u_+(t) + u_-(t), \quad u_\pm(t) = \sum_{j=1}^{m_\pm} c_j^\pm \exp(\tau_j^\pm t). \tag{1.8}$$

It is clear that $u_+(t)$ decays exponentially as $t \to +\infty$. Moreover, if $u_- \not\equiv 0$, then $u_-(t)$ grows exponentially as $t \to +\infty$. Therefore, if a solution of (1.1) decays as $t \to +\infty$, then u_- must be zero identically. Hence, a solution of (1.1) that decays exponentially on the positive half-line depends on m_+ arbitrary constants c_j^+. Thus, to single out such a solution, we need to impose m_+ initial conditions. Simple calculations show that it suffices to specify the values of a solution and its first $m_+ - 1$ derivatives at zero:

$$D_t^j u(0) = u_j, \quad j = 1, \ldots, m_+. \tag{1.9}$$

The higher-order derivatives of $u(t)$ at zero can be expressed as linear functionals of $[u_0, \ldots, u_{m_+-1}]$:

$$D_t^j u(0) = \sum_{k=0}^{m_+-1} r_{jk}^+ u_k, \quad j = m_+, \ldots, m-1, \tag{1.10}$$

where $r_{jk}^+ \in \mathbb{C}$ are some constants. Defining $\mathbb{E}^+$ by formula (1.4_+) in which $r_{jk}^+(\varepsilon) \equiv r_{jk}^+$, we see that $[u_0, \ldots, u_{m_+-1}] \in \mathbb{E}^+$ if and only if the solution of (1.1), (1.2) decays exponentially as $t \to +\infty$. The subspace E^- can be constructed in a similar way. The fact that $\mathbb{C}^m$ can be represented as the direct sum of $\mathbb{E}^+$ and $\mathbb{E}^-$ follows easily from (1.8).

Step 3. We now consider the general case. Let us rewrite (1.1) in the form

$$P^-(D_t)P^+(D_t)u + \varepsilon Q(t, D_t)u = 0, \tag{1.11}$$

where $P^-(\tau)$ and $P^+(\tau)$ are polynomials corresponding to the roots of $P(\tau)$ that belong to the half-planes $\operatorname{Im}\tau < 0$ and $\operatorname{Im}\tau > 0$, respectively. We claim that, for $|\varepsilon| \ll 1$, Eq. (1.11) has a unique solution decaying exponentially as $t \to +\infty$ and satisfying the initial conditions (1.9).

Indeed, let $v_+(t)$ be the solution the equation $P^+(D_t)u = 0$ supplemented with the initial conditions (1.9). Clearly, it belongs to the space $C_\sigma^m(\mathbb{R}_+)$, where $\sigma > 0$ is the constant in inequality (1.5). We fix an arbitrary μ, $0 \leq \mu < \sigma$, and seek a solution of (1.11), (1.9) in the form

$$u(t) = v_+(t) + (G^+G^-h)(t), \tag{1.12}$$

where $h \in C_\mu(\mathbb{R}_+)$ is an unknown function, $G^+ : C_\mu^{m_-}(\mathbb{R}_+) \to C_\mu^m(\mathbb{R}_+)$ is the resolving operator for the problem (1.6) with $m = m_+$ and $P = P^+$, and $G^- : C_\mu(\mathbb{R}_+) \to C_\mu^{m_-}(\mathbb{R}_+)$ is the resolving operator for (1.7) with $P = P^-$. It is clear that $u(t)$ satisfies (1.9).

Substitution of (1.12) into (1.11) results in

$$h + \varepsilon Bh = -\varepsilon Q(t, D_t)v_+, \quad B = Q(t, D_t)G^+G^-. \tag{1.13}$$

The operator $Q(t,d_t)G^+G^-$ is bounded in the space $C_\mu(\mathbb{R}_+)$, and the right-hand side of (1.13) belongs to $C_\mu(\mathbb{R}_+)$. Thus, for sufficiently small $\varepsilon \in \mathbb{C}$, Eq. (1.13) has a unique solution, which can be represented as a Neumann series.

We have thus constructed an exponentially decaying solution $u(t)$ of the problem (1.1), (1.9). The higher-order derivatives of $u(t)$ at zero can be expressed as linear functionals of the initial data (cf. (1.10)):

$$D_t^j u(0) = \sum_{k=0}^{m_+-1} r_{jk}^+(\varepsilon) u_k, \quad j = m_+, \dots, m-1.$$

It is not difficult to show that $r_{jk}^+(\varepsilon) = r_{jk}^+ + \varepsilon q_{jk}^+(\varepsilon)$, where r_{jk}^+ are the constants in (1.10) and $q_{jk}^+(\varepsilon)$ are uniformly bounded functions of $\varepsilon \in \mathbb{C}$, $|\varepsilon| \ll 1$. Defining $\mathbb{E}^+$ by (1.4_+), we can easily verify assertions (i) and (iii) in the case of index $+$.

Similar constructions can be carried out for solutions on the negative half-line $\mathbb{R}_-$. The fact that $\mathbb{C}^m$ is representable as the direct sum of $\mathbb{E}^+$ and $\mathbb{E}^-$ follows from a similar property in the case of constant coefficients and some simple perturbation arguments. □

The scheme used in the proof of Theorem 1.1 can be applied to construct stable and unstable subspaces for linear hyperbolic equations of the form

$$P(D)u + \varepsilon Q(t,x,D)u = 0, \quad (t,x) \in \mathbb{R}^{n+1}. \tag{1.14}$$

However, there is a substantial difference between ODE's and hyperbolic PDE's. To show it, we denote by $P^+(D)$ and $P^-(D)$ the operators corresponding to the roots of $P(\tau,\xi)$ that lie in the upper and lower half-planes, respectively, and rewrite (1.14) as

$$P^-(D)P^+(D)u + \varepsilon Q u = 0, \quad Q = Q(t,x,D). \tag{1.15}$$

Now note that, to represent the solution of (1.13) as a Neumann series, we used the fact that operators $P^+(D_t)$ and $P^-(D_t)$ are of order m_+ and m_-, respectively, and the inverse operators $G^\pm$ "gain" $m_\pm$ derivatives. In the case of a hyperbolic PDE, one derivative is lost, and the inverse of $P^\pm$ gains only $m_\pm - 1$ derivatives. Therefore, if we retain the same notation for analogs of v_+, G^+, and G^- in the case of Eq. (1.15) and seek its solution in the form (1.12), we obtain (cf. (1.13))

$$h + \varepsilon B h = -\varepsilon Q v_+, \quad B = QG^+G^-. \tag{1.16}$$

Since $\operatorname{ord} Q = m$, the operator B is now of order 2, and the formal Neumann series diverges because of the loss of smoothness.

To overcome the above difficulty, we need to refine representation (1.15), so that the remainder term (i. e. Q) has order $m-2$. This idea can be realized in the case of linear equations (see [21]), and the corresponding result is formulated in Section 4.2.

The case in which the operator Q in (1.14) is a nonlinear function of u and its derivatives up to order m is even more difficult. To handle this case, we used Petrovskii's idea for reducing a fully nonlinear equation to a system of quasilinear equations (see [20]). The main ingredients of Petrovskii approach are presented in the next subsection, using the example of a fully nonlinear perturbation of the damped Klein–Gordon equation.

1.2. **Nonlinear perturbations of the damped Klein–Gordon equation.** Let us consider the problem

$$\partial_t^2 u + \gamma\partial_t u - (\Delta - 1)u + \varepsilon Q(\varepsilon, t, x, \partial^2 u) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n, \tag{1.17}$$

$$u(0,x) = u_0(x) \in H^{(s+1)}(\mathbb{R}^n), \quad \partial_t u(0,x) = u_1(x) \in H^{(s)}(\mathbb{R}^n), \tag{1.18}$$

where $\gamma > 0$ is a constant, $\varepsilon \in [-1,1]$ is a small parameter, $s \geq 0$ is an integer, ∂^2 is the set of all partial derivatives up to the second order, and Q is a smooth function of its arguments. More exactly, we assume that the function $Q(\varepsilon, t, x, p)$, which is obtained from the operator $Q(\varepsilon, t, x, \partial^2 u)$ on replacing the set of partial derivatives $\partial^2 u = (\partial^\alpha u, |\alpha| \leq 2)$ by the variables $p = (p_\alpha, |\alpha| \leq 2) \in \mathbb{R}^d$, belongs to the space $C_b^\infty([-1,1] \times \mathbb{R}^{n+1}_{t,x} \times B)$ for any closed ball $B \subset \mathbb{R}^d$ and satisfies the relation $Q(\varepsilon, t, x, 0) \equiv 0$. Under these conditions, we have the following result:

Theorem 1.2. *For sufficiently large integers $s > 0$ and an arbitrary $R > 0$ there is $\varepsilon_0 = \varepsilon_0(s, R) > 0$ such that for any initial functions $u_0 \in H^{(s+1)}$ and $u_1 \in H^{(s)}$ satisfying the condition*

$$\|u_0\|_{(s+1)} + \|u_1\|_{(s)} \leq R \tag{1.19}$$

the problem (1.17), (1.18) *has a unique solution $u(t,x)$ such that*

$$\partial_t^j u \in C_b(\mathbb{R}_+, H^{(s+1-j)}), \quad j = 0, \dots, s+1. \tag{1.20}$$

Moreover, there are positive constants $\sigma = \sigma(s,R)$ and $C = C(s,R)$ such that

$$\sum_{j=0}^{s+1} \|\partial_t^j u(t,\cdot)\|_{(s+1-j)} \leq C\, e^{-\sigma t}\big(\|u_0\|_{(s+1)} + \|u_1\|_{(s)}\big), \quad t \geq 0. \tag{1.21}$$

Proof. We will only give the main ideas of the proof, which consists of the following three steps:

1. A priori estimates for solutions of the problem (1.17), (1.18) in the case when Q is a linear differential operator:
$$Q(\varepsilon, t, x, \partial^2 u) = Q(\varepsilon, t, x, \partial)u = \sum_{|\alpha| \leq 2} q_\alpha(\varepsilon, t, x)\partial^\alpha u; \tag{1.22}$$
2. Reduction of the fully nonlinear equation (1.17) to a quasilinear system;
3. Construction of a solution of the reduced problem, using the Leray–Schauder fixed point theorem and a priori estimates of Step 1.

Steps 1 and 3 will only be sketched, since they are rather well-known, whereas Step 2 will be discussed in more details.

Step 1. We assume that the operator Q has the form (1.22) and rewrite Eq. (1.17) as

$$\partial_t^2 u + \gamma\partial_t u - (\Delta - 1)u + \varepsilon \sum_{|\alpha| \leq 2} q_\alpha(\varepsilon, t, x)\partial^\alpha u = 0. \tag{1.23}$$

The main observation is that for any integer $s \geq 0$ the solution of (1.23), (1.18) satisfies inequality (1.21), where σ and C are positive constants depending on s and some appropriate norms of the coefficients q_α. The proof of this assertion in the case of linear high-order hyperbolic operators is based on Leray's separating operator technique. In the case of second-order equations, the corresponding argument is rather simple, and we will now establish (1.21) for $s = 0$.

