SOME MATHEMATICAL PROBLEMS OF STATISTICAL HYDRODYNAMICS

ARMEN SHIRIKYAN
Laboratoire de Mathématiques, Université de Paris-Sud XI
Bâtiment 425, 91405 Orsay Cedex, France
E-mail: Armen.Shirikyan@math.u-psud.fr

We give a survey of recent advances in the problem of ergodicity for a class of dissipative PDE's perturbed by an external random force. One of the main results in this direction asserts that, if the external force is sufficiently non-degenerate, then there is a unique stationary measure, which is exponentially mixing. Moreover, a strong law of large numbers and a central limit theorem hold for solutions of the problem in question. The results obtained apply, for instance, to the 2D Navier–Stokes system and to the complex Ginzburg–Landau equation.

1. Introduction

Let us consider the two-dimensional Navier–Stokes system in a bounded domain $D \subset \mathbb{R}^2$ with smooth boundary ∂D:

$$\dot{u} + (u, \nabla)u - \nu\Delta u + \nabla p = \eta(t, x), \quad \operatorname{div} u = 0, \quad x \in D. \tag{1}$$

Here $u = (u_1, u_2)$ is the velocity field of the fluid, $p = p(t, x)$ is the pressure, $\nu > 0$ is the viscosity, and η is an external force. Equation (1) is supplemented with the Dirichlet boundary condition

$$u\Big|_{\partial D} = 0. \tag{2}$$

It is well known that, under some mild regularity assumptions on the right-hand side η, the Cauchy problem for (1), (2) has a unique solution defined on the half-line $t \geq 0$. We shall describe some recent results on asymptotic behaviour of solutions as $t \to +\infty$ in the case when η is a random process.

Let us give a more precise formulation of the problems we are dealing with. Suppose that the right-hand side $\eta(t, x)$ depends on an additional random parameter ω. Thus, it can be considered as a stochastic process valued in a functional space, and we shall assume that η is stationary in time and smooth in x. In this case, solutions of (1), (2) are also random processes, and we can consider their distributions, which are probability measures on the (infinite-dimensional) phase space of the problem. We are interested in the following questions:

- **(i)** Existence of stationary solutions, i.e., solutions whose distribution does not depend on time.
- **(ii)** Uniqueness of distribution of stationary solutions.

(iii) Large-time asymptotics of distributions of solutions.
(iv) Large-time asymptotics of the time average of some observables.

The aim of this paper is to give a survey of some recent results concerning the above questions and to formulate some open problems.

The paper is organised as follows. Section 2 is devoted to the initial value problem for (1), (2). In Sec. 3, we recall the concept of stationary measure and exponential mixing and formulate a fundamental result on stationary measures for problem (1), (2). Section 4 is devoted to the law of large numbers (LLN) and central limit theorem (CLT) for solutions. Finally, in Sec. 5, we describe two open problems. Most of the results of this paper are obtained in collaboration with Sergei Kuksin.

2. Cauchy problem

We fix a bounded domain $D \subset \mathbb{R}^2$ with smooth boundary ∂D and introduce the space

$$H = \big\{u \in L^2(D, \mathbb{R}^2) : \operatorname{div} u = 0, (u, \boldsymbol{n})\big|_{\partial D} = 0\big\},$$

where $\boldsymbol{n}$ is the outward normal to ∂D. Let $\Pi\colon L^2(D, \mathbb{R}^2) \to H$ be the orthogonal projection onto H. Applying formally the operator Π to (1), we obtain the following evolution equation in H:

$$\dot{u} + \nu L u + B(u) = \eta(t), \tag{3}$$

where $L = -\Pi\Delta$ is the Stokes operator, $B(u) = \Pi(u, \nabla)u$ is the bilinear form resulting from the nonlinear term in (1), and we retained the notation for the right-hand side.

We shall assume that the right-hand side of (3) is a random process white in time and smooth in the space variables. Namely, let us denote by $\{e_j\}$ an orthonormal basis in H formed of the eigenvectors of the Stokes operator. We assume that

$$\eta(t) = \frac{\partial}{\partial t}\zeta(t), \quad \zeta(t) = \sum_{j=1}^{\infty} b_j \beta_j(t) e_j, \tag{4}$$

where $\{\beta_j\}$ is a sequence of independent standard Brownian motions defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $b_j \geq 0$ are some constants such that

$$B := \sum_{j=1}^{\infty} b_j^2 < \infty. \tag{5}$$

Condition (5) ensures that a.e. realisation of $\zeta(t)$ is a continuous function of time with range in H.

