On dissipative systems perturbed by bounded random kick-forces

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Dedicated to the memory of Jürgen Moser

Abstract

In this paper, we continue our investigation of dissipative PDE's forced by random bounded kick-forces and of the corresponding random dynamical system (RDS) in function spaces. It was proved in [KS] that the domain of attainability from zero $\mathcal{A}$ (which is a compact subset of a function space) is invariant for the RDS associated with the original equation and carries a stationary measure μ, which is unique among all measures supported by $\mathcal{A}$. Here we show that μ is the unique stationary measure for the RDS in the whole space and study its ergodic properties.

1 Main result

We first recall the main result of the paper [KS]. Let H be a separable Hilbert space with norm $\|\cdot\|$ and orthonormal basis $\{e_1, e_2, \dots\}$ and let S be a locally Lipschitz transformation of H satisfying the condition $S(0) = 0$ and the three hypotheses below (see Remark 2.3 in [KS] for a discussion of their meaning).

(A) *For any $R > r > 0$ there exist positive constants $a = a(R, r) < 1$ and $C = C(R)$ and an integer $n_0 = n_0(R, r) \geq 1$ such that*

$$\|S(u_1) - S(u_2)\| \leq C(R)\|u_1 - u_2\| \quad \textit{for all} \quad u_1, u_2 \in B_H(R), \tag{1.1}$$

$$\|S^n(u)\| \leq \max\{a\|u\|, r\} \quad \textit{for} \quad u \in B_H(R), \quad n \geq n_0, \tag{1.2}$$

where $B_H(R)$ denotes the ball in H of radius R centred at zero.

The next condition expresses the property of the existence of a bounded absorbing set for the system in question. To formulate it, we introduce some notations.

For any $R > 0$ and a compact set $K \subset H$, we define a sequence of sets $\mathcal{A}_k(R, K) \subset H$ by the rule

$$\mathcal{A}_0(R, K) = B_H(R), \quad \mathcal{A}_k(R, K) = S\big(\mathcal{A}_{k-1}(R, K)\big) + K, \quad k \geq 1. \tag{1.3}$$

Let us set

$$\mathcal{A}(R, K) = \overline{\bigcup_{k=0}^{\infty} \mathcal{A}_k(R, K)}. \tag{1.4}$$

(B′) *For any* $R>0$ *and any compact set* $K\subset H$, *the set* $\mathcal{A}(R,K)\subset H$ *is bounded. Moreover, for any compact set* K, *there is a* $\rho>0$ *such that* $\mathcal{A}_k(R,K)\subset B_H(\rho)$ *for* $k\ge k_0$, *where* k_0 *is an integer depending on* R *and* K.

We note that, in [KS], it was required that $\mathcal{A}(R,K)$ is bounded only for $R=0$, and this is why we denoted this condition here by (B′) rather than (B).

Clearly, inequality (1.2) and condition (B′) hold true if we assume the following stronger hypothesis, which is satisfied for the Navier-Stokes system (see Example 1.1 below), but not for the Schrödinger equation (Example 1.2):

(A′) There is a positive constant $\gamma<1$ such that $\|S(u)\|\le\gamma\|u\|$ for all $u\in H$.

For a subspace $E\subset H$, we denote by $E^\perp$ its orthogonal complement in H. For an integer $N\ge 1$, let H_N be the finite-dimensional subspace generated by the vectors $e_1,\dots,e_N$ and let P_N and Q_N be the orthogonal projections onto H_N and $H_N^\perp$, respectively.