Let us fix sufficiently small $\lambda > 0$ and set

$$E(t,u) := \|\partial_t u\|^2 + \|\nabla u\|^2 + \|u\|^2 + 2\lambda(\partial_t u, u),$$

where $\|\cdot\|$ and $(\cdot,\cdot)$ are the norm and scalar product in $L^2(\mathbb{R}^n)$. It is clear that, for sufficiently small $\lambda > 0$, the functions $E(t,u)$ can be estimated from below and from above by the expression $\|\partial_t u\|^2 + \|u\|_1^2$. Taking the scalar product of (1.23) and $2(\partial_t + \lambda)u$ and performing some simple transformations, we obtain

$$\partial_t E(t,u) + \lambda E(t,u) + 2\varepsilon(Q(\varepsilon,t,\cdot,\partial)u,(\partial_t+\lambda)u) = \\ = -(2\gamma - 3\lambda)\|\partial_t u\|^2 - \lambda\|\nabla u\|^2 - \lambda\|u\|^2 - \lambda(2\gamma-1)(\partial_t u, u). \quad (1.24)$$

The right-hand side of (1.24) is nonpositive for $0 < \lambda \ll 1$. Moreover, it is not difficult to show that the third term on the left-hand side can be represented in the form $\varepsilon\big(\partial_t F_1(t,u) + F_2(t,u)\big)$, where F_1 and F_2 are some quadratic forms with respect to u that can be estimated from above by $E(t,u)$. Therefore, for sufficiently small $\varepsilon > 0$, the required estimate (1.21) with $\sigma = \lambda/4$ and $s = 0$ follows from the Gronwall inequality.

To establish (1.21) for $s \geq 1$, it suffices to apply the above argument to equations resulting from differentiation of (1.23) with respect to t and x.

Step 2. We follow Petrovskii's argument [15] to reduce the problem (1.17), (1.18) to a quasilinear system.

Let us set

$$P(\partial) = \partial_t^2 + \gamma\partial_t - (\Delta - 1), \quad q_\alpha(z,p) = \partial_{p_\alpha} Q(z,p), \quad Q_i(z,p) = \partial_i Q(z,p),$$

where $z = (\varepsilon, t, x)$, $\partial_0 = \partial_t$, and $\partial_j = \partial_{x_j}$ for $j = 1,\dots,n$. Differentiating (1.17) with respect t and x_j, we obtain the following system for the functions $v_i = \partial_i u$, $i = 0,\dots,n$:

$$P(\partial)v_i + \varepsilon \sum_{|\alpha|\leq 2} q_\alpha(z,\partial^2 u)\partial^\alpha v_i + \varepsilon Q_i(z,\partial^2 u) = 0. \quad (1.25)$$

Let $B_0,\dots,B_n$ be some sets of multi-indices of length ≤ 1 such that

$$\partial^2 u = \big(\partial^{B_0}(\partial_0 u),\dots,\partial^{B_n}(\partial_n u), u\big) = \big(\partial^{B_0} v_0,\dots,\partial^{B_n} v_n, u\big), \quad (1.26)$$

where we used the notation $\partial^{B_i} = (\partial^\beta, \beta \in B_i)$. Replacing $\partial^2 u$ in Eqs. (1.17) and (1.25) by the right-hand side of (1.26) and setting $v_{n+1} = u$, we obtain the following system of $n+2$ quasilinear equations of the second order for the vector function $V = [v_0,\dots,v_{n+1}]$:

$$P(\partial)v_i + \varepsilon \sum_{|\alpha|\leq 2} q_\alpha(z,\partial^B V)\partial^\alpha v_i + \varepsilon Q_i(z,\partial^B V) = 0, \quad i = 0,\dots,n, \quad (1.27)$$

$$P(\partial)v_{n+1} + \varepsilon Q(z,\partial^B V) = 0, \quad (1.28)$$

where we set $B = (B_0,\dots,B_n)$ and $\partial^B V = (\partial^{B_0} v_0,\dots,\partial^{B_n} v_n, v_{n+1})$.

Let us consider the Cauchy problem for system (1.27), (1.28):

$$v_i(0,x) = v_{i0}(x), \quad \partial_t v_i(0,x) = v_{i1}(x), \quad i = 0,\dots,n+1. \quad (1.29)$$

We will say that the set of initial functions (v_{i0}, v_{i1}) satisfies the *compatibility condition* if

$$v_{00} = v_{n+1,1}, \quad v_{j0} = \partial_j v_{n+1,0}, \quad v_{j1} = \partial_j v_{n+1,1}, \quad j = 1,\dots,n,$$
$$v_{01} + \gamma v_{00} - (\Delta - 1)v_{n+1,0} + \varepsilon Q(\varepsilon, 0, x, (\partial^B V)(0,x)) = 0.$$

Roughly speaking, the above relations mean that the initial conditions for v_j, $j = 1,\dots,n$, are obtained by differentiation of (1.18), and Eq. (1.17) is satisfied for $t = 0$.

The proposition below establishes link between the problems (1.27)–(1.29) and (1.17), (1.18). (To simplify the presentation, we do not give exact formulation.)

Proposition 1.3. *Suppose that* $v_{n+1,0} = u_0$ *and* $v_{n+1,1} = u_1$ *and that the initial functions* (v_{i0}, v_{i1}) *satisfies the compatibility condition. Then for any smooth solution* $V = [v_0, \dots, v_{n+1}]$ *of* (1.27)–(1.29) *the function* $u = v_{n+1}$ *is a solution of* (1.17), (1.18).

For the proof of this assertion in the case of general hyperbolic systems, see [15]; the case of almost periodic solutions is analyzed in [20].

Thus, it suffices to construct a solution for (1.27)–(1.29), and this is done in the next step.

Step 3. Let us take an arbitrary smooth vector function $\widetilde{V} = [\tilde{v}_0, \dots, \tilde{v}_{n+1}]$ and denote by $V = [v_0, \dots, v_{n+1}]$ the solution of the linear system

$$P(\partial)v_i + \varepsilon \sum_{|\alpha|\le 2} q_\alpha(z, \partial^B \widetilde{V})\partial^\alpha v_i = -\varepsilon Q_i(z, \partial^B \widetilde{V}), \quad i = 0, \dots, n,$$

$$P(\partial)v_{n+1} = -\varepsilon Q(z, \partial^B \widetilde{V}),$$

supplemented with the initial conditions (1.29). The function V is well-defined for sufficiently small ε. It is not difficult to choose a compact convex set $\mathcal{B}$ in the space of exponentially decaying smooth functions, with an appropriate topology, such that the operator $\mathcal{F} : \widetilde{V} \mapsto V$ is continuous from $\mathcal{B}$ into itself. In view of the Leray–Schauder theorem [16, Theorem V.19], the operator $\mathcal{F}$ has a fixed point, which is the required solution of (1.27)–(1.29). □

2. Hyperbolic equations and systems: Cauchy problem.

In this section, we recall some important results on well-posedness of the Cauchy problem for linear and nonlinear hyperbolic equations and systems. We will use the same notation for spaces of scalar and vector functions; the context will tell us which space is meant.

2.1. High-order hyperbolic equations.

We begin with the case of linear equations. Let us consider the problem

$$P(y, D_t, D_x)u \equiv \sum_{|\alpha|\le m} p_\alpha(y) D_t^{\alpha_0} D_x^{\alpha'} u = f(y), \quad y = (t, x) \in \mathbb{R}^{n+1}, \tag{2.1}$$

$$D_t^j u(0, x) = u_j(x), \quad j = 0, \dots, m-1, \tag{2.2}$$

where $\alpha = (\alpha_0, \alpha')$, and $p_\alpha(y)$ are smooth functions, bounded together with all their derivatives. We assume that the operator $P(y, D)$ is *uniformly strictly hyperbolic*, i.e., the coefficient of D_t^m is identically equal to 1, and the roots $\tau_j^0(y, \xi)$, $j = 1, \dots, m$, of the principal symbol

$$P^0(y, \tau, \xi) := \sum_{|\alpha|=m} p_\alpha(y) \tau^{\alpha_0} \xi^{\alpha'}$$

are real and satisfy the inequalities

$$|\tau_j^0(y, \xi) - \tau_k^0(y, \xi)| \ge \varkappa |\xi|, \quad j \ne k, \quad (y, \xi) \in \mathbb{R}_y^{n+1} \times \mathbb{R}_\xi^n,$$

where $\varkappa > 0$ is a constant not depending on (y, ξ). The proof of the following result on well-posedness of the Cauchy problem for strictly hyperbolic equations can be found in [15, 11, 7, 4].

Theorem 2.1. *For any $s \in \mathbb{R}$, any right-hand side $f \in L^1_{\mathrm{loc}}(\mathbb{R}, H^{(s)})$, and an arbitrary set of initial functions $u_j \in H^{(m-1+s-j)}$, $j = 0, \dots, m-1$, the problem* (2.1), (2.2) *has a unique solution $u(t,x)$ that satisfies the inclusions*

$$D_t^j u \in C(\mathbb{R}, H^{(m-1+s-j)}), \quad j = 0, \dots, m-1. \tag{2.3}$$

Moreover, for any $T > 0$ we have the estimate

$$\sum_{j=0}^{m-1} \|D_t^j u(t,\cdot)\|_{(m-1+s-j)} \le C\biggl(\sum_{j=0}^{m-1} \|u_j\|_{(m-1+s-j)} + \|f\|_{L^1(J_T, H^{(s)})}\biggr), \quad |t| \le T, \tag{2.4}$$

where $J_T = [-T, T]$, and the constant $C = C(s,T) > 0$ does not depend on f and u_j.

The Cauchy problem is also well-posed for *nonlinear* strictly hyperbolic equations, although the existence can only be guaranteed on a sufficiently small time interval. For simplicity, we will formulate the corresponding result for small nonlinear perturbations of Eq. (2.1). More exactly, let us consider the equation[2]

$$P(y, \partial_t, \partial_x)u + \varepsilon Q(\varepsilon, y, \partial^m u) = f(y), \quad y = (t,x), \tag{2.5}$$

where $\partial = (\partial_t, \partial_x)$, ∂^m is the set of all partial derivatives of order $\le m$, P is a strictly hyperbolic operator of order m, and $\varepsilon \in [-1, 1]$ is a small parameter. Concerning the nonlinear term, we assume that the function $Q(\varepsilon, y, p)$, obtained from the nonlinear operator $Q(\varepsilon, y, \partial^m u)$ on replacing the partial derivatives $\partial^m u = (\partial^\alpha u, |\alpha| \le m)$ by the variables $p = (p_\alpha, |\alpha| \le m) \in \mathbb{R}^d$, belongs to the space $C_b^\infty([-1,1] \times \mathbb{R}_y^{n+1} \times B)$ for any closed ball $B \subset \mathbb{R}^d$. Moreover, $Q(\varepsilon, y, 0) \equiv 0$. We have the following result (see [15, 3]).

Theorem 2.2. *Under the above conditions, for any sufficiently large integer $s > 0$ and an arbitrary $R > 0$ there are positive constants $\varepsilon_0 = \varepsilon_0(s, R)$ and $T = T(s, R)$ such that the following assertion holds: for any right-hand side $f(t,x)$ satisfying the conditions*

$$\partial_t f \in C([-T, T], H^{(s-k)}), \quad \sup_{|t| \le T} \|\partial_t^k f(t,\cdot)\|_{(s-k)} \le R, \quad k = 0, \dots, s, \tag{2.6}$$

and any set of initial functions

$$u_j \in H^{(m-1+s-j)}, \quad \|u_j\|_{(m-1+s-j)} \le R, \quad j = 0, \dots, m-1,$$

the problem (2.5), (2.2) *has a unique solution $u(t,x)$ defined on $(-T, T)$ and satisfying the inclusions*

$$\partial_t^j u \in C\bigl((-T, T), H^{(m-1+s-j)}\bigr), \quad j = 0, \dots, m-1+s.$$

2.2. **First-order hyperbolic systems.** We now turn to the first-order linear system

$$\mathcal{P}(y, D)u := D_t u - P(y, D_x)u = f(y), \quad y = (t,x) \in \mathbb{R}^{n+1}, \tag{2.7}$$

where $D = (D_t, D_x)$, $P(y, D_x)$ is an $m \times m$ matrix of first-order operators with coefficients in $C_b^\infty(\mathbb{R}_y^{n+1})$. Let $P^0(y, \xi)$ be the principal symbol for $P(y, D_x)$. We recall that P is said to be *strongly hyperbolic* if the two properties below hold for all $(y, \omega) \in \mathbb{R}_y^{n+1} \times \Sigma_{n-1}$:

[2] In our study of nonlinear equations we write the derivatives $\partial = (\partial_t, \partial_x)$ without the complex factor $-i$, because all the functions here are assumed to be real-valued. In this case, the symbol of an operator is obtained on replacing the derivatives ∂_t and ∂_x by $i\tau$ and $i\xi$, respectively.