Consider the initial value problem for (3), (4):

$$u(0) = u_0, \tag{6}$$

where u_0 is an H-valued random variable independent of ζ. Let us set $V = H \cap H_0^1$, where $H_0^1 = H_0^1(D, \mathbb{R}^2)$ is the space of vector functions that belong to the Sobolev space of order 1 and vanish at ∂D.

Definition 2.1. A random process $u(t)$ defined for $t \geq 0$ is called a solution of (3), (4), (6) if it possesses the following properties:

(i) The process $u(t)$ is progressively measurable with respect to the filtration $\mathcal{F}_t$ generated by u_0 and ζ, and its almost every realisation belongs to the space $C(\mathbb{R}_+, H) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+, V)$.

(ii) With probability one, for any $t \ge 0$ we have

$$u(t) - u_0 + \int_0^t \bigl(\nu L u + B(u)\bigr)\, ds = \zeta(t),$$

where the left- and right-hand sides of this relation are regarded as elements of the dual space of H_0^1.

The following theorem, which gives, in particular, the existence and uniqueness of a solution for the problem (3), (4), (6), is established in [29] (see also [8, 3]).

Theorem 2.1. *Suppose that condition* (5) *is satisfied, and let* u_0 *be an* H*-valued random variable independent of* ζ*. Then the following assertions hold:*

(i) *The problem* (3), (4), (6) *has a unique solution* $u(t, u_0)$ *in the sense of Def. 2.1.*

(ii) *The set of solutions* $u(t, v)$ *corresponding to all deterministic initial functions* $v \in H$ *form a Markov process in the space* H*.*

Let $\mathcal{B}(H)$ be the family of Borel subsets of H, let $\mathcal{P}(H)$ be the set of probability measures on $(H, \mathcal{B}(H))$, and let $C_b(H)$ be the space of bounded continuous functionals $f : H \to \mathbb{R}$. We shall denote by $P_t(v, \Gamma)$ the transition function of the Markov process constructed in Theorem 2.1:

$$P_t(v, \Gamma) = \mathbb{P}\bigl\{u(t, v) \in \Gamma\bigr\}, \quad v \in H, \quad \Gamma \in \mathcal{B}(H).$$

Finally, let us recall the definition of Markov semi-groups $\mathfrak{P}_t$ and $\mathfrak{P}_t^*$ corresponding to the transition function $P_t(v, \Gamma)$. Namely, for $f \in C_b(H)$ and $\mu \in \mathcal{P}(H)$, we set

$$\mathfrak{P}_t f(v) = \int_H P_t(v, dz) f(z), \quad \mathfrak{P}_t^* \mu(\Gamma) = \int_H P_t(v, \Gamma)\mu(dv).$$

3. Stationary measures: Existence, uniqueness, and exponential mixing

Let us recall that a measure $\mu \in \mathcal{P}(H)$ is said to be *stationary* for (3), (4) if

$$\mathfrak{P}_t^* \mu = \mu \quad \text{for all } t \ge 0.$$

We denote by $\mathcal{L}(H)$ the space of functionals $f \in C_b(H)$ such that

$$\|f\|_{\mathcal{L}} := \sup_{u \in H} |f(u)| + \sup_{u \ne v} \frac{|f(u) - f(v)|}{\|u - v\|} < \infty,$$

where $\|\cdot\|$ is the natural norm in L^2.

Definition 3.1. A stationary measure μ is said to be *exponentially mixing* if there is a constant $\beta > 0$ and an increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $v \in H$ and $f \in \mathcal{L}(H)$, we have

$$\bigl|\mathfrak{P}_t f(v) - (f, \mu)\bigr| \le h(\|v\|)\, \|f\|_{\mathcal{L}} e^{-\beta t}, \quad t \ge 0, \tag{7}$$

where (f, μ) is the mean value of f with respect to μ.

The following result is of fundamental importance for applications.