(C) *For any* $R>0$ *there is a decreasing sequence* $\gamma_N(R)>0$ *tending to zero as* $N\to\infty$ *such that*

$$\big\|\mathsf{Q}_N\big(S(u_1)-S(u_2)\big)\big\|\le\gamma_N(R)\|u_1-u_2\|\quad \textit{for all}\quad u_1,u_2\in B_H(R). \tag{1.5}$$

We consider the following random dynamical system in H:

$$u_k=S(u_{k-1})+\eta_k,\quad k\ge 1. \tag{1.6}$$

Here η_k, $k\ge 1$, is a sequence of independent H-valued random variables of the form

$$\eta_k=\sum_{j=1}^\infty b_j\xi_{jk}e_j, \tag{1.7}$$

where $b_j\ge 0$ are some constants such that

$$\sum_{j=1}^\infty b_j^2<\infty, \tag{1.8}$$

and $\{\xi_{jk}\}$ is a family of independent real-valued random variables satisfying the following condition:

(D) *For any* j, *the random variables* ξ_{jk}, $k\ge 1$, *have the same distribution* π_j *such that* $\pi_j(dr)=p_j(r)\,dr$, *where the densities* $p_j(r)$ *are Lipschitz continuous and, moreover,* $p_j(0)>0$ *and* $\operatorname{supp}p_j\subset[-1,1]$.

Equation (1.6) defines a discrete-time random dynamical system in H. Let $\mathcal{B}(H)$ be the σ-algebra of Borel subsets of H, let $\mathcal{P}(H)$ be the set of probability measures on $\big(H,\mathcal{B}(H)\big)$, and let $C_b(H)$ be the space of bounded continuous functions on H. We denote by $P(k,v,\Gamma)$, where $k \geq 0$, $v \in H$, and $\Gamma \in \mathcal{B}(H)$, the corresponding transition function:

$$P(k,v,\Gamma) = \mathbb{P}\{u_k \in \Gamma\},$$

where $(u_k, k \geq 0)$ is a solution of (1.6) such that $u_0 = v$. Let

$$P_k \colon C_b(H) \to C_b(H), \quad P_k^* \colon \mathcal{P}(H) \to \mathcal{P}(H)$$

be the Markov semigroups associated with $P(k,v,\Gamma)$.

Random dynamical systems of the form (1.6) naturally arise in the theory of dissipative PDE's forced by random kick-forces (see [KS]):

Example 1.1. Let us consider the 2D Navier-Stokes (NS) equations

$$\dot{u} - \nu\Delta u + (u,\nabla)u + \nabla p = \eta(t,x) \equiv \sum_{k=1}^{\infty} \eta_k(x)\delta(t-k), \quad \operatorname{div} u = 0. \tag{1.9}$$

The system (1.9) is supplemented with either periodic or Dirichlet boundary conditions. In the first case, we also assume that $\langle u(t,x)\rangle = \langle \eta(t,x)\rangle = 0$ for $t \geq 0$, where we set $\langle v\rangle := \int_{\mathbb{T}^2} v(x)\,dx$, and $\mathbb{T}^2$ denotes the 2D torus. To be precise, we confine ourselves to the case of periodic boundary conditions.

Let H be the space of divergence-free vector functions $u \in L^2(\mathbb{T}^2, \mathbb{R}^2)$ such that $\langle u(x)\rangle = 0$, and $\{e_j\}$ be the trigonometric basis in H. Assuming that that the kicks η_k have the form (1.7) and normalising the solution $u(t)$ by the condition of continuity from the right, we observe that (1.9) can be written in the form (1.6), where $u_k = u(k)$ and $S \colon H \to H$ is the time-one shift along trajectories of the free NS system. As is shown in [KS, Section 7], the operator S satisfies conditions (A), (B′), (C) (as well as (A′)).

Example 1.2. The theory developed in [KS] applies also to the Schrödinger equation

$$\dot{u} - \nu(\Delta - 1)u + i|u|^2 u = \sum_{k=1}^{\infty} \eta_k(x)\delta(t-k), \quad x \in \mathbb{T}^n. \tag{1.10}$$

In this case, the corresponding random dynamical system is defined in the Sobolev space $H^s(\mathbb{T}^2, \mathbb{R}^2)$ with $s > n/2$.