(a) the matrix $P^0(y,\omega)$ has elementary divisors of degree no higher than 1, the number of coinciding elementary divisors does not depend on the choice of the point (y,ω);

(b) the characteristic roots $\tau_j^0(y,\omega)$, $j=1,\dots,m$, of $P^0(y,\omega)$ are real.

The class of systems satisfying properties (a) and (b) was introduced by Petrovskii [15]; the term *strongly hyperbolic* is taken from [10].

The above properties imply, in particular, that the number of pairwise distinct roots of $P^0(y,\omega)$ and their multiplicities do not depend on (y,ω). Let us denote these roots by $\sigma_j(y,\omega)$, $j=1,\dots,l$. We say that P is *uniformly strongly hyperbolic* if

$$\bigl|\sigma_j(y,\omega)-\sigma_k(y,\omega)\bigr|\ge\varkappa \quad\text{for}\quad (y,\omega)\in\mathbb{R}_y^{n+1}\times\Sigma_{n-1},\quad j\ne k, \tag{2.8}$$

where $\varkappa>0$ does not depend on (y,ω).

Let us consider the Cauchy problem for a uniformly strongly hyperbolic system of the form (2.7):

$$u(0,x)=u_0(x), \tag{2.9}$$

where $u_0\in H^{(s)}$. The result below can be derived using the methods of [10, 4].

Theorem 2.3. *For any $s\in\mathbb{R}$, any right-hand side $f\in L^1_{\text{loc}}(\mathbb{R},H^{(s)})$, and an arbitrary initial function $u_0\in H^{(s)}$ the problem* (2.7), (2.9) *has a unique solution $u(y)\in C(\mathbb{R},H^{(s)})$, which satisfies the following inequality for any $T>0$:*

$$\|u(t,\cdot)\|_{(s)}\le C\bigl(\|u_0\|_{(s)}+\|f\|_{L^1(J_T,H^{(s)})}\bigr),\quad |t|\le T, \tag{2.10}$$

where $J_T=[-T,T]$, and the constant $C=C(s,T)>0$ does not depend on f and u_0.

Finally, we consider a quasilinear perturbation of (2.7):

$$\partial_t u-P(y,\partial_x)u-\varepsilon Q(\varepsilon,y,u,\partial_x)u=f(y), \tag{2.11}$$

where $\varepsilon\in[-1,1]$ is a small parameter and $Q(\varepsilon,y,u,\partial_x)$ is an $m\times m$ matrix of first-order linear operators with smooth coefficients that depend on ε, t, x, and u. In contrast to strictly hyperbolic equations, strongly hyperbolic systems are not stable under small perturbations. Therefore, to ensure that the Cauchy problem is well-posed for (2.11), we have to require that the perturbed (nonlinear) operator $P+\varepsilon Q$ be strongly hyperbolic. Namely, we assume that the matrix function $P_\varepsilon:=P(y,i\omega)+\varepsilon Q(\varepsilon,y,u,i\omega)$ satisfies conditions (a) and (b) for all (ε,y,u,ω) and that the pairwise distinct roots of P_ε are separated from one another uniformly with respect to $\varepsilon\in[-1,1]$, $y\in\mathbb{R}^{n+1}$, $\omega\in\Sigma_{n-1}$, and $u\in B$, where B is an arbitrary bounded ball in $\mathbb{R}^m_u$ (cf. (2.8)).

Under the above conditions, we have the following theorem on existence and uniqueness of solutions. Its proof can be carried out using the results in the linear case and some standard fixed point arguments (for instance, see [15, 3, 10]).

Theorem 2.4. *For any sufficiently large integer $s>0$ and any $R>0$ there are positive constants $\varepsilon_0=\varepsilon_0(s,R)$ and $T=T(s,R)$ such that the following assertion holds: for any right-hand side $f(t,x)$ satisfying* (2.6) *and any initial function*

$$u_0\in H^{(s)},\quad \|u_0\|_{(s)}\le R,$$

the problem (2.11), (2.9) *has a unique solution $u(t,x)$ defined on $(-T,T)$ and satisfying the inclusions*

$$\partial_t^j u\in C\bigl((-T,T),H^{(s-j)}\bigr),\quad j=0,\dots,s.$$

3. **Energy estimates.** In this section, we present some energy estimates that serve as a basis for analysis of linear equations and systems. We begin with the case of stable characteristic roots and show that a two-sided estimate for an appropriate quadratic form related to the problem in question implies the exponential decay of solutions for homogeneous equations. We next turn to the case when there are both stable and unstable roots. The proofs of the energy estimates presented here can be found in [17, 18]. We conclude this section by some examples to which the results can be applied.

3.1. **The case of stable characteristic roots.** We begin with the case of scalar equations of the form

$$P_\varepsilon(y,D)u := P(D)u + \varepsilon Q_\varepsilon(y,D)u = f(y), \quad y=(t,x)\in\mathbb{R}^{n+1}, \tag{3.1}$$

where $D=(D_t,D_x)$, $\varepsilon\in[-1,1]$ is a small parameter, $P(D)$ is a strictly hyperbolic operator of order m with constant coefficients, and $Q_\varepsilon(y,D)$ is an operator of order m with real principal part. We will assume that the coefficients of Q_ε, which are functions of (ε,y), belong to the space $C_b^\infty(\mathbb{R}^{n+1})$. The fact that we restrict ourselves to small perturbations of an operator with constant coefficients is essential: counterexamples show that for general hyperbolic equations the results are no longer true.

We note that the class of strictly hyperbolic operators is stable under small perturbation by an operator with real principal part, and therefore the operator P_ε is also strictly hyperbolic. Thus, the Cauchy problem is well-posed for (3.1).

Our first result concerns the case of stable characteristic roots. More exactly, we assume that the following condition is satisfied:

Condition 3.1. There is $\delta>0$ such that

$$P(\tau,\xi)\neq 0 \quad \text{for} \quad (\operatorname{Re}\tau,\xi)\in\mathbb{R}^{n+1}, \quad \operatorname{Im}\tau<\delta, \tag{3.2}$$

where τ and ξ are the variables dual to t and x, respectively.

For any two operators $R_i(y,D)$, $i=1,2$, a function $u(y)$, and real numbers $s\in\mathbb{R}$ and $\mu>0$, we set

$$J^-_{s,\mu,R_1,R_2}(u,t) = \int_{-\infty}^t e^{-2\mu(t-\theta)}(R_1(\theta,\cdot,D)u(\theta,\cdot), R_2(\theta,\cdot,D)u(\theta,\cdot))_s\, d\theta. \tag{3.3}$$

We also introduce the seminorm

$$E^2_{m-1,s,\mu}(u,\mathbb{R}_-(t)) = E^2_{m-1,s}(u,t) + \int_{-\infty}^t e^{-2\mu(t-\theta)}E^2_{m-1,s}(u,\theta)\,d\theta, \tag{3.4}$$

where $\mathbb{R}_-(t)=(-\infty,t]$ and

$$E^2_{m-1,s}(u,t) = \sum_{j=0}^{m-1}\|D_t^j u(t,\cdot)\|^2_{(m-1+s-j)}. \tag{3.5}$$

Theorem 3.2. *Under the above conditions, for any μ, $0<\mu<\delta$, and $s_0>0$ there are positive constants K and ε_0 such that*

$$K^{-1}E^2_{m-1,s,\mu}(u,\mathbb{R}_-(t)) \le \operatorname{Im} J^-_{s,\mu,P_\varepsilon,R}(u,t) \le K\,E^2_{m-1,s,\mu}(u,\mathbb{R}_-(t)), \quad t\in\mathbb{R}, \tag{3.6}$$

where $|\varepsilon|\le\varepsilon_0$, $|s|\le s_0$, $R(\tau,\xi)=-\partial_\tau P(\tau,\xi)$ is the Leray separating polynomial, and $u\in C_b^\infty\big(\mathbb{R},H^{(\infty)}\big)$.

Inequality (3.6) implies that

$$E^2_{m-1,s}(u,t) \le C \sup_{\theta\le t} \|P_\varepsilon u(\theta,\cdot)\|_{(s)} \int_{-\infty}^{t} e^{-2\mu(t-\theta)}\, \|P_\varepsilon u(\theta,\cdot)\|_{(s)}\, d\theta. \tag{3.7}$$

In particular, if $P_\varepsilon u(t,x) = 0$ for $t \ge 0$, then the energy $E^2_{m-1,s}(u,t)$ decays exponentially as $t \to +\infty$.

A similar result is true for first-order hyperbolic systems with nearly constant coefficients. Namely, let us consider the the problem

$$\mathcal{P}_\varepsilon(y,D)u := D_t u - P_\varepsilon(y,D_x)u = f(y), \quad y = (t,x) \in \mathbb{R}^{n+1}, \tag{3.8}$$

where u is now a vector function with m components, $\varepsilon \in [-1,1]$ is a small parameter, and $P_\varepsilon(y,D_x)$ is a strongly hyperbolic matrix operator whose coefficients do not depend on t and x for $\varepsilon = 0$. We assume that P_ε satisfies the following assumption (cf. Condition 3.1):

Condition 3.3. There is $\delta > 0$ such that

$$\operatorname{Im} \tau_j(\varepsilon,y,\xi) \ge \delta \quad \text{for} \quad (\varepsilon,y,\xi) \in [-1,1]\times\mathbb{R}^{n+1}_y\times\mathbb{R}^n_\xi, \quad j = 1,\dots,m,$$

where $\tau_j(\varepsilon,y,\xi)$ are the characteristic roots of $P_\varepsilon(y,\xi)$.

In contrast to scalar equations, an appropriate quadratic form for systems involves pseudodifferential operators (ΨDO's), and we now introduce the necessary class of symbols.

We denote by S^j the space of functions $p(\varepsilon,y,\xi) \in C^\infty([-1,1]\times\mathbb{R}^{n+1}_y\times\mathbb{R}^n_\xi)$ such that

$$\sup_{z\in\Omega,\,\xi\in\mathbb{R}^n} |\partial_\xi^\alpha \partial_z^\beta p(z,\xi)| \langle\xi\rangle^{|\alpha|-j} < \infty \quad \text{for any multi-indices } \alpha \text{ and } \beta,$$

where $z = (\varepsilon,t,x)$, $\Omega = [-1,1]\times\mathbb{R}^{n+1}$, $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, $|\xi|^2 = (\xi,\xi)$, and $(\cdot,\cdot)$ is the standard scalar product in $\mathbb{R}^n$. Let $S^j(m)$ be the space of matrix symbols $P(z,\xi) = \|p_{ik}(z,\xi)\|_{i,k=1}^m$ whose elements belong to S^j.

As was mentioned in Section 2.2, the class of strongly hyperbolic systems is not stable under small perturbations. This results in that an analog of the operator $R(D)$ (see Theorem 3.2) is not a ΨDO with constant symbol. However, we can choose it to be close to a constant symbol. More exactly, we say that a symbol $R(z,\xi)$ is nearly constant if it does not depend on (t,x) for $\varepsilon = 0$.

