Controllability of three-dimensional Navier–Stokes equations and applications

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Abstract

We formulate two results on controllability properties of the 3D Navier–Stokes (NS) system. They concern the approximate controllability and exact controllability in finite-dimensional projections of the problem in question. As a consequence, we obtain the existence of a strong solution of the Cauchy problem for the 3D NS system with an arbitrary initial function and a large class of right-hand sides. We also discuss some qualitative properties of admissible weak solutions for randomly forced NS equations.

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1 Main results

Let $D \subset \mathbb{R}^3$ be a bounded domain with $C^2$-smooth boundary $\partial D$. Consider 3D Navier–Stokes (NS) equations

$$
\dot{u} + (u, \nabla)u - \nu \Delta u + \nabla p = f(t, x), \quad \text{div } u = 0, \quad x \in D,
$$

(1)

where $u = (u_1, u_2, u_3)$ and $p$ are unknown velocity and pressure fields, $\nu > 0$ is the viscosity, and $f(t, x)$ is an external force. We introduce the spaces

$$
H = \{ u \in L^2(D, \mathbb{R}^3) : \text{div } u = 0 \text{ in } D, \langle u, n \rangle|_{\partial D} = 0 \},
$$

$$
V = H^1_0(D, \mathbb{R}^3) \cap H, \quad U = H^2(D, \mathbb{R}^3) \cap V,
$$

where $n$ stands for the outward unit normal to $\partial D$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^3$. It is well known (e.g., see [Tem79]) that $H$ is a closed vector...
space in $L^2(D, \mathbb{R}^3)$, and we denote by $\Pi$ the orthogonal projection in $L^2(D, \mathbb{R}^3)$ onto $H$. Equations (1) are equivalent to the following evolution equation in $H$:

$$\dot{u} + \nu Lu + B(u) = f.$$  

(2)

Here $L = -\Pi \Delta$, $B(u) = B(u, u)$, $B(u, v) = \Pi\{(u, \nabla)v\}$, and we use the same notation for the right-hand side of (1) and its projection to $H$. Equation (2) is supplemented with the initial condition

$$u(0) = u_0,$$  

(3)

where $u_0 \in V$. Let us assume that the right-hand side of (2) is represented in the form

$$f(t, x) = h(t, x) + \eta(t, x),$$  

(4)

where $h \in L^2_{\text{loc}}(\mathbb{R}_+, H)$ is a given function and $\eta$ is a control taking on values in a finite-dimensional subspace. To formulate the main results, we introduce some notation.

Define the space $X_T = C(J_T, V) \cap L^2(J_T, U)$, where $J_T = [0, T]$. For any $T > 0$, $h \in L^2(J_T, H)$, and $u_0 \in V$, we denote by $\Theta_T(h, u_0)$ the set of functions $\eta \in L^2(J_T, H)$ for which problem (2) – (4) has a unique solution $u \in X_T$. It follows from the implicit function theorem that

$$D_T := \{(u_0, \eta) \in V \times L^2(J_T, H) : \eta \in \Theta_T(h, u_0)\}$$  

(5)

is an open subset of $V \times L^2(J_T, H)$, and the operator $\mathcal{R}$ taking $(u_0, \eta) \in D_T$ to the solution $u \in X_T$ of (2) – (4) is locally Lipschitz continuous. We denote by $\mathcal{R}_t$ the restriction of $\mathcal{R}$ to the time $t \in J_T$. Let $E \subset U$ and $F \subset H$ be finite-dimensional subspaces, let $P_F : H \to H$ be the orthogonal projection onto $F$, and let $X \subset L^2(J_T, E)$ be a vector space, not necessarily closed. We denote by $B_F(R)$ the closed ball in $F$ of radius $R$ centred at origin.