Theorem 3.1. *Suppose that condition* (5) *is satisfied. Then the assertions below take place for any* $\nu > 0$.

1. Existence and a priori estimate: *There exists a stationary measure* $\mu \in \mathcal{P}(H)$. *Moreover, there are positive constants* c *and* C *not depending on* ν *such that any stationary measure* μ *satisfies the inequality*

$$\int_H e^{c\nu\|u\|^2}\mu(du) \le C. \tag{8}$$

2. Uniqueness: *There is an integer* $N = N_\nu \ge 1$ *depending on the constant* B *in* (5) *such that, if*

$$b_j \ne 0 \quad \textit{for} \quad j = 1, \dots, N, \tag{9}$$

then the stationary measure μ *is unique.*

3. Exponential mixing: *If* (9) *holds, then the stationary measure* μ *is exponentially mixing in the sense of Def. 3.1 with* $h(r) = \mathrm{const}(1 + r)$.

Theorem 3.1 implies two important corollaries. The first of them concerns the asymptotic behaviour of the distributions of solutions to the Cauchy problem.

Corollary 3.1. *Under the conditions of Theorem 3.1, for any* H*-valued random variable* u_0 *that is independent of* ζ *and has a finite mean value, we have*

$$\mathcal{D}\big(u(t, u_0)\big) \rightharpoonup \mu \quad \textit{as } t \to +\infty, \tag{10}$$

where $\mathcal{D}(\xi)$ *is the distribution of the random variable* ξ*, and the convergence in* (10) *is understood in the weak** *topology of* $\mathcal{P}(H)$.

To formulate the second corollary, we introduce a space of Hölder continuous functionals with at most exponential growth at infinity. Namely, for $\alpha \in (0, 1]$ and $\varepsilon > 0$ we denote by $C^\alpha(H, \varepsilon)$ the space of continuous functionals $f\colon H \to \mathbb{R}$ such that

$$\|f\|_{\alpha,\varepsilon} := \sup_{u\in H} \frac{|f(u)|}{\exp(\varepsilon\|u\|^2)} + \sup_{u\ne v} \frac{|f(u) - f(v)|}{\big(\exp(\varepsilon\|u\|^2) + \exp(\varepsilon\|v\|^2)\big)\|u - v\|^\alpha} < \infty.$$

Corollary 3.2. *Under the conditions of Theorem 3.1, for any* $\alpha \in (0, 1]$ *and sufficiently small* $\varepsilon > 0$ *there are positive constants* C *and* γ *depending on* ν *such that*

$$\big|\mathbb{E}f(u(t, u_0)) - (f, \mu)\big| \le C\, e^{-\gamma t}\mathbb{E}\exp(\varepsilon\|u_0\|^2), \quad t \ge 0, \tag{11}$$

where $f \in C^\alpha(H, \varepsilon)$ *is any functional with norm* $\|f\|_{\alpha,\varepsilon} \le 1$ *and* u_0 *is an arbitrary* H*-valued random variable independent of* ζ.

Let us make some comments on the history of the problem of ergodicity for the 2D Navier–Stokes system. The existence of a stationary measure is established in [29] with the help of the Bogolyubov–Krylov argument (see also [8]). A proof of the a priori estimate (8) can be found in [1, 25]. The problem of uniqueness and exponential mixing, which is much more delicate, was in the focus of attention during the last few years. The first result in this direction was obtained by Flandoli and Maslowski [9]. Under the condition that the right-hand side η of Eq. (3) is sufficiently irregular with respect to the space variables, they established the uniqueness of stationary measure and convergence to it in the variational norm. Their results were refined later by Ferrario [7]. We also mention Mattingly's paper [22] devoted to the case $\nu \gg 1$.