We now introduce the compact set

$$K = \left\{ u = \sum_{j=1}^{\infty} x_j e_j : x_j \in \operatorname{supp} p_j \text{ for all } j \geq 1 \right\}, \tag{1.11}$$

and note that the distribution $\mathcal{D}(\eta_k)$ of the random variable η_k does not depend on k and is supported by K. If we define the sets $\mathcal{A}_k$ and $\mathcal{A}$ by relations (1.3) and (1.4) in which $R = 0$ and K is given by (1.11), then according to condition (B′), the set $\mathcal{A}$ is bounded in H. Moreover, it follows from (C) that $\mathcal{A}$ is compact in H. We also note that

(i) $\mathcal{A}_l$ is the support of the measure $\mathcal{D}(u_l)$, where $(u_k, k \geq 0)$ is the solution of (1.6) with $u_0 = 0$;

(ii) $\mathcal{A}$ is invariant for (1.6), that is, if $v \in \mathcal{A}$ and $(u_k, k \geq l)$ is the solution of (1.6) with $u_l = v$, then $u_k \in \mathcal{A}$ almost surely for all $k \geq l$.

Thus, we can regard (1.6) as a random dynamical system in $\mathcal{A}$. We shall use the same notation for the corresponding transition function and the Markov semigroups.

Let us recall that $\mu \in \mathcal{P}(H)$ is called a *stationary measure* for a Markov chain if $P_k^* \mu = \mu$ for $k \geq 1$. The main result of [KS] is the following theorem.

Theorem 1.3. *Suppose that conditions* (A) – (D) *are satisfied. There is an integer $N_0 \geq 1$ such that if*

$$b_j \neq 0 \quad \textit{for} \quad 1 \leq j \leq N_0, \tag{1.12}$$

then the restriction of the Markov chain (1.6) *to $\mathcal{A}$ has a unique stationary measure μ. Moreover, for any continuous function $f \in C(H)$,*

$$P_k f(u) \to (\mu, f) := \int_H f(u)\mu(du) \quad \textit{as} \quad k \to \infty, \tag{1.13}$$

uniformly in $u \in \mathcal{A}$.

In some rigorous sense explained in [KS], the measure μ is equivalent to an abstract 1D Gibbs measure. The aim of this paper is to prove the following result concerning the Markov chain (1.6) in the whole space H.

Theorem 1.4. *Under the conditions of Theorem 1.3, there exists an integer $N_1 \geq N_0$ such that if* (1.12) *is satisfied with N_0 replaced by N_1, then* (1.13) *holds uniformly in $u \in B_H(R)$ for any $f \in C(H)$ and $R > 0$. In particular, μ is the unique stationary measure for the Markov chain* (1.6) *in H.*

This theorem readily implies that if μ_0 is an arbitrary initial distribution, then $P_k^* \mu_0 \rightharpoonup \mu$ as $k \to \infty$.

In terms of the unique stationary measure $\mu = \mu_\nu$ for Equation (1.9) (or Equation (1.10)), the *turbulence problem* for this equation can be stated as understanding of the limiting properties of the measure as $\nu \to 0$. For some non-trivial properties of a stationary measure for Equation (1.10) with $\nu \ll 1$, see [K].

After the work [KS], E, Mattingly, Sinai [EMS] and Bricmont, Kupiainen, Lefevere [BKL] considered the 2D Navier–Stokes equations perturbed by a white noise force

$$\eta(t, x) = \sum_{j=1}^{\infty} b_j \dot{w}_j(t) e_j(x),$$

where $w_1, w_2, \ldots$ are independent standard Brownian motions. Under the assumptions that $b_j \neq 0$ for $1 \leq j \leq N$ and $b_j = 0$ for $j \geq N'$ with sufficiently large integers $N' > N$, they obtained uniqueness of a stationary measure μ. Moreover, it is shown in [BKL] that for any Lipschitz function f the expression $P_k f(u)$ converges exponentially fast to (μ, f) for μ-almost all initial conditions $u \in H$.

We note that the main difficulty in proving the uniqueness of a stationary measure is related to the fact that the force $\eta(t,x)$ can be smooth in x. The non-smooth case was treated earlier in [FM].

2 Proof of the main result

To prove Theorem 1.4, we first establish an auxiliary assertion, which is of independent interest, and then show that it implies convergence (1.13) and uniqueness of a stationary measure for the Markov chain in H.