**Definition 1.** Equations (2), (4) with $\eta \in X$ are said to be *approximately controllable in time $T$* if for any $u_0, \tilde{u} \in V$ and any $\varepsilon > 0$ there is a control $\eta \in \Theta_T(h, u_0) \cap X$ such that

$$\|\mathcal{R}_T(u_0, \eta) - \tilde{u}\|_V < \varepsilon.$$  

(6)

Equations (2), (4) with $\eta \in X$ are said to be *$F$-controllable in time $T$* if for any $u_0 \in V$ and $\tilde{u} \in F$ there is $\eta \in \Theta_T(h, u_0) \cap X$ such that

$$P_F \mathcal{R}_T(u_0, \eta) = \tilde{u}.$$  

(7)

Equations (2), (4) with $\eta \in X$ are said to be *solidly $F$-controllable in time $T$* if for any $u_0 \in V$ and any $R > 0$ there is a constant $\delta > 0$ and a compact set $C$ in a finite-dimensional subspace $Y \subset X$ such that $C \subset \Theta_T(h, u_0)$, and for any continuous mapping $\Phi : C \to F$ satisfying the inequality

$$\sup_{\eta \in C} \|\Phi(\eta) - P_F \mathcal{R}_T(u_0, \eta)\|_F \leq \delta,$$  

(8)

we have $\Phi(C) \supset B_F(R)$. 

2
For any finite-dimensional subspace $G \subset U$, we denote by $F(G)$ the largest vector space $G_1 \subset U$ such that any element $\eta_1 \in G_1$ is representable in the form

$$\eta_1 = \eta - \sum_{j=1}^{k} \lambda_j B(\zeta_j),$$

where $\eta, \zeta_1, \ldots, \zeta_k \in G$ are some vectors and $\lambda_1, \ldots, \lambda_k$ are non-negative constants. Since $B$ is a quadratic operator continuous from $U$ to $V$, we see that $F(G) \subset U$ is a well-defined vector space of finite dimension. Also note that $F(G) \supset G$.

We now define a sequence of subspaces $E_k \subset U$ by the rule

$$E_0 = E, \quad E_k = F(E_{k-1}) \quad \text{for} \quad k \geq 1, \quad E_\infty = \bigcup_{k=1}^{\infty} E_k. \quad (9)$$

The following theorem established in [Shi06a, Shi06b].

**Theorem 2.** Let $E \subset U$ be a finite-dimensional subspace such that $E_\infty$ is dense in $H$. Then the following assertions take place for any $T > 0, \nu > 0$, and $h \in L^2(J_T, H)$.

(i) Equations (2), (4) with $\eta \in C^\infty(J_T, E)$ are approximately controllable in time $T$.

(ii) Equations (2), (4) with $\eta \in C^\infty(J_T, E)$ are solidly $F$-controllable in time $T$ for any finite-dimensional subspace $F \subset H$.

In the general case, it is difficult to verify whether a subspace $E \subset U$ satisfies the conditions of Theorem 2. However, if $D$ is a torus in $\mathbb{R}^3$, then one can obtain a sufficient condition under which $E_\infty$ is dense in $H$.

## 2 Case of a torus

In this subsection, we study controlled Navier–Stokes equations with periodic boundary conditions. More precisely, let us fix a vector $q = (q_1, q_2, q_3)$ with positive components and set $T_q = \mathbb{R}^3/2\pi Z^3_q$, where

$$Z^3_q = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_i/q_i \in \mathbb{Z} \text{ for } i = 1, 2, 3 \}.$$ 

Consider the Navier–Stokes system on $T_q^3$. In other words, we consider Eqs. (1) with $D = \mathbb{R}^3$ and assume that all functions are periodic of period $2\pi q_i$ with respect to $x_i$, $i = 1, 2, 3$. To simplify notation, we shall assume, without loss of generality, that the mean values of $u$, $h$, and $\eta$ with respect to $x \in T_q^3$ are zero. As in the case of a bounded domain with Dirichlet boundary condition, one can reduce (1) to an evolution equation in an appropriate Hilbert space. Namely, we set

$$H = \left\{ u \in L^2(T_q^3, \mathbb{R}^3) : \text{div} \ u \equiv 0, \int_{T_q^3} u(x) \, dx = 0 \right\}$$
and denote by $\Pi : L^2(T^3_q, \mathbb{R}^3) \rightarrow H$ the orthogonal projection in $L^2(T^3_q, \mathbb{R}^3)$ onto the closed subspace $H$. Define the spaces

$$V = H^1(T^3_q, \mathbb{R}^3) \cap H, \quad U = H^2(T^3_q, \mathbb{R}^3) \cap H.$$ 

Projecting (1) to the space $H$, we obtain Eq. (2) in which $L = -\Delta$ is the Stokes operator with the domain $D(L) = U$ and $B(u) = \Pi\{(u, \nabla)u\}$. Theorem 2, which was formulated for the Dirichlet boundary condition, remains valid in this case as well. Our aim is to describe explicitly a finite-dimensional subspace $E \subset U$ for which the hypothesis of Theorem 2 is fulfilled.