The first uniqueness result that allows the right-hand side to be infinitely smooth with respect to the space variables and applies to all $\nu > 0$ was established by Kuksin and the author [13] (see also [14, 17]). We considered the case in which the right-hand side η is a random process of the form

$$\eta(t) = \sum_{k=1}^{\infty} \eta_k(x)\delta(t-k), \tag{12}$$

where $\{\eta_k\}$ is a sequence of i.i.d. random variables in H with sufficiently non-degenerate distribution. The proof in [13] is based on a new approach involving a Lyapunov–Schmidt type reduction and a version of the Ruelle–Perron–Frobenius theorem. E, Mattingly, Sinai [4] and Bricmont, Kupiainen, Lefevere [2] studied later the case when the space variables x belong to the torus and the right-hand side η is white noise in time and trigonometric polynomial in x. They showed that there is at most one stationary measure. Moreover, it was established in [2] that, for μ-a.e. initial function (where μ is the stationary measure), the corresponding solution converges to μ in distribution exponentially fast. We also mention the paper [6] by Eckmann and Hairer in which an infinite-dimensional version of the Malliavin calculus is developed to study the problem of ergodicity for the real Ginzburg–Landau equation perturbed by a rough degenerate forcing.

In the papers [15, 16, 11, 18, 26] another approach based on coupling of solutions was developed to establish uniqueness of stationary measure and exponential convergence to it for all initial functions. The idea of coupling was also used in the papers by Mattingly [23], Masmoudi and Young [21], and Hairer [10]. Some further properties of random dynamical systems generated by stochastic PDE's are studied in [19, 20]. In particular, it is proved in [20] that the support of the Markov disintegration of a mixing stationary measure is a minimal random point attractor. Finally, as is shown in [12, 27], Theorem 3.1 implies a strong law of large numbers and central limit theorem for solutions of the problem (3), (4). The corresponding results are discussed in the next subsection.

The approach developed in the papers [15, 16, 11, 18, 26] applies not only to the 2D Navier–Stokes system, but also to a large class of dissipative PDE's perturbed by a random force of the form (4) or (12). For instance, an analogue of Theorem 3.1 is valid for the complex Ginzburg–Landau equation

$$\dot{u} - (\nu + i\alpha)\Delta u + i\lambda|u|^2 u = \eta(t). \tag{13}$$

We shall not dwell on the details in this paper.

4. Law of large numbers and central limit theorem

To simplify the presentation, we shall confine ourselves to bounded uniformly Lipschitz functionals. The following result on strong law of large numbers for solutions of the problem (3), (4) is established in [12, 27].

Theorem 4.1. *Suppose that* (5) *holds and the non-degeneracy condition* (9) *is satisfied with sufficiently large* $N \geq 1$. *Then for any* $\varepsilon \in (0, \frac{1}{3})$ *there is a constant* $D > 0$ *such that, for an arbitrary initial function* $u_0 \in H$ *and any functional* $f \in \mathcal{L}(H)$, *the following statements hold:*

(i) *There is a random variable* $T(\omega) \geq 1$ *depending on* ε, v, *and* f *such that*

$$\left| t^{-1} \int_0^t f(u(s, u_0))\, ds - (f, \mu) \right| \leq D \, \|f\|_{\mathcal{L}} t^{-\frac{1}{3}+\varepsilon} \quad \text{for} \quad t \geq T(\omega).$$

(ii) *The random variable* T *is almost surely finite. Moreover, if* $0 < r \leq 2\varepsilon$, *then*

$$\mathbb{E}\, T^r \leq D \left(1 + \|u_0\|^2\right) \|f\|_{\mathcal{L}}^2.$$

We note that Theorem 4.1 remains valid for functionals of the class $C^\alpha(H, \varepsilon)$ defined in Sec. 3 (see [27]). Moreover, if the problem is studied on the two-dimensional torus $\mathbb{T}^2$, and the right-hand side $\eta(t, x)$ is infinitely smooth with respect to the space variables, then in Theorem 4.1 we can take Hölder continuous functionals on any Sobolev space H^s. For instance, one can consider m-point correlation tensors of the form $f(u) = u(x^1) \cdots u(x^m)$, where $x^1, \ldots, x^m \in \mathbb{T}^2$ are given points.

We now discuss the CLT for the problem (3), (4). For any $f \in \mathcal{L}(H)$ satisfying the condition $(f, \mu) = 0$, we introduce the functional

$$g(u) = \int_0^\infty \mathfrak{P}_s f(u)\, ds, \quad u \in H.$$

Inequality (7) implies that g is well defined. Furthermore, we introduce a non-negative constant σ_f such that

$$\sigma_f^2 = 2(gf, \mu). \tag{14}$$

It can be shown that

$$2(gf, \mu) = \lim_{t \to +\infty} \mathbb{E} \left(t^{-\frac{1}{2}} \int_0^t f(u(s))\, ds \right)^2 \geq 0,$$

where $u(s)$ is a stationary solution with distribution μ. Hence, the constant $\sigma_f \geq 0$ is well defined by relation (14).