For any $R \in S^j(m)$, $s \in \mathbb{R}$, and $\mu > 0$, we set (cf. (3.3) and (3.4))

$$\mathcal{J}^-_{s,\mu,R}(u,t) = -\int_{-\infty}^{t} e^{-2\mu(t-\theta)} \big(R(\theta,\cdot,D_x)(\mathcal{P}_\varepsilon u)(\theta,\cdot), R(\theta,\cdot,D_x)u(\theta,\cdot)\big)_s d\theta,$$

$$\mathcal{E}^2_{s,\mu}(u,\mathbb{R}_-(t)) = \|u(t,\cdot)\|^2_{(s)} + \int_{-\infty}^{t} e^{-2\mu(t-\theta)} \|u(\theta,\cdot)\|^2_{(s)} d\theta.$$

We can now formulate an analog of Theorem 3.2 for first-order systems.

Theorem 3.4. *Under the above conditions, for any $\delta' < \delta$ there is a Hermitian symbol $R(z,\xi) \in S^0(m)$ depending on δ' such that the following assertions hold:*

(i) *The symbol $P^*R^2 - R^2P$ belongs to $S^0(m)$, and for all $(z,\xi) \in \Omega\times\mathbb{R}^n_\xi$, we have*

$$i\big(P^*(z,\xi)R^2(z,\xi) - R^2(z,\xi)P(z,\xi)\big) \ge 2\delta' R^2(z,\xi),$$
$$M^{-1}I \le R^2(z,\xi) \le MI,$$

where $M > 1$ is a constant depending on δ'.

(ii) *For any μ, $0<\mu<\delta'$, and $s_0>0$ there are positive constants K and ε_0 such that*

$$K^{-1}\mathcal{E}^2_{s,\mu}(u,\mathbb{R}_-(t))\le \operatorname{Im}\mathcal{J}^-_{s,\mu,R}(u,t)\le K\mathcal{E}^2_{s,\mu}(u,\mathbb{R}_-(t)),\quad t\in\mathbb{R}, \tag{3.9}$$

where $|\varepsilon|\le\varepsilon_0$, $|s|\le s_0$, and $u\in C_b^\infty(\mathbb{R},H^{(\infty)})$.

As in the case of scalar equations, the a priori estimate (3.9) implies the exponential decay of solutions as $t\to+\infty$.

3.2. **The general case.** We now turn to the case when there are both stable and unstable characteristic roots. We begin with the case of scalar equations and assume that the following condition is satisfied:

Condition 3.5. There is $\delta>0$ such that

$$P(\tau,\xi)\neq 0 \quad\text{for}\quad (\operatorname{Re}\tau,\xi)\in\mathbb{R}^{n+1},\quad |\operatorname{Im}\tau|<\delta. \tag{3.10}$$

Let m_+ and m_- be the number of roots of the symbol $P(\tau,\xi)$ that lie in the half-planes $\operatorname{Im}\tau\ge\delta$ and $\operatorname{Im}\tau\le-\delta$, respectively. Condition 3.5 implies that m_+ and m_- do not depend on ξ and that P is representable in the form

$$P(\tau,\xi)=P_+(\tau,\xi)P_-(\tau,\xi), \tag{3.11}$$

where P_+ (P_-) is a polynomial of degree m_+ (m_-) corresponding to the roots in the half-plane $\operatorname{Im}\tau\ge\delta$ ($\operatorname{Im}\tau\le-\delta$):

$$P_\pm(\tau,\xi)=\tau^{m_\pm}+\sum_{j=1}^{m_\pm}p_j^\pm(\xi)\tau^{m-j}. \tag{3.12}$$

It is not difficult to show that $p_j^\pm(\xi)$ are smooth functions belonging to the class S^j.

Let us set (cf. (3.3) and (3.4))

$$J^+_{s,\mu,R_1,R_2}(u,t)=\int_t^{+\infty}e^{-2\mu|t-\theta|}(R_1(\theta,\cdot,D)u(\theta,\cdot),R_2(\theta,\cdot,D)u(\theta,\cdot))_s\,d\theta,$$

$$E_{m-1,s}(u)=\sup_{t\in\mathbb{R}}E_{m-1,s}(u,t).$$

Theorem 3.6. *Under the above conditions, let*

$$R_+(\eta)=-P_-(\eta)\,\partial_\tau P_+(\eta),\quad R_-(\eta)=-P_+(\eta)\,\partial_\tau P_-(\eta),$$

where $\eta=(\tau,\xi)$. Then for any μ, $0<\mu<\delta$, and $s_0>0$ there are positive constants K and ε_0 such that

$$K^{-1}E_{m-1,s}(u)\le\sup_{t\in\mathbb{R}}\operatorname{Im}\big(J^-_{s,\mu,P_\varepsilon,R_+}(u,t)+J^+_{s,\mu,P_\varepsilon,R_-}(u,t)\big)\le K\,E_{m-1,s}(u), \tag{3.13}$$

where $|\varepsilon|\le\varepsilon_0$, $|s|\le s_0$, and $u\in C_b^\infty\big(\mathbb{R},H^{(\infty)}\big)$.

It follows immediately from (3.13) that

$$E_{m-1,s}(u)\le K\sup_{t\in\mathbb{R}}\big\|(P_\varepsilon u)(t,\cdot)\big\|_{(s)}. \tag{3.14}$$

In particular, the homogeneous equations has no time-bounded solutions.

Finally, we consider a first-order system whose characteristic roots satisfy the following assumption (cf. Condition 3.3):

Condition 3.7. There is $\delta>0$ such that

$$|\operatorname{Im}\tau_j(\varepsilon,y,\xi)|\ge\delta\quad\text{for}\quad(\varepsilon,y,\xi)\in[-1,1]\times\mathbb{R}^{n+1}_y\times\mathbb{R}^n_\xi,\quad j=1,\dots,m. \tag{3.15}$$

For any $R \in S^j(m)$, $s \in \mathbb{R}$, and $\mu > 0$, we set

$$\mathcal{J}^+_{s,\mu,R}(u,t) = \int_t^{+\infty} e^{-2\mu|t-\theta|}\big(R(\theta,\cdot,D_x)(\mathcal{P}_\varepsilon u)(\theta,\cdot), R(\theta,\cdot,D_x)u(\theta,\cdot)\big)_s d\theta.$$

The assertion below is an analog of Theorem 3.6 for systems.

Theorem 3.8. *Under the above conditions, for any δ', $0 < \delta' < \delta$, there are nonnegative Hermitian symbols $R_\pm(z,\xi) \in S^0(m)$ depending on δ' such that the following assertions hold:*

(i) *The symbols $P^* R^2_\pm - R^2_\pm P$ belong to $S^0(m)$, and for all (z,ξ) we have*

$$i\big(P^*(z,\xi)R^2_+(z,\xi) - R^2_+(z,\xi)P(z,\xi)\big) \geq 2\delta' R^2_+(z,\xi), \tag{3.16}$$

$$-i\big(P^*(z,\xi)R^2_-(z,\xi) - R^2_-(z,\xi)P(z,\xi)\big) \geq 2\delta' R^2_-(z,\xi). \tag{3.17}$$

(ii) *There is a constant $M = M(\delta') > 1$ such that*

$$M^{-1} I \leq R^2_+(z,\xi) + R^2_-(z,\xi) \leq M\, I. \tag{3.18}$$

(iii) *For any μ, $0 < \mu < \delta'$, and $s_0 > 0$ there are positive constants K and ε_0 such that*

$$K^{-1} \sup_{t\in\mathbb{R}} \|u(t,\cdot)\|^2_{(s)} \leq \sup_{t\in\mathbb{R}} \operatorname{Im}\big(\mathcal{J}^-_{s,\mu,R_+}(u,t) + \mathcal{J}^+_{s,\mu,R_-}(u,t)\big) \leq \leq K \sup_{t\in\mathbb{R}} \|u(t,\cdot)\|^2_{(s)} \tag{3.19}$$

where $|\varepsilon| \leq \varepsilon_0$, $|s| \leq s_0$, and $u \in C^\infty_b(\mathbb{R}, H^{(\infty)})$.

3.3. **Examples.** We now construct some examples of hyperbolic operators satisfying Conditions 3.1, 3.3, 3.5, and 3.7.

Example 3.9. Let $a > 0$, b, and γ be real constants and let $c \in \mathbb{R}^n$. Then the operators

$$P_{b,c}(D) = D_t + (c, D_x) - ib, \quad P_{a,\gamma}(D) = D_t^2 - 2i\gamma D_t - a^2\big(|D_x|^2 + 1\big)$$

are strictly hyperbolic. Moreover, the root of the polynomial $P_{b,c}(\tau,\xi)$ lies on the line $\operatorname{Im}\tau = b$ for any $\xi \in \mathbb{R}^n$, while the roots of $P_{a,\gamma}(\tau,\xi)$ satisfy the inequality $|\operatorname{Im}\tau - \gamma| \leq \sqrt{\rho} < |\gamma|$, where $\rho = \max\{0, \gamma^2 - a^2\}$. Therefore, if $b > 0$ and $\gamma > 0$, then the operators $P_{b,c}(D)$ and $P_{a,\gamma}(D)$ satisfy Condition 3.1.

Example 3.10. Taking products of the operators in the foregoing example, we can construct high-order operators satisfying Conditions 3.1 and 3.5. Namely, let $a_j > 0$, b_j, γ_j, $j = 1, \dots, k$, be nonzero real constants, let $c \in \mathbb{R}^n$, and let $a_j \neq a_l$ for $j \neq l$ and $|c| < a_j$ for $j = 1, \dots, k$. In this case the operators

$$\prod_{j=1}^k P_{a_j,\gamma_j}(D), \quad P_{b,c}(D)\prod_{j=1}^k P_{a_j,\gamma_j}(D)$$

are strictly hyperbolic and satisfy Condition 3.5. Moreover, if the constants γ_j and b are positive, then Condition 3.1 holds.

Example 3.11. Let $m_\pm$ be positive integers and let $P^\pm(D)$ be strictly hyperbolic polynomials of order $m_\pm$ whose principal symbols have no common roots for $\xi \neq 0$. For instance, if $P(\tau,\xi)$ is a strictly hyperbolic polynomial of order m, then $P^+ = P$

and $P^- = \partial_\tau P$ satisfy this condition with $m_+ = m$ and $m_- = m-1$. It is well known that (e.g., see [17, Section 2.2]) there are real numbers $\gamma_\pm$ such that

$$P^\pm(\tau,\xi) \neq 0 \quad \text{for} \quad (\operatorname{Re}\tau,\xi) \in \mathbb{R}^{n+1}, \quad \pm(\operatorname{Im}\tau - \gamma_\pm) < 0.$$

It follows that the operator

$$P(D) = P^+(D_t + i(\gamma_+ - \delta), D_x)\, P^-(D_t + i(\gamma_- - \delta), D_x)$$

is strictly hyperbolic and satisfies Condition 3.5.

Example 3.12. The operators constructed in Examples 3.10 and 3.11 are products of lower–order operators. The existence of strictly hyperbolic operators that satisfy Condition 3.1 or 3.5 and do not admit such a factorization follows from the stability of these conditions under small perturbations (see [17, Section 3.3]). Namely, let $P(D)$ be a strictly hyperbolic operator of order m for which (3.10) holds and let $Q(D)$ be an operator of the same order with real principal part. Then, for sufficiently small real ν, the operator $P + \nu Q$ satisfies Condition 3.5.

Example 3.13. Using arguments similar to those in Example 3.11, it is not difficult to construct matrix operators satisfying Condition 3.3 or 3.3. For instance, let $P_\varepsilon^+(t,x,D_x)$ and $P_\varepsilon^-(t,x,D_x)$ be first-order strongly hyperbolic matrix operators. Then, as is known (see [18, Section 5.2]), the characteristic roots of their symbols lie in a strip $|\operatorname{Im}\tau| \leq \gamma$. It follows that the matrix operator below is strongly hyperbolic and satisfies Condition 3.7:

$$\begin{pmatrix} P_\varepsilon^+ - i(\gamma+\delta)I_+ & 0 \\ 0 & P_\varepsilon^- + i(\gamma+\delta)I_- \end{pmatrix}$$

where I_+ and I_- are identity matrices whose sizes coincide with those of P_ε^+ and P_ε^-, respectively.