2.1 Markov semigroups with a squeezing property

Let X be a complete separable metric space with the Borel σ-algebra $\mathcal{B}(X)$ and let $P(k,u,\Gamma)$ be a Feller transition function (see [Kr, § V.5] for the definition). We shall denote by P_k and P_k^* the corresponding Markov semigroups. We recall that a Borel set $\mathcal{A} \subset X$ is said to be *invariant* for $P(k,u,\Gamma)$ if $P(1,u,\mathcal{A}) = 1$ for $u \in \mathcal{A}$.

Proposition 2.1. *Suppose that there exists a decreasing sequence of Borel subsets of X, $X_1 \supset X_2 \supset \cdots$, and an integer $M \geq 1$ such that*

$$\inf_{u \in X} P(M,u,X_1) \geq \varepsilon, \tag{2.1}$$

$$\inf_{u \in X_k} P(1,u,X_{k+1}) \geq 1 - \varepsilon_k \quad \text{for all} \quad k \geq 1, \tag{2.2}$$

where ε_k and ε are positive constants such that $\varepsilon_k < 1$ and $\sum_{k=1}^{\infty} \varepsilon_k < \infty$. Then

$$\sup_{u \in X} P(k,u,X \setminus X_n) \leq \zeta_n/\delta + (1-\delta)^{m_k - 1}, \tag{2.3}$$

where $k \geq M + n - 1$, m_k is the largest integer not exceeding $k/(M+n-1)$, and

$$\delta = \varepsilon \prod_{j=1}^{\infty} (1 - \varepsilon_j) > 0, \quad \zeta_n = 1 - \prod_{j=n}^{M+2(n-1)} (1 - \varepsilon_j).$$

Moreover, the set $\mathcal{A} = \cap_{j \geq 1} X_j$ is invariant, and any stationary measure μ for P_k^* is concentrated on it.

We note that $\zeta_n \to 0$ as $n \to \infty$. This follows immediately from the convergence of the series $\sum \varepsilon_k$.

Proof of Proposition 2.1. 1) We first show that

$$\inf_{u\in X_n} P(l,u,X_{n+l}) \geq \prod_{j=n}^{l+n-1}(1-\varepsilon_j) \quad \text{for} \quad n,l\geq 1, \tag{2.4}$$

$$\inf_{u\in X} P(M+i-1,u,X_i) \geq \varepsilon \prod_{j=1}^{i-1}(1-\varepsilon_j) \geq \delta > 0 \quad \text{for} \quad i\geq 1. \tag{2.5}$$

To this end, note that, in view of inequality (2.2) with $k=n+l-1$ and the Chapman-Kolmogorov (CK) relation [Fe, § X.1], we have

$$\begin{aligned} P(l,u,X_{n+l}) &\geq \int_{X_{n+l-1}} P(l-1,u,dv)P(1,v,X_{n+l}) \\ &\geq P(l-1,u,X_{n+l-1}) \inf_{v\in X_{n+l-1}} P(1,v,X_{n+l}) \\ &\geq (1-\varepsilon_{n+l-1})P(l-1,u,X_{n+l-1}). \end{aligned}$$

Taking the infimum over $u\in X_n$ and arguing by induction, we obtain (2.4).

Inequality (2.5) follows from (2.1), (2.4) (with $n=1$ and $l=i-1$), and the CK relation. The fact that $\delta>0$ is a simple consequence of the convergence of the series $\sum \varepsilon_k$.

2) We can now prove (2.3). Let us note that, in view of the CK relation and the invariance of $\mathcal{A}$, we have

$$\begin{aligned} \sup_{u\in\mathcal{A}} P(l,u,Y) &= \sup_{u\in\mathcal{A}} \int_{\mathcal{A}} P(l-k,u,dv)P(k,v,Y) \\ &\leq \sup_{u\in\mathcal{A}} P(k,u,Y) \quad \text{for} \quad k\leq l, \quad Y\in\mathcal{B}(X). \end{aligned}$$

Therefore, it suffices to establish (2.3) for $k=mk_n$, where $k_n=M+n-1$ and $m\geq 1$ is an arbitrary integer.