To this end, we first construct an orthogonal basis in $H$ formed of the eigenfunctions of $L$. For $x, y \in \mathbb{R}^3$, let

$$\langle x, y \rangle_q = 3 \sum_{i=1}^{3} q_i^{-1} x_i y_i, \quad \langle x, y \rangle = \sum_{i=1}^{3} x_i y_i, \quad |x| = \sum_{i=1}^{3} |x_i|.$$ 

We set $\mathbb{Z}^3_* = \mathbb{Z}^3 \setminus \{0\}$ and $\mathbb{R}^3_* = \mathbb{R}^3 \setminus \{0\}$. For $a \in \mathbb{R}^3_*$, denote by $a^\perp$ the two-dimensional subspace in $\mathbb{R}^3$ defined by the equation $\langle x, a \rangle_q = 0$. Note that $a^\perp = (-a)^\perp$. For any $m \in \mathbb{Z}^3_*$, let us choose a vector $\ell(m) \in m^\perp$ so that $\{\ell(m), \ell(-m)\}$ is an orthonormal basis in $m^\perp$ with respect to the scalar product $\langle \cdot, \cdot \rangle$. We now set

$$c_m(x) = \ell(m) \cos(m, x)_q, \quad s_m(x) = \ell(m) \sin(m, x)_q \quad \text{for } m \in \mathbb{Z}^3_*.$$ 

It is a matter of direct verification to show that $c_m$ and $s_m$ are eigenfunctions of $L$ and that $\{c_m, s_m, m \in \mathbb{Z}^3_*\}$ is an orthogonal basis in $H$. For a finite family of functions $\mathcal{A}$, we denote by span$\mathcal{A}$ the vector space spanned by $\mathcal{A}$.

**Theorem 3.** For any vector $q = (q_1, q_2, q_3)$ with positive components there is an integer $d \geq 4$ such that if

$$E = \text{span}\{c_m, s_m, |m| \leq d\},$$ 

then the vector space $E_\infty$ defined in (9) is dense in $H$.

Theorems 2 and 3 imply the following result on controllability of the NS system by a force of finite dimension.

**Corollary 4.** Let $E \subset U$ be the subspace defined in Theorem 3. Then for any finite-dimensional subspace $F \subset H$ and arbitrary constants $T > 0$ and $\nu > 0$ the Navier–Stokes equations (2), (4) with $\eta \in C^\infty(J_T, E)$ are approximately controllable and solidly $F$-controllable in time $T$.

The proofs of the above results are based on a development of a general approach introduced by Agrachev and Sarychev in the case of 2D Navier–Stokes equations (see [AS05, AS06]).
3 Applications

Our first application concerns the Cauchy problem for (2). Let \( G \subset H \) be a closed vector space. For any \( u_0 \in V, T > 0, \) and \( \nu > 0, \) let \( \Xi_{T,\nu}(G, u_0) \) be the set of functions \( f \in L^2(J_T, G) \) for which problem (2), (3) has a unique solution \( u \in X_T. \) If \( E \subset G \) is a closed subspace, then we denote by \( G \ominus E \) the orthogonal complement of \( E \) in \( G \) and by \( Q(T, G, E) \) the orthogonal projection in \( L^2(J_T, G) \) onto the subspace \( L^2(J_T, G \ominus E). \) The following result is established in [Shi06a].

**Theorem 5.** Let \( E \subset U \) be a finite-dimensional subspace such that \( E_\infty \) is dense in \( H \) and let \( G \subset H \) be a closed subspace containing \( E. \) Then \( \Xi_{T,\nu}(G, u_0) \) is a non-empty open subset of \( L^2(J_T, G) \) such that

\[
Q(T, G, E)\Xi_{T,\nu}(G, u_0) = L^2(J_T, G \ominus E) \quad \text{for any} \ T > 0, \nu > 0, u_0 \in V.
\]

Our second application concerns the case in which Navier–Stokes equations are perturbed by a random force. Namely, suppose that

\[
f(t, x) = h(x) + \eta(t, x), \quad (10)
\]

where \( h \in H \) is a deterministic function and \( \eta \) is an \( H \)-valued random process satisfying the following condition.