4. **Linear equations and systems: stability, dichotomy, and AP solutions.**

In this section, we present a series of results on asymptotic behavior of solutions for linear hyperbolic PDE's. We begin with the case of stable characteristic roots and study the solvability of inhomogeneous equations and systems on the positive or negative half-line. We next turn to the case when there are both stable and unstable roots and investigate the property of exponential dichotomy for homogeneous problems. Finally, we discuss the invertibility of hyperbolic operators in spaces of time-bounded and almost periodic functions. As a rule, we will confine ourselves to the case of systems, since for scalar equations the results are almost identical.

4.1. **Exponential stability.**

Let us consider the inhomogeneous problem

$$D_t u - P_\varepsilon(t,x,D_x)u = f(t,x), \tag{4.1}$$

$$u(0,x) = u_0(x), \tag{4.2}$$

where $P_\varepsilon(t,x,D_x)$ is a uniformly strongly hyperbolic matrix operator with nearly constant coefficients (i. e., the coefficients do not depend on (t,x) for $\varepsilon = 0$.) We will need the following functional spaces.

Let $\mathbb{R}_- = (-\infty, 0]$ and $\mathbb{R}_+ = [0,\infty)$. For any real numbers μ and s, we denote by $\mathbb{F}_{s,[\mu]}(\mathbb{R}_\pm)$ and $\mathbb{L}_{s,[\mu]}(\mathbb{R}_\pm)$ the spaces of functions $u(y) \in C(\mathbb{R}_\pm, H^{(s)})$ and $f(y) \in L^1_{\mathrm{loc}}(\mathbb{R}_\pm, H^{(s)})$, respectively, such that

$$F_{s,[\mu]}(u,\mathbb{R}_\pm) := \sup_{\pm t \geq 0}\Big(e^{\mu t}\|u(t,\cdot)\|_s\Big) < \infty, \quad L_{s,[\mu]}(f,\mathbb{R}_\pm) < \infty,$$

where, for any interval $J \subset \mathbb{R}$, we set

$$L_{s,[\mu]}(f,J) := \sup_{t\in J}\Big(e^{\mu t}\int_{[t,t+1]\cap J} \|f(\theta,\cdot)\|_s d\theta < \infty\Big).$$

Theorem 4.1. *Suppose that a strongly hyperbolic matrix operator P_ε satisfies Condition 3.3. Then for any $s_0 > 0$ and $\mu < \delta$ there are positive constants ε_0 and C such that the following statements hold for $|\varepsilon| \le \varepsilon_0$ and $|s| \le s_0$.*

(i) *For any right-hand side $f \in \mathbb{L}_{s,[\mu]}(\mathbb{R}_+)$ and initial function $u_0 \in H^{(s)}$ the Cauchy problem* (4.1), (4.2) *has a unique solution $u \in \mathbb{F}_{s,[\mu]}(\mathbb{R}_+)$, which satisfies the inequality*

$$\|u(t,\cdot)\|_{(s)} \le C\, e^{-\mu t}\big(\|u_0\|_{(s)} + L_{s,[\mu]}(f,[0,t])\big), \quad t \ge 1. \tag{4.3}$$

In particular, the zero solution of the homogeneous equation is exponentially asymptotically stable as $t \to +\infty$.

(ii) *For any right-hand side $f \in \mathbb{L}_{s,[\mu]}(\mathbb{R}_-)$, Eq.* (4.1) *has a unique solution $u \in \mathbb{F}_{s,[\mu]}(\mathbb{R}_-)$, which satisfies the inequality*

$$\|u(t,\cdot)\|_{(s)} \le C\, e^{-\mu t} L_{s,[\mu]}(f,\mathbb{R}_-), \quad t \le 0. \tag{4.4}$$

In particular, the homogeneous equation has no nontrivial solution bounded for $t \le 0$.

A similar result holds in the case of scalar equations. See [17, Section 6] and [18, Sections 2–4] for proofs.

We note that the zero solution of the homogeneous problem may be unstable if the coefficients of the hyperbolic operator are not close to being constant. A counterexample proving this observation is constructed in [17, Section 6.3].

4.2. **Exponential dichotomy.** We now consider the problem of exponential dichotomy (ED) for homogeneous problems. Let us begin with the case of systems.

Theorem 4.2. *Suppose that a strongly hyperbolic matrix operator P_ε satisfies Condition 3.7. Then for any $s_0 > 0$ and μ, $0 < \mu < \delta$, there are positive constants ε_0 and C and closed subspaces $\mathbb{E}_s^+(\varepsilon), \mathbb{E}_s^-(\varepsilon) \subset H^{(s)}$ not depending on μ such that the following assertions hold for $|s| \le s_0$ and $|\varepsilon| \le \varepsilon_0$.*

(i) *A vector function $u_0 \in H^{(s)}$ belongs to $\mathbb{E}_s^\pm(\varepsilon)$ if and only if the solution of the problem* (4.1), (4.2) *with $f \equiv 0$ satisfies the inequality*

$$\|u(t,\cdot)\|_{(s)} \le C\, e^{-\mu|t|}\|u_0\|_{(s)}, \quad \pm t \ge 0. \tag{4.5}$$

(ii) *The phase space $H^{(s)}$ can be represented as the direct sum*

$$H^{(s)} = \mathbb{E}_s^+(\varepsilon) \dot{+} \mathbb{E}_s^-(\varepsilon). \tag{4.6}$$

Moreover, the norms of the projections in $H^{(s)}$ that correspond to the direct decomposition (4.6) *are bounded uniformly with respect to ε and s.*

For a proof of Theorem 4.2, see [18, Section 1].

We note that a similar result holds for strictly hyperbolic scalar equations:

$$P_\varepsilon(y,D)u := P(D)u + \varepsilon Q_\varepsilon(y,D)u = 0, \tag{4.7}$$

$$D_t^j u(0,x) = u_j(x), \quad j = 0,\dots,m-1, \tag{4.8}$$

Moreover, in this case the stable and unstable subspaces $\mathbb{E}_s^+(\varepsilon)$ and $\mathbb{E}_s^-(\varepsilon)$ admit an "explicit" description as graphs of some operators acting in the phase space

$$\mathbb{E}_{m-1,s} := \prod_{j=0}^{m-1} H^{(m-1+s-j)}(\mathbb{R}^n). \tag{4.9}$$

To formulate the corresponding result, we introduce some notations.

We first note that if $P(\tau,\xi)$ is a strictly hyperbolic polynomial satisfying condition (3.10), then there are $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for $|\varepsilon| \le \varepsilon_1$ the roots of $P_\varepsilon(y,\tau,\xi)$ lie outside the strip $|\operatorname{Im}\tau| < \delta_1$. Let m_+ and m_- be the number of roots in the half-planes $\operatorname{Im}\tau \ge \delta_1$ and $\operatorname{Im}\tau \le -\delta_1$, respectively. We have the factorization (cf. (3.11) and (3.12))

$$P_\varepsilon(y,\tau,\xi) = P_\varepsilon^+(y,\tau,\xi)P_\varepsilon^-(y,\tau,\xi), \tag{4.10}$$
$$P_\varepsilon^\pm(y,\tau,\xi) = \tau^{m_\pm} + \sum_{j=1}^{m_\pm} p_j^\pm(\varepsilon,y,\xi)\tau^{m-j}.$$

where P_+ (P_-) is a polynomial of degree m_+ (m_-) with roots in the half-plane $\operatorname{Im}\tau \ge \delta_1$ ($\operatorname{Im}\tau \le -\delta_1$). It is a matter of direct verification to show that the coefficients p_j belong to S^j.[3]

Let us denote by

$$C_j^\pm(\varepsilon,y,\tau,\xi) = \sum_{k=0}^{m_\pm-1} c_{jk}^\pm(\varepsilon,y,\xi)\tau^k, \quad j = m_\pm,\dots,m-1,$$

the remainder after division of τ^j by $P_\varepsilon^\pm(t,x,\tau,\xi)$. It is not difficult to see that

$$c_{jk}^\pm(\varepsilon,y,\xi) \in S^{j-k}, \quad j = m_\pm,\dots,m-1, \quad k = 0,\dots,m_\pm - 1.$$

In particular, the ΨDO's $c_{jk}^\pm(\varepsilon,y,D_x)$ are continuous from $H^{(r)}$ to $H^{(r-j+k)}$ for any ε and t. Recall that the energy $E_{m-1,s}(u,t)$ of a solution for (4.7) is defined by formula (3.5).

Theorem 4.3. *Suppose that $P(D)$ is a strictly hyperbolic operator of order m and that Condition 3.5 is satisfied. Then for any $s_0 > 0$ and μ, $0 < \mu < \delta$, there are positive constants ε_0 and C depending on s_0 and μ and continuous operators*

$$\mathcal{R}_{jk}^\pm(\varepsilon) : H^{(m-1+s-k)} \to H^{(m-1+s-j)}, \quad j = m_\pm,\dots,m-1, \ \ k = 0,\dots,m_\pm - 1.$$

such that the following assertions hold for $|s| \le s_0$ and $|\varepsilon| \le \varepsilon_0$.

(i) *The energy $E_{m-1,s}(u,t)$ of a solution of the Cauchy problem* (4.7), (4.8) *with $[u_0,\dots,u_{m-1}] \in \mathbb{E}_{m-1,s}$ is bounded for $\pm t \ge 0$ if and only if*

$$u_j = \sum_{k=0}^{m_\pm-1} \mathcal{R}_{jk}^\pm(\varepsilon)u_k, \quad j = m_\pm,\dots,m-1. \tag{4.11}$$

In this case, we have

$$E_{m-1,s}(u,t) \le Ce^{-\mu|t|} \sum_{j=0}^{m-1} \|u_j\|_{(m-1+s-j)}, \quad \pm t \ge 0.$$

[3] This assertion is not quite accurate since in the definition of S^j it was assumed that the domain of variation of the parameter ε is the interval $[-1,1]$. Factorization (4.10) holds for $|\varepsilon| \le \varepsilon_1$. The above statement will be correct if we modify the definition of S^j, assuming that ε varies in the interval $[-\varepsilon_1,\varepsilon_1]$.

(ii) *The phase space* $\mathbb{E}_{m-1,s}$ *can be represented as the direct sum*

$$\mathbb{E}_{m-1,s} = \mathbb{E}^{+}_{m-1,s}(\varepsilon) \dotplus \mathbb{E}^{-}_{m-1,s}(\varepsilon), \tag{4.12}$$

where we set

$$\mathbb{E}^{\pm}_{m-1,s}(\varepsilon) = \big\{[u_0, \dots, u_{m-1}] \in \mathbb{E}_{m-1,s} : \text{relations } (4.11_{\pm}) \text{ hold}\big\}.$$

Moreover, the norms of the projections in $\mathbb{E}_{m-1,s}$ *that correspond to the direct decomposition* (4.12) *are bounded uniformly with respect to* ε *and* s.

(iii) *There are continuous linear operators*

$$d^{\pm}_{jk}(\varepsilon) : H^{(m-1+s-k)} \to H^{(m+s-j)}, \quad j = m_{\pm}, \dots, m-1, \;\; k = 0, \dots, m_{\pm} - 1,$$

whose norms are bounded uniformly with respect to ε *and* s *such that*

$$\mathcal{R}^{\pm}_{jk}(\varepsilon) = c^{\pm}_{jk}(\varepsilon, 0, x, D_x) + \varepsilon d^{\pm}_{jk}(\varepsilon).$$

For a proof of Theorem 4.3, see [21].