To this end, we fix a point $u\in X$ and arbitrary positive integers m and n and estimate the expression

$$\begin{aligned} p_n^{(m)}(u) := P(mk_n,u,X\setminus X_n) &= \int_X P\big((m-1)k_n,u,dv\big)P(k_n,v,X\setminus X_n) \\ &= \int_{X_n} + \int_{X\setminus X_n}. \end{aligned} \tag{2.6}$$

In view of inequalities (2.4) and (2.5) with $l=k_n$ and $i=n$, we have

$$\begin{aligned} &\sup_{v\in X_n} P(k_n,v,X\setminus X_n) \leq \sup_{v\in X_n} P(k_n,v,X\setminus X_{n+k_n}) \leq \zeta_n, \\ &\sup_{v\in X} P(k_n,v,X\setminus X_n) \leq 1-\delta. \end{aligned}$$

Substituting these estimates into (2.6), we obtain

$$p_n^{(m)}(u) \leq \zeta_n + (1-\delta)p_n^{(m-1)}(u).$$

Iterating this inequality and using the trivial estimate $p_n^{(0)}(u) \le 1$, we derive

$$p_n^{(m)}(u) \le \zeta_n/\delta + (1-\delta)^{m-1},$$

which completes the proof of (2.3).

3) The fact that $\mathcal{A}$ is an invariant set follows from (2.2):

$$P(1,u,\mathcal{A}) = \lim_{k\to\infty} P(1,u,X_k) \ge \lim_{k\to\infty}(1-\varepsilon_k) = 1, \quad u \in \mathcal{A}.$$

Furthermore, if $P_1^*\mu = \mu$, then, by (2.3), for any $m \ge 1$ we have

$$\begin{aligned}\mu(\mathcal{A}) &= \lim_{n\to\infty}\mu(X_n) = \lim_{n\to\infty}\int_X P(mk_n,u,X_n)\mu(du)\\ &\ge \lim_{n\to\infty}\bigl(1-\zeta_n/\delta-(1-\delta)^{m-1}\bigr) = 1-(1-\delta)^{m-1}, \end{aligned} \tag{2.7}$$

where we used the fact that $\zeta_n \to 0$ as $n \to \infty$. It follows from (2.7) that $\mu(\mathcal{A}) = 1$. ☐

2.2 Proof of Theorem 1.4

1) The fact that convergence (1.13) implies the uniqueness of a stationary measure is well-known. Thus, it suffices to show that (1.13) holds uniformly with respect to $u \in B_H(R)$. To this end, we apply Proposition 2.1. Let us fix a small constant $d > 0$ (which will be chosen later) and set[1]

$$X = S\bigl(\mathcal{A}(\rho,K)\bigr) + K, \quad X_k = \bigl\{u \in X : \operatorname{dist}(u,\mathcal{A}) \le 2^{-k}d\bigr\}, \tag{2.8}$$

where ρ is the constant defined in condition (B′), K is the support of $\mathcal{D}(\eta_k)$ (see (1.11)), and $\mathcal{A}$ is the domain of attainability from zero.

It is clear that X is a compact set in H and $\cap_{k\ge1}X_k = \mathcal{A}$. If we show that the conditions of Proposition 2.1 are satisfied, then the required assertion will follow from Theorem 1.3. Indeed, suppose that the assertion of Proposition 2.1 holds and let $f \in C(H)$. Condition (B′) implies that

$$\inf_{\|u\|\le R} P(k,u,X) = 1 \quad \text{for} \quad k \ge k_0 + 1.$$

Therefore, to prove the theorem, it suffices to show that (1.13) holds uniformly in $u \in X$.