**(C)** There is an orthonormal basis \( \{f_k\} \) in \( V \) and a sequence of standard independent Brownian motions \( \{\beta_j(t), t \geq 0\} \) defined on a filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \) such that

\[
\eta(t) = \frac{\partial}{\partial t} \zeta(t), \quad \zeta(t) = \sum_{j,k=1}^{\infty} b_{jk} \beta_j(t)f_k,
\]

where \( \{b_{jk}\} \) is a family of real constants satisfying the condition

\[
B := \sum_{j,k=1}^{\infty} b_{jk}^2 < \infty.
\]

Let us recall the concepts of an admissible weak solution and of a stationary measure for (2), (10). Define an Ornstein–Uhlenbeck process by the formula

\[
z(t) = \int_0^t e^{-\nu(t-s)L} d\zeta(t).
\]

It is well known that if Condition (C) is fulfilled, then \( z \) is a Gaussian process whose almost every trajectory belongs to the space \( C(\mathbb{R}_+, V) \cap L^2_{\text{loc}}(\mathbb{R}_+, U) \) and satisfies the Stokes equation

\[
\dot{u} + \nu Lu = \eta(t).
\]
Definition 6. An $H$-valued random process $u(t)$ is called an admissible weak solution for (2), (10) if it is representable in the form

$$u(t) = v(t) + z(t),$$

where $v(t)$ is an $H$-valued $\mathcal{F}_t$-adapted random process whose almost every trajectory belongs to the space $L^2_{\text{loc}}(\mathbb{R}_+, V) \cap L^\infty_{\text{loc}}(\mathbb{R}_+, H)$ and satisfies the equation

$$\dot{v} + \nu L v + B(v + z) = h$$
in the sense of distributions and the energy inequality

$$\frac{1}{2} \|v(t)\|^2 + \nu \int_0^t \|v(s)\|_V^2 ds + \int_0^t (B(v + z, v), v) ds \leq \frac{1}{2} \|v(0)\|^2 + \int_0^t (h, v) ds, \quad t \geq 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $H$.

Definition 7. An admissible weak solution $u(t)$ for (2), (10) is said to be stationary if its distribution does not depend on $t$:

$$\mathcal{D}(u(t)) = \mu \quad \text{for all} \quad t \geq 0.$$In this case, $\mu$ is called a stationary measure for (2), (10).

Existence of admissible weak stationary solutions for 3D Navier–Stokes equations was established in [VF88, FG95]. Moreover, the construction of these works implies that

$$\int_H \|v\|_V^2 \mu(dv) < \infty. \quad (11)$$

Let us denote by $Q$ the vector space of functions $v \in V$ that are representable in the form

$$v = \sum_{j,k=1}^{\infty} b_{jk} u_j f_k,$$

where $\{u_j\}$ is a sequence of real numbers such that $\sum_j u_j^2 < \infty$. Recall that the vector space $E_\infty$ is defined in (9). For a finite-dimensional space $F$, denote by $\ell_F$ the Lebesgue measure on $F$. The following theorem established in [Shi06c] provides some qualitative properties of stationary measures for (2), (10) (see also [AKSS06]).

Theorem 8. Let $\eta$ be a stationary process satisfying Condition (C), let $E \subset U$ be a finite-dimensional vector space for which $E_\infty$ is dense in $H$, and let $\mu$ be a stationary measure for (2), (10) such that (11) holds. Suppose that $Q \supset E$. Then the following assertions take place.

(i) The support of $\mu$ coincides with $H$.

(ii) Let $F \subset H$ be a finite-dimensional subspace and let $\mu_F$ be the projection of $\mu$ to $F$. Then there is a function $\rho_F \in C(F)$ such that $\mu_F \geq \rho_F \ell_F$ and $\rho_F(x) > 0$ for $\ell_F$-almost every $x \in F$. 


References


[FG95] F. Flandoli and D. Ga