4.3. **Hyperbolic operators in spaces of time-bounded and almost periodic functions.** We will confine ourselves to hyperbolic systems, since the results in the case of scalar equations are exactly the same.

Let us introduce a linear manifold

$$\mathcal{D}_s = \big\{u \in C_b(\mathbb{R}, H^{(s)}) : D_t u - P_\varepsilon(y, D_x)u \in C_b(\mathbb{R}, H^{(s)})\big\} \tag{4.13}$$

and consider the closed linear operator

$$\mathcal{P}_\varepsilon : \mathcal{D}_s \to C_b(\mathbb{R}, H^{(s)}), \quad u \mapsto D_t u - P_\varepsilon(y, D_x)u. \tag{4.14}$$

Theorem 4.4. *Under the conditions of Theorem 4.2, for any* $s_0 > 0$ *there is a constant* $\varepsilon_0 > 0$ *such that, for* $|s| \le s_0$ *and* $|\varepsilon| \le \varepsilon_0$*, the operator* $\mathcal{P}_\varepsilon$ *has a continuous inverse whose norm is bounded uniformly in* s *and* ε.

A trivial consequence of Theorem 4.4 is the following result: if the coefficients of the operator $P_\varepsilon(t, x, D_x)$ and the right-hand side $f \in C_b(\mathbb{R}, H^{(s)})$ are periodic in time with period $T > 0$, then, for sufficiently small ε, Eq. (4.1) has a unique periodic solution $u \in C_b(\mathbb{R}, H^{(s)})$.

Assertions similar to Theorem 4.4 hold true if the space $C_b(\mathbb{R}, H^{(s)})$ is replaced by some spaces of almost periodic (AP) functions in time. To formulate the corresponding results, we introduce some notations.

Let $\mathfrak{M} = \{\lambda_k\} \subset \mathbb{R}$ be a countable module, i.e., a countable subset of the real line that is a group with respect to addition. We assume that $\mathfrak{M}$ does not coincide with any set of the form $\{k\lambda : k \in \mathbb{Z}\}$. Let us introduce a new metric on $\mathbb{R}$ by the formula

$$d_{\mathfrak{M}}(t_1, t_2) = \sum_{k=1}^{\infty} 2^{-k} |\exp(i\lambda_k(t_1 - t_2)) - 1|, \quad t_1, t_2 \in \mathbb{R},$$

and denote by $\mathbb{R}_{\mathfrak{M}}$ the set of real numbers endowed with the metric $d_{\mathfrak{M}}$.

Let X be a metric space and let $f(t) \in C(\mathbb{R}, X)$. Since $\mathbb{R}_{\mathfrak{M}}$ and $\mathbb{R}$ coincide in the set-theoretic sense, $f(t)$ can be regarded as a function from $\mathbb{R}_{\mathfrak{M}}$ into X.

Definition 4.5. A continuous function $f : \mathbb{R} \to X$ is said to be *Bohr AP with module contained in* $\mathfrak{M}$ if $f(t)$ is a uniformly continuous function from $\mathbb{R}_{\mathfrak{M}}$ into X. The space of these functions is denoted by $AP(X, \mathfrak{M})$.

Definition 4.6. A continuous function $f : \mathbb{R} \to X$ is said to be *Levitan AP with module contained in* $\mathfrak{M}$ if $f(t)$ is a continuous function from $\mathbb{R}_{\mathfrak{M}}$ into X. The space of these functions is denoted by $LAP(X, \mathfrak{M})$.

In particular, since $C_b^\infty(\mathbb{R}_x^n)$ is a polynormed space, we can define the classes $AP(C_b^\infty(\mathbb{R}_x^n), \mathfrak{M})$ and $LAP(C_b^\infty(\mathbb{R}_x^n), \mathfrak{M})$ of almost periodic functions.

Theorem 4.7. (i) *Suppose that the conditions of Theorem 4.2 are satisfied and that, for any fixed $\varepsilon \in [-1, 1]$, the coefficients of the operator $P_\varepsilon(t, x, D_x)$, regarded as functions of t with range in $C_b^\infty(\mathbb{R}_x^n)$, belong to $AP(C_b^\infty(\mathbb{R}_x^n), \mathfrak{M})$ for some countable module $\mathfrak{M} \subset \mathbb{R}$. Let $\mathcal{R}_\varepsilon$ be the inverse of $\mathcal{P}_\varepsilon$ constructed in Theorem 4.4. Then $\mathcal{R}_\varepsilon$ maps the space $AP(H^{(s)}, \mathfrak{M})$ into itself.*

(ii) *Assertion* (i) *remains valid if we replace the spaces $AP(C_b^\infty(\mathbb{R}_x^n), \mathfrak{M})$ and $AP(H^{(s)}, \mathfrak{M})$ by $LAP(C_b^\infty(\mathbb{R}_x^n), \mathfrak{M})$ and $LAP(H^{(s)}, \mathfrak{M})$, respectively.*

The proofs of Theorems 4.4 and 4.7 and of their analogs for scalar equations can be found in [17, 18].

5. **Nonlinear equations and systems: stable, unstable, and center manifolds, linearization, and AP solutions.** This section is devoted to nonlinear problems. We first study the existence of stable and unstable manifolds for homogeneous first-order quasilinear systems and high-order fully nonlinear scalar equations. We next turn to inhomogeneous problems and discuss their solvability in spaces of time-bounded and almost periodic functions. For semilinear equations and systems, we formulate two results on linearization of the phase portrait. The sections is concluded by a theorem on the existence and different properties of a center manifold for scalar equations.

5.1. **Stable and unstable manifolds and almost periodic solutions.** Let us begin with the quasilinear system

$$\partial_t u = P_\varepsilon(y, u, \partial_x)u, \tag{5.1}$$

where P_ε is an $m \times m$ matrix of first-order partial differential operators in x whose coefficients depend on $\varepsilon \in [-1, 1]$, $y = (t, x) \in \mathbb{R}^{n+1}$, and $u \in \mathbb{R}^m$. We assume that P_ε satisfies the following three conditions.

(H$_1$) The operator P_ε is a perturbation of an operator with constant coefficients, i.e., it can be represented in the form

$$P_\varepsilon(y, u, \partial_x) = P(\partial_x) + \varepsilon Q_\varepsilon(y, u, \partial_x),$$

where the coefficients of Q_ε (which are functions of (ε, y, u)) belong to the space $C_b^\infty([-1, 1] \times \mathbb{R}_y^{n+1} \times B)$ for any bounded ball $B \subset \mathbb{R}_u^m$.

(H$_2$) The operator P_ε is uniformly strongly hyperbolic in the sense specified before Theorem 2.4.

(H$_3$) For any $R > 0$ there is $\delta = \delta_R > 0$ such that the characteristic roots of the full symbol $P_\varepsilon(y, u, i\xi)$ are outside the strip $|\operatorname{Re} \tau| < \delta$ for $\varepsilon \in [-1, 1]$, $y \in \mathbb{R}^{n+1}$, $u \in B_R$, $\xi \in \mathbb{R}^n$, where $B_R \subset \mathbb{R}^m$ is a closed ball of radius R centered at zero (cf. Condition 3.7).

Let us consider the Cauchy problem for Eq. (5.1):

$$u(0, x) = u_0(x). \tag{5.2}$$

By Theorem 2.4, the problem (5.1), (5.2) is locally well-posed, i. e., for any $u_0 \in H^{(s)}$, $s \gg 1$, and $|\varepsilon| \ll 1$ it has a unique solution defined on a small time-interval. We wish to study the existence of global solutions.

Let us consider the linearization of (5.1) on the function $u \equiv 0$:

$$\partial_t u = P_\varepsilon(t, x, 0, \partial_x)u. \tag{5.3}$$

We note that (5.3) satisfies the conditions of Theorem 4.2. Therefore, for any integer $s \in \mathbb{R}$ and sufficiently small values of ε we can construct stable and unstable subspaces $\mathbb{E}_s^+(\varepsilon)$ and $\mathbb{E}_s^-(\varepsilon)$. We denote by $B_s(R)$ a closed ball in $H^{(s)}$ of radius R centered at zero and set $\mathbb{E}_s^\pm(\varepsilon, R) = \mathbb{E}_s^\pm(\varepsilon) \cap B_s(R)$.

Theorem 5.1. *Suppose that conditions* (H$_1$) – (H$_3$) *are satisfied. Then for an arbitrary $R > 0$, sufficiently large integers $s > 0$, and any μ, $0 < \mu < \delta$, there are positive constants ε_0 and C such that the following statements hold for $|\varepsilon| \le \varepsilon_0$.*

(i) *There are weakly continuous injective operators*

$$\mathcal{R}_s^\pm(\varepsilon, \cdot)\colon \mathbb{E}_s^\pm(\varepsilon, R) \to H^{(s)} \tag{5.4}$$

such that for any $u_0 \in \mathcal{M}_s^\pm(\varepsilon, R) := \operatorname{Image}(\mathcal{R}_s^\pm(\varepsilon))$ the solution $u(t,x)$ of the problem (5.1), (5.2) *is defined on the half-line $\mathbb{R}_\pm$ and satisfies the inequality*

$$\sum_{j=0}^{s} \|\partial_t^j u(t,\cdot)\|_{(s-j)} \le C\, e^{-\mu|t|}\|u_0\|_{(s)}, \quad \pm t \ge 0. \tag{5.5}$$

(ii) *The sets $\mathcal{M}_s^+(\varepsilon, R)$ and $\mathcal{M}_s^-(\varepsilon, R)$ intersect only at zero. Moreover, the operators $\mathcal{R}_s^+(\varepsilon)$ and $\mathcal{R}_s^-(\varepsilon)$ are weakly differentiable at zero, and their derivatives are equal to the identity operators in $\mathbb{E}_s^+(\varepsilon)$ and $\mathbb{E}_s^-(\varepsilon)$, respectively. In particular, the manifold $\mathcal{M}_s^\pm(\varepsilon, R)$ has a tangent space $T\mathcal{M}_s^\pm(\varepsilon)$ at zero, which coincides with $\mathbb{E}_s^\pm(\varepsilon)$, and hence $H^{(s)} = T\mathcal{M}_s^+(\varepsilon) \dot{+} T\mathcal{M}_s^-(\varepsilon)$.*

The sets $\mathcal{M}_s^+(\varepsilon, R)$ and $\mathcal{M}_s^-(\varepsilon, R)$ are called stable and unstable manifolds for system (5.1). The proof of the above theorem is sketched in [18, Section 4.1]

Similar results are true for fully nonlinear scalar equations. Moreover, in this case the stable and unstable manifolds admit an “explicit” description as graphs of some nonlinear operators in the phase space (cf. Theorem 4.3).

More exactly, let us consider the homogeneous equation

$$P(\partial)u + \varepsilon Q_\varepsilon(t, x, \partial^m u) = 0, \quad (t,x) \in \mathbb{R}^{n+1}, \tag{5.6}$$

supplemented by the initial conditions

$$\partial_t^j u(0,x) = u_j(x), \quad j = 0, \ldots, m-1. \tag{5.7}$$

Here $P(\partial)$ is a strictly hyperbolic operator of order m with constant coefficients and $Q_\varepsilon(t, x, \partial^m u)$ is a nonlinear operator that satisfies the conditions mentioned after formula (2.5). We also assume that the symbol $P(i\tau, i\xi)$ of the operator $P(\partial_t, \partial_x)$ satisfies Condition 3.5 and denote by m_+ and m_- the number of its roots that lie in the half-planes $\operatorname{Im}\tau \ge \delta$ and $\operatorname{Im}\tau \le -\delta$, respectively.