Without loss of generality, it can be assumed that $(\mu, f) = 0$. Let $\omega(s)$ be the modulus of continuity for the restriction of f to X. Then

$$\bigl|f(u_1) - f(u_2)\bigr| \le \omega(s) \quad \text{if} \quad u_1, u_2 \in X, \quad \|u_1 - u_2\| \le s,$$

[1] We do not take $X = \mathcal{A}(\rho, K)$, since it is more convenient to deal with a compact phase space.

and $\omega(s)$ goes to zero with s. It follows from (1.1) that if v_k and v_k' are two solutions of (1.6) starting from deterministic points $v \in X$ and $v' \in X$, respectively, then $\|v_k - v_k'\| \le L^k \|v - v'\|$, where $L > 0$ does not depend on v, v', and k. Since $P_k f(v) = \mathbb{E} f(v_k)$, we conclude that

$$\bigl|P_k f(v) - P_k f(v')\bigr| \le \omega\bigl(L^k \|v - v'\|\bigr), \quad v, v' \in X. \tag{2.9}$$

We now fix an integer $n \ge 1$ and note that

$$|P_{k+l} f(u)| \le \int_X P(k, u, dv) |P_l f(v)| = \int_{X \setminus X_n} + \int_{X_n}. \tag{2.10}$$

By (2.3), the first integral on the right-hand side of (2.10) does not exceed

$$\bigl\|f|_X\bigr\|_\infty \sup_{u \in X} P(k, u, X \setminus X_n) \le \bigl\|f|_X\bigr\|_\infty \bigl(\zeta_n/\delta + (1-\delta)^{m_k - 1}\bigr).$$

To estimate the second integral, we note that, in view of Theorem 1.3,

$$\sup_{v' \in \mathcal{A}} \bigl|P_l f(v')\bigr| := \beta_l \to 0 \quad \text{as} \quad l \to \infty. \tag{2.11}$$

Let us fix arbitrary $v \in X_n$ and choose $v' \in \mathcal{A}$ such that $\|v - v'\| \le 2^{-n} d$. Using (2.9) and (2.11), we derive

$$\bigl|P_l f(v)\bigr| \le \bigl|P_l f(v')\bigr| + \omega\bigl(L^l \|v - v'\|\bigr) \le \beta_l + \omega(2^{-n} L^l d).$$

Substitution of the above estimates into (2.10) results in

$$\sup_{u \in X} |P_{k+l} f(u)| \le \bigl\|f|_X\bigr\|_\infty \bigl(\zeta_n/\delta + (1-\delta)^{m_k - 1}\bigr) + \beta_l + \omega(2^{-n} L^l d). \tag{2.12}$$

Since $\zeta_n \to 0$ as $n \to \infty$ and $\omega(s) \to 0$ as $s \to 0$, the right-hand side of (2.12) can be made arbitrary small if we fix sufficiently large integers n and l, $n \gg l$, and then require that $k \gg 1$. This completes the proof of (1.13).

Thus, it remains to show that the conditions of Proposition 2.1 are satisfied.

2) To prove (2.1), we take arbitrary $u \in X$ and, for any sequence $\xi_j \in K$, denote by $\theta_k = \theta_k(u; \xi_1, \dots, \xi_k)$ the solution of (1.6) such that $\theta_0 = u$ and $\eta_j = \xi_j$, $j \ge 1$. It follows from (1.2) that if $\xi_j = 0$ for all $j \ge 1$, then there is an integer $M \ge 1$, not depending on $u \in X$, such that $\|\theta_M\| \le d/4$, where d is the constant in the definition of X_k. By continuity, there is $\gamma > 0$ such that if $\|\xi_j\| \le \gamma$ for $1 \le j \le M$, then $\|\theta_M(u; \xi_1, \dots, \xi_M)\| \le d/2$. It follows from (1.8) that the event $\|\eta_k\| \le \gamma$ has a positive k-independent probability. Since the random variables η_k, $k \ge 1$, are independent, we conclude that

$$\mathbb{P}\bigl\{\|\theta_M\| \le d/2\bigr\} \ge \mathbb{P}\bigl\{\|\eta_j\| \le \gamma,\ 1 \le j \le M\bigr\} \ge \varepsilon := \gamma^M.$$

To complete the proof of inequality (2.1), it remains to note that, since $0 \in X_1$, we have $\{\|\theta_M\| \le d/2\} \subset \{\theta_M \in X_1\}$.