For any $\sigma > 0$, we denote by $\Phi_\sigma(r)$ the one-dimensional centred Gaussian distribution with variance σ:

$$\Phi_\sigma(r) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^r e^{-s^2/2\sigma^2} ds.$$

Finally, for $\sigma = 0$, we set

$$\Phi_0(r) = \begin{cases} 1, & r \geq 0, \\ 0, & r < 0. \end{cases}$$

The following result on central limit theorem for solutions of the problem (3), (4) is established in [12, 27].

Theorem 4.2. *Suppose that* (5) *holds and the non-degeneracy condition* (9) *is satisfied with sufficiently large* $N \geq 1$. *Then the following statements hold:*

(i) *For any $\bar\sigma > 0$ there is a function $h_{\bar\sigma}(r_1, r_2) \geq 0$ defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and increasing in both arguments such that, for any $f \in \mathcal{L}(H)$ satisfying the conditions $\sigma_f \geq \bar\sigma$ and $(f, \mu) = 0$, we have*

$$\sup_{z\in\mathbb{R}} \Big|\mathbb{P}\Big\{t^{-\frac{1}{2}} \int_0^t f(u(s,u_0))\,ds \leq z\Big\} - \varPhi_{\sigma_f}(z)\Big| \leq h_{\bar\sigma}\big(\|u_0\|, \|f\|_{\mathcal{L}}\big)\, t^{-\frac{1}{5}},$$

where $t \geq 1$ and $u_0 \in H$.

(ii) *There is a function $h(r_1, r_2) \geq 0$ defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and increasing in both arguments such that, for any $f \in \mathcal{L}(H)$ satisfying the conditions $\sigma_f = 0$ and $(f, \mu) = 0$, we have*

$$\sup_{z\in\mathbb{R}} \bigg(\big(|z| \wedge 1\big)\Big|\mathbb{P}\Big\{t^{-\frac{1}{2}} \int_0^t f(u(s,u_0))\,ds \leq z\Big\} - \varPhi_0(z)\Big|\bigg) \leq h\big(\|u_0\|, \|f\|_{\mathcal{L}}\big)\, t^{-\frac{1}{4}},$$

where $t \geq 1$ and $u_0 \in H$.

As in the case of LLN, the above theorem remains valid for a class of Hölder continuous functionals growing at infinity. Furthermore, for the problem with periodic boundary conditions and smooth (in x) right-hand side, we can consider continuous functionals on Sobolev spaces of arbitrarily large order.

5. Open problems

Many questions remain unsolved in the theory of randomly forced 2D Navier–Stokes equations. Let us formulate two of them.

5.1. *Uniform condition for ergodicity*

Let us consider the NS system (1) either in a bounded domain $D \subset \mathbb{R}^2$, with Dirichlet boundary condition, or on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Theorem 3.1 ensures that, for any $\nu > 0$, there is an integer $N_\nu \geq 1$ such that, if condition (9) is satisfied with $N = N_\nu$, then (1) has a unique stationary measure, which is exponentially mixing. In particular, if $b_j \neq 0$ for all $j \geq 1$, then a stationary measure is unique for any $\nu > 0$.

Problem 1. *Prove the following assertion: there is a finite integer $N \geq 1$ not depending on ν such that, if condition* (9) *is satisfied, then for any $\nu > 0$ there is a unique stationary measure for* (1)*, which is exponentially mixing.*

5.2. *Problems with continuous spectrum*

As was mentioned in Section 3, Theorem 3.1 remains valid for a large class of type (3) dissipative PDE's perturbed by a sufficiently non-degenerate random force of the form (4) or (12), provided that the problem is considered in a bounded domain (and therefore the spectrum of the linear operator L is discrete). On the other hand, the Bogolyubov–Krylov argument enables one to construct stationary measures for some equations in unbounded domains (e.g., see [5, 24]).

Problem 2. *Find a sufficient condition that guarantees uniqueness and ergodicity of a stationary measure for equations in unbounded domains.*

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