Let us recall that the space $\mathbb{E}_{m-1,s}$ is defined by (4.9) and (3.5). We endow $\mathbb{E}_{m-1,s}$ by the metric

$$\|U\|_{(m-1,s)} = \Big(\sum_{j=0}^{m-1} \|u_j\|^2_{(m-1+s-j)}\Big)^{1/2}, \quad U = [u_0, \ldots, u_{m-1}] \in \mathbb{E}_{m-1,s},$$

and denote by $\mathbb{B}_{m-1,s}(R)$ a closed ball in $\mathbb{E}_{m-1,s}$ of radius R centered at zero. We also set

$$E_s^2(u,t) = \sum_{j=0}^{s} \|\partial_t^j u(t,\cdot)\|_{(s-j)}^2.$$

The assertion below is an analog of Theorem 4.3 for nonlinear equations.

Theorem 5.2. *Under the above conditions, for any μ, $0 < \mu < \delta$, sufficiently large integers $s > 0$, and an arbitrary $R > 0$ there are positive constants ε_0 and $C = C_{s,\mu}(R)$ and nonlinear weakly continuous operators*

$$\mathcal{R}_j^{\pm}(\varepsilon;\cdot)\colon \mathbb{B}_{m_\pm-1,s+m_\mp}(R) \to H^{(m-1+s-j)}, \quad j = m_\pm, \dots, m-1,$$

such that the following assertions hold for $|\varepsilon| \le \varepsilon_0$.

(i) *The solution $u(t,x)$ of the Cauchy problem* (5.6), (5.7) *with initial set of functions $[u_0,\dots,u_{m-1}] \in \mathbb{B}_{m-1,s}(R)$ satisfies the inequality*

$$E_{m-1+s}(u,t) \le C_{s,\mu}(R), \quad \pm t \ge 0,$$

if and only if

$$u_j = \mathcal{R}_j^{\pm}(\varepsilon; u_0,\dots,u_{m_\pm-1}), \quad j = m_\pm,\dots,m-1. \tag{5.8}$$

In this case, we also have the estimate

$$E_{m-1+s}(u,t) \le Ce^{-\mu|t|} \sum_{j=0}^{m-1} \|u_j\|_{(m-1+s-j)}, \quad \pm t \ge 0.$$

(ii) *The manifolds*

$$\mathcal{M}_s^{\pm}(\varepsilon,R) = \bigl\{[u_0,\dots,u_{m-1}] \in \mathbb{B}_{m-1,s}(R) : \text{relations } (5.8_\pm) \text{ hold}\bigr\}$$

intersect only at zero and are weakly differentiable at that point. Moreover, the corresponding tangent spaces $T\mathcal{M}_s^+(\varepsilon)$ and $T\mathcal{M}_s^-(\varepsilon)$ coincide, accordingly, with the stable and unstable subspaces constructed for the linearized equation

$$P(\partial)u + \varepsilon \sum_{|\alpha|\le m} (\partial_{p_\alpha} Q_\varepsilon)(t,x,0)\partial^\alpha u = 0.$$

In particular, in view of (4.12), *we have the direct decomposition*

$$\mathbb{E}_{m-1,s} = T\mathcal{M}_s^+(\varepsilon) \dotplus T\mathcal{M}_s^-(\varepsilon).$$

For a proof of the above theorem in the quasilinear case, see [21, Section 7].

We now turn to inhomogeneous problems. We will confine ourselves to the case of systems:

$$\partial_t u - P_\varepsilon(t,x,u,\partial_x)u = f(t,x). \tag{5.9}$$

For any integer $s \ge 0$ and any countable module $\mathfrak{M} \subset \mathbb{R}$ we set

$$\mathbb{AP}_s(\mathfrak{M}) = \bigl\{u(t,x) : \partial_t^j u \in AP(H^{(s-j)},\mathfrak{M}),\ j = 0,\dots,s\bigr\},$$
$$\mathbb{LAP}_s(\mathfrak{M}) = \bigl\{u(t,x) : \partial_t^j u \in LAP(H^{(s-j)},\mathfrak{M}),\ j = 0,\dots,s\bigr\},$$

where the spaces $AP(X,\mathfrak{M})$ and $LAP(X,\mathfrak{M})$ are defined in Section 4.3. The following result is an analog of Theorems 4.4 and 4.7 in the nonlinear case.

Theorem 5.3. *Suppose that conditions* (H_1) – (H_3) *are fulfilled. Then for any* $R > 0$ *and sufficiently large integers* $s > 0$ *there are positive constants* C *and* ε_0 *such that the following assertions hold for* $|\varepsilon| \le \varepsilon_0$ *and any right-hand side* $f(t,x)$ *satisfying the conditions*

$$\partial_t^j f \in C_b(\mathbb{R}, H^{(s-j)}), \quad \sup_{t\in\mathbb{R}} \|\partial_t^j f(t,\cdot)\|_{(s-j)} \le R, \quad j = 0,\dots,s.$$

(i) *System* (5.9) *has a unique solution* $u(t,x)$ *such that*

$$\partial_t^j u \in C_b(\mathbb{R}, H^{(s-j)}), \quad \sup_{t\in\mathbb{R}} \|\partial_t^j f(t,\cdot)\|_{(s-j)} \le C, \quad j = 0,\dots,s.$$

If the coefficients of P_ε *and the function* f *are periodic in time with period* $T > 0$, *then so is the solutions* $u(t,x)$.

(ii) *Let* $f \in \mathbb{AP}_{s-1}(\mathfrak{M})$ *for some countable module* $\mathfrak{M}$ *and let the coefficients of* P_ε *belong to the space* $AP(C_b^\infty(\mathbb{R}^{n+1}_{t,x} \times B), \mathfrak{M})$ *for any closed ball* $B \subset \mathbb{R}^m_u$. *Then* $u \in \mathbb{AP}_{s-1}(\mathfrak{M})$. *This assertion remains true if we replace the spaces* $AP(C_b^\infty(\mathbb{R}^{n+1}_{t,x} \times B), \mathfrak{M})$ *and* $\mathbb{AP}_{s-1}(\mathfrak{M})$ *by* $LAP(C_b^\infty(\mathbb{R}^{n+1}_{t,x} \times B), \mathfrak{M})$ *and* $\mathbb{LAP}_{s-1}(\mathfrak{M})$, *respectively.*

A similar result holds for fully nonlinear scalar equations; see [20, Sections 5 and 6] for details.

5.2. **Grobman–Hartman type theorems.** Theorems 5.1 and 5.2 describe the asymptotic behavior of solutions for homogeneous problems in the case when the initial conditions lie on the stable or unstable manifolds. We now formulate two results concerning the whole phase portrait in the neighborhood of the zero solution. To simplify the presentation, we will confine ourselves to the autonomous case.

We begin with a semilinear strongly hyperbolic system of the form

$$\partial_t u = P_\varepsilon(x, \partial_x)u + \varepsilon Q_\varepsilon(x,u), \tag{5.10}$$

where P_ε is an $m \times m$ matrix of first-order differential operator with nearly constant coefficients and Q_ε is a real-valued function belonging to the space $C_b^\infty([-1,1] \times \mathbb{R}^n \times B)$ for any ball $B \subset \mathbb{R}^m$. We denote by $\tau_j(\varepsilon, x, \xi)$ the characteristic roots of the full symbol $P_\varepsilon(x, i\xi)$ and assume that they are outside an open strip $|\operatorname{Re}\tau| < \delta$, $\delta > 0$. (cf. Condition 3.7).

We wish to compare the solutions of (5.10) with those of the linear system

$$\partial_t v = P_\varepsilon(x, \partial_x)v. \tag{5.11}$$

We denote by

$$\mathcal{U}_\varepsilon(t,\cdot) : H^{(s)} \to H^{(s)}, \quad u_0 \mapsto u(t,\cdot),$$
$$\mathcal{V}_\varepsilon(t) : H^{(s)} \to H^{(s)}, \quad u_0 \mapsto v(t,\cdot),$$

the resolving operators of the Cauchy problem for Eqs. (5.10) and (5.11), respectively. Here $u(t,x)$ and $v(t,x)$ are the solutions of (5.10) and (5.11) starting from the initial function $u_0 \in H^{(s)}$. Thus, $\mathcal{U}_\varepsilon(t,\cdot)$ is defined in a small neighborhood of zero, while $\mathcal{V}_\varepsilon(t)$ exists for all $t \in \mathbb{R}$ (see Theorems 2.2 and 2.4).

Let us set

$$\sigma_{\max} = \sup|\operatorname{Re}\tau_j(\varepsilon,x,\xi)|, \quad \sigma_{\min} = \inf|\operatorname{Re}\tau_j(\varepsilon,x,\xi)|,$$

where the supremum and infimum extend over $\varepsilon \in [-1,1]$, $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, $j = 1,\dots,m$. It is clear that $\sigma_{\max} \ge \sigma_{\min} > 0$.

Let $\dot{B}_s(R) \subset H^{(s)}$ be an open ball of radius R centered at zero. The following result is an analog of the Grobman–Hartman theorem for hyperbolic systems.

Theorem 5.4. *Under the above conditions, for any γ, $0<\gamma<\sigma_{\min}/\sigma_{\max}$, sufficiently large integers $s>0$, and an arbitrary $R>0$ there are positive constants C and ε_0 such that the following statements hold for $|\varepsilon|\le\varepsilon_0$.*

(i) *There exists an open neighborhood of zero $O_R(\varepsilon)\subset H^{(s)}$ and a homeomorphism $\Phi_\varepsilon:\dot B_s(R)\to O_R$ such that for any $u_0\in\dot B_s(R)$ we have*

$$\Phi_\varepsilon\big(\mathcal{U}_\varepsilon(t,u_0)\big)=\mathcal{V}_\varepsilon(t)\Phi_\varepsilon(u_0)\quad\text{as long as}\quad \mathcal{U}_\varepsilon(t,u_0)\in\dot B_s(R).$$

(ii) *The operator Φ_ε and its inverse Φ_ε^{-1} are uniformly Hölder continuous with exponent γ, i. e.,*

$$\big\|\Phi_\varepsilon(u_0)-\Phi_\varepsilon(v_0)\big\|_{(s)}\le C\|u_0-v_0\|_{(s)}^\gamma,\qquad u_0,v_0\in\dot B_s(R),$$
$$\big\|\Phi_\varepsilon^{-1}(u_0)-\Phi_\varepsilon^{-1}(v_0)\big\|_{(s)}\le C\|u_0-v_0\|_{(s)}^\gamma,\qquad u_0,v_0\in O_R(\varepsilon).$$

A similar result holds for semilinear scalar equations. Moreover, in this case it can be proved that the phase portrait is homeomorphic to that of the unperturbed equation with constant coefficients.

More exactly, let us consider the linear equation (with real constant coefficients)

$$P(\partial)u(t,x)=0,\quad (t,x)\in\mathbb{R}^{n+1},\tag{5.12}$$

and its semilinear perturbation

$$P(\partial)u+\varepsilon\sum_{|\alpha|\le m}q_\alpha(\varepsilon,x)\partial^\alpha u+\varepsilon q(\varepsilon,x,\partial^{m-1}u)=0,\quad (t,x)\in\mathbb{R}^{n+1}.\tag{5.13}$$

Here $\varepsilon\in[-1,1]$ is a parameter, $P(\partial)$ is a strictly hyperbolic operator satisfying Condition 3.5, and $q_\alpha(\varepsilon,x)$ and $q(\varepsilon,x,p)$, $p=(p_\alpha,|\alpha|\le m-1)\in\mathbb{R}^d$, are real-valued functions that belong to the spaces $C_b^\infty([-1,1]\times\mathbb{R}_x^n)$ and $C_b^\infty([-1,1]\times\mathbb{R}_x^n\times B)$, respectively, for any closed ball $B\subset\mathbb{R}^d$. Moreover, we assume that $q(\varepsilon,x,0)\equiv 0$, so that the function $u\equiv 0$ is a solution of (5.13).