3) To prove (2.2), we need the following auxiliary assertion.

Lemma 2.2. *Suppose that* $b_j \neq 0$ *for* $j = 1, \dots, N$. *Then there is a constant* $C > 0$ *such that*

$$\mathbb{P}\bigl\{v + \mathsf{P}_N \eta_1 \in \mathsf{P}_N K\bigr\} \geq 1 - C\|v\| \quad \text{for} \quad v \in H_N. \tag{2.13}$$

Taking this lemma for granted, let us complete the verification of (2.2). By conditions (A) and (C), there is a constant $L > 0$ and an integer $N \geq 1$ such that

$$\bigl\|S(u_1) - S(u_2)\bigr\| \leq L\|u_1 - u_2\|, \tag{2.14}$$

$$\bigl\|\mathsf{Q}_N\bigl(S(u_1) - S(u_2)\bigr)\bigr\| \leq \|u_1 - u_2\|/2, \tag{2.15}$$

where $u_1, u_2 \in X$. We claim that if $b_j \neq 0$ for $1 \leq j \leq N$, then (2.2) holds. Indeed, let $u \in X_k$ for an integer $k \geq 1$. Since $\mathcal{A}$ is a compact subset of H, there is $u' \in \mathcal{A}$ such that $\|u - u'\| \leq 2^{-k}d$. By (2.15),

$$\bigl\|\mathsf{Q}_N\bigl(S(u) - S(u')\bigr)\bigr\| \leq \|u - u'\|/2 \leq 2^{-(k+1)}d. \tag{2.16}$$

Since $S(u') + K \in \mathcal{A}$ and $K = \mathsf{P}_N K \times \mathsf{Q}_N K$, it follows from (2.16) that

$$\begin{aligned}\operatorname{dist}\bigl(S(u) + \eta_1, \mathcal{A}\bigr) &\leq \operatorname{dist}\bigl(S(u) + \eta_1, S(u') + K\bigr)\\ &\leq \operatorname{dist}\bigl(\mathsf{P}_N(S(u) + \eta_1), \mathsf{P}_N(S(u') + K)\bigr) + 2^{-(k+1)}d\end{aligned}$$

and, therefore,

$$\begin{aligned}P(1, u, X_{k+1}) &= \mathbb{P}\bigl\{S(u) + \eta_1 \in X_{k+1}\bigr\}\\ &= \mathbb{P}\bigl\{\operatorname{dist}\bigl(S(u) + \eta_1, \mathcal{A}\bigr) \leq 2^{-(k+1)}d\bigr\}\\ &\geq \mathbb{P}\bigl\{\mathsf{P}_N(S(u) + \eta_1) \in \mathsf{P}_N(S(u') + K)\bigr\}.\end{aligned} \tag{2.17}$$

Applying Lemma 2.2 with $v = \mathsf{P}_N(S(u) - S(u'))$ and taking into account (2.14), we conclude that the right-hand side of (2.17) can be estimated from below by the expression

$$1 - C\bigl\|S(u) - S(u')\bigr\| \geq 1 - CL\|u - u'\| \geq 1 - 2^{-k}dCL.$$

Thus, inequality (2.2) is satisfied with $\varepsilon_k = 2^{-k}dCL$. It is clear that $\varepsilon_k < 1$ if $dCL \leq 1$ and that $\sum \varepsilon_k < \infty$.

To complete the proof of Theorem 1.4, it remains to establish (2.13).

Proof of Lemma 2.2. By condition (D), the random variable $\mathsf{P}_N \eta_1$ has a Lipschitz-continuous density $D(x)$ with respect to the Lebesgue measure on H_N. Therefore,

$$\mathbb{P}\bigl\{v + \mathsf{P}_N \eta_1 \in \mathsf{P}_N K\bigr\} = \int_{\mathsf{P}_N K} D(x + v)\, dx. \tag{2.18}$$

It remains to note that the right-hand side of (2.18) is uniformly Lipschitz-continuous with respect to v and is equal to 1 for $v = 0$. □

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