By Theorem 2.2, the Cauchy problem for Eq. (5.13) is well-posed. Let

$$\mathcal{U}_\varepsilon(t,\cdot)\colon\mathbb{E}_{m-1,s}\to\mathbb{E}_{m-1,s},\quad [u_0,\dots,u_{m-1}]\mapsto[u(t,\cdot),\partial_t u(t,\cdot),\dots,\partial_t^{m-1}u(t,\cdot)],$$

be the corresponding resolving operator. Here $u(t,x)$ is the solution of (5.13), (5.7), and t varies in some interval $(-T,T)$ on which the solution is defined. For $\varepsilon=0$, Eq. (5.13) coincides with the linear equation (5.12), and therefore $\mathcal{U}_0(t)$ is a bounded linear operator in $\mathbb{E}_{m-1,s}$ defined for all $t\in\mathbb{R}$.

Let $\dot{\mathbb{B}}_{m-1,s}(R)$ be an open ball in $\mathbb{E}_{m-1,s}$ of radius R centered at zero. The following theorem establishes conjugacy between $\mathcal{U}_\varepsilon(t,\cdot)$ and $\mathcal{U}_0(t)$ in the neighborhood of zero.

Theorem 5.5. *Under the above conditions, for sufficiently large integers $s>0$ and an arbitrary $R>0$ there is a constant $\varepsilon_0>0$, an open neighborhood of zero $\mathbb{O}_R(\varepsilon)\subset\mathbb{E}_{m-1,s}$, and a homeomorphism*

$$\Phi_\varepsilon:\dot{\mathbb{B}}_{m-1,s}(R)\to\mathbb{O}_R(\varepsilon),\quad |\varepsilon|\le\varepsilon_0,$$

such that for $|\varepsilon|\le\varepsilon_0$ and any $U_0\in\dot{\mathbb{B}}_{m-1,s}(R)$ we have

$$\Phi_\varepsilon\big(\mathcal{U}_\varepsilon(t,U_0)\big)=\mathcal{U}_0(t)\Phi_\varepsilon(U_0)\quad\text{as long as}\quad \mathcal{U}_\varepsilon(t,U_0)\in\dot{\mathbb{B}}_{m-1,s}(R).$$

We refer the reader to [23, Chapter I] for the proof of Theorems 5.4 and 5.5 in the case of nonautonomous semilinear scalar equations and for some further discussion and examples concerning the Grobman–Hartman theorem.

5.3. **Center manifold theorem.** Up to now, we have discussed the situation when the characteristic roots of the equation or system under study are either stable or unstable. In this subsection, we will consider the case when some of the roots are neutral. For brevity, we will confine ourselves to scalar equations.

We consider the semilinear equation (5.13), where $P(\partial_t, \partial_x)$ is a strictly hyperbolic operator of order m whose full symbol $P(i\tau, i\xi)$ satisfies the following assumption (cf. Conditions 3.1 and 3.5):

Condition 5.6. There are $\delta > \nu \geq 0$ such that

$$P(i\tau, i\xi) \neq 0 \quad \text{for} \quad (\operatorname{Re}\tau, \xi) \in \mathbb{R}^{n+1}, \quad \nu < |\operatorname{Im}\tau| < \delta.$$

We will denote by m_c the number of roots of $P(i\tau, i\xi)$ that lie in the strip $|\operatorname{Im}\tau| \leq \nu$ and assume $1 \leq m_c \leq m-1$. Let $m_h = m - m_c$. Concerning the functions q_α and q, we suppose that they satisfy the conditions formulated after (5.13).

Let X and Y be Banach spaces and let $\Omega \subset X$ be an open subset. For $\gamma \in (0,1)$ and any integer $l \geq 1$, we denote by $C^{l,\gamma}(\Omega, Y)$ the space of l times continuously differentiable functions from Ω to Y whose lth derivative is uniformly Hölder continuous with exponent γ.

The following theorem provides some information on the behavior of solutions in the neighborhood of the stationary point $u \equiv 0$.

Theorem 5.7. *Suppose that the above conditions are fulfilled. Let $\gamma \in (0,1)$ and an integer $l \geq 1$ satisfy the inequality $l\nu + \gamma \leq \delta$. Then for any $R > 0$, $\mu \in (\nu, \delta/l)$, and sufficiently large integers $s > 0$ there are positive constants ε and C and continuous operators*

$$\mathcal{R}_j(\varepsilon; u_0, \dots, u_{m_c-1}) \colon \mathbb{B}_{m_c-1, s+m_h}(R) \to H^{(m-1+s-j)}, \quad j = m_c, \dots, m-1,$$

such that $\mathcal{R}_j(\varepsilon; 0) = 0$, and the following assertions are true for $|\varepsilon| \leq \varepsilon_0$.

(i) *Local invariance. The manifold*

$$\mathcal{M}_c(\varepsilon, R) = \bigl\{[u_0, \dots, u_{m-1}] \in \mathbb{B}_{m-1,s}(R) : u_j = \mathcal{R}_j(\varepsilon; u_0, \dots, u_{m_c-1}),\, j \geq m_c\bigr\}$$

is invariant under the action of the resolving operator $\mathcal{U}_\varepsilon(t,\cdot)$, i.e., if $U_0 \in \mathcal{M}_c(\varepsilon, R)$, then $\mathcal{U}_\varepsilon(t, U_0) \in \mathcal{M}_c(\varepsilon, R)$ as long as $\mathcal{U}_\varepsilon(t, U_0) \in \mathbb{B}_{m-1,s}(R)$.

(ii) *Attraction property. Suppose that $\mathcal{U}_\varepsilon(t, U_0) \in \mathbb{B}_{m-1,s}(\rho)$ for $t \geq 0$ and some $\rho < R$. Then there is a constant $T \geq 0$ and a vector function $V \in \mathcal{M}_c(\varepsilon, R)$ such that*

$$\bigl\|\mathcal{U}_\varepsilon(t, U_0) - \mathcal{U}_\varepsilon(t, V_0)\bigr\|_{(m-1,s)} \leq Ce^{-\mu t}, \quad t \geq T.$$

A similar assertion holds for solutions defined on the half-line $t \geq 0$. Moreover, if the phase trajectory $U(t) = [u(t,\cdot), \dots, \partial_t^{m-1} u(t,\cdot)]$ of a solution $u(t,x)$ belongs to the ball $\mathbb{B}_{m-1,s}(R)$ for all $t \in \mathbb{R}$, then $U(t) \in \mathcal{M}_c(\varepsilon, R)$ for $t \in \mathbb{R}$.

(iii) *Smoothness. For any fixed ε, the operators $\mathcal{R}_j(\varepsilon; \cdot)$ belong to the class $C^{l,\gamma}(\mathbb{B}_{m_c-1, s+m_h}(R), H^{(m-1+s-j)})$.*

The infinite-dimensional surface $\mathcal{M}_c(\varepsilon, R)$, which is embedded into the phase space $\mathbb{E}_{m-1,s}$, is called a *center manifold* for Eq. (5.13). It can be proved that the operators $\mathcal{R}_j$, defining $\mathcal{M}_c(\varepsilon, R)$, are small perturbations of some pseudodifferential operators whose symbols can be expressed in terms of the symbol of the original equation. Moreover, the dynamics on the invariant manifold $\mathcal{M}_c(\varepsilon, R)$ is described by a nonlocal perturbation of a hyperbolic equation. We refer the reader to [23, Chapter II] for proofs and further details.

REFERENCES

[1] E. A. Coddington and N. Levinson, "Theory of Ordinary Differential Equations," McGraw-Hill Book Company, New York–Toronto–London, 1955.

[2] Ju. L. Daleckiĭ and M. G. Kreĭn, "Stability of Solutions of Differential Equations in Banach Space," Translations of Mathematical Monographs, Vol. 43, AMS, Providence, RI, 1974.

[3] P.-A. Dionne, Sur les problèmes de Cauchy hyperboliques bien posés, *J. Analyse Math.* **10** (1962/1963), 1–90.

[4] S. G. Gindikin and L. R. Volevich, "Mixed Problem for Partial Differential Equations with Quasihomogeneous Principal Part," Translations of Mathematical Monographs, 147, American Mathematical Society, Providence, RI, 1996.

[5] P. Hartman, "Ordinary Differential Equations," John Wiley & Sons, New York–London–Sydney, 1964.

[6] D. Henry, "Geometric Theory of Semilinear Parabolic Equations," Lecture Notes in Mathematics, 840, Springer-Verlag, Berlin–New York, 1981.

[7] L. Hörmander, "The Analysis of Linear Partial Differential Operators III. Pseudo-Differential Operators," Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1985.

[8] U. Kirchgraber and K. J. Palmer, "Geometry in the Neighborhood of Invariant Manifolds of Maps and Flows and Linearization," John Wiley & Sons, New York, 1990.

[9] M. A. Krasnoselskiĭ, V. Sh. Burd, and Yu. S. Kolesov, "Nonlinear Almost Periodic Oscillations," Halsted Press, New York–Toronto, 1973.

[10] H.-O. Kreiss and J. Lorenz, "Initial Boundary Value Problems and the Navier–Stokes Equations," Pure and Applied Mathematics, vol. 136, Academic Press, Boston, 1989.

[11] J. Leray, "Hyperbolic Differential Equations," The Institute for Advanced Study, Princeton, N.J., 1953.

[12] B. M. Levitan and V. V. Zhikov, "Almost Periodic Functions and Differential Equations," Cambridge University Press, Cambridge–New York, 1982.

[13] J. L. Massera and J. J. Schäffer, "Linear Differential Equations and Function Spaces," Academic Press, New York–London, 1966.

[14] A. A. Pankov, "Bounded and Almost Periodic Solutions of Nonlinear Operator Differential Equations," Mathematics and its Applications, vol. 55, Kluwer, Dordrecht, 1990.

[15] I. G. Petrovskii, On Cauchy's problem for systems of partial differential equations in the domain of non-analytic functions; On Cauchy's problem for a system of partial differential equations, *in* "Selected Works. Part I: Systems of Partial Differential Equations and Algebraic Geometry," Gordon & Breach, Amsterdam, 1996.

[16] M. Reed and B. Simon, "Methods of Modern Mathematical Physics I. Functional Analysis," Academic Press, New York, 1980.

[17] A. R. Shirikyan and L. R. Volevich, Bounded and almost periodic solutions of linear high-order hyperbolic equations, *Math. Nachr.* **193** (1998), 137–197.

[18] A. R. Shirikyan and L. R. Volevich, Exponential dichotomy and time-bounded solutions for first-order hyperbolic systems, *J. Dynam. Differential Equations* **14** (2002), no. 4, 777–827.

[19] L. R. Volevich and A. R. Shirikyan, Quasilinear hyperbolic equations: solutions bounded in time and almost periodic in time, *Russian J. Math. Phys.* **4** (1996), no. 4, 527–538.

[20] L. R. Volevich and A. R. Shirikyan, Bounded and almost periodic in time solutions to nonlinear high-order hyperbolic equations, *Trans. Moscow Math. Soc.* **58** (1997), 89–135.

[21] L. R. Volevich and A. R. Shirikyan, Exponential dichotomy and exponential splitting for hyperbolic equations, *Trans. Moscow Math. Soc.* **59** (1998), 95–133.

[22] L. R. Volevich and A. R. Shirikyan, Some problems for strictly hyperbolic equations on the entire time-axis, Appendix in the Russian edition of the book: L. R. Volevich, S. G. Gindikin, Mixed Problems for Partial Differential Equations with Quasihomogeneous Principal Part, URSS, Moscow, 1999, 227–266.

[23] L. R. Volevich and A. R. Shirikyan, Local dynamics for semilinear hyperbolic equations of high order, *Izv. Math.* **64** (2000), no. 3, 439–485.

Received July 2001; revised May 2002.

E-mail address: Armen.Shirikyan@math.u-psud.fr, Volevich@spp.keldysh.ru