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Local dynamics for high-order semilinear hyperbolic equations

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Abstract. This paper is devoted to studying high-order semilinear hyperbolic equations. It is assumed that the equation is a small perturbation of an equation with real constant coefficients and that the roots of the full symbol of the unperturbed equation with respect to the variable τ dual to time are either separated from the imaginary axis or lie outside the domain $\nu < |\operatorname{Re}\tau| < \delta$, where $\delta > \nu \geqslant 0$. In the first case, it is proved that the phase diagram of the perturbed equation can be linearized in the neighbourhood of zero using a time-preserving family of homeomorphisms and that the constructed homeomorphisms and their inverses are Hölder continuous. In the other case, it is proved that the neighbourhood of zero in the phase space of the equation contains a locally invariant smooth manifold $\mathcal{M}$ which includes all solutions uniformly bounded on the entire time axis and exponentially attracts the solutions bounded on the half-axis. The manifold $\mathcal{M}$ can be represented as the graph of a non-linear operator that acts on the phase space and is a small perturbation of a pseudo-differential operator whose symbol can be written explicitly. In this case, the dynamics on the invariant manifold $\mathcal{M}$ is described by a hyperbolic equation whose order coincides with the number of roots of the full symbol that lie in the strip $|\operatorname{Re}\tau| \leqslant \nu$.

Introduction

In the theory of ordinary differential equations (ODE), the behaviour of solutions in the neighbourhood of a stationary point has been rather thoroughly studied . We mention the well-known result of Grobman–Hartman on the linearization of the phase diagram in the neighbourhood of a hyperbolic stationary point (see [4], [20], [2], [10]).

Let us consider the system

$$\dot{u}(t) = Pu(t) + \varepsilon Q(u(t)), \quad u(t) \in \mathbb{E}, \tag{0.1}$$

where $\mathbb{E} = \mathbb{R}^m$ is the phase space, $\varepsilon \in \mathbb{R}$ is a small parameter, P is an $m \times m$ matrix with real entries, and $Q(u)\colon \mathbb{R}^m \to \mathbb{R}^m$ is a smooth compactly supported function vanishing at $u = 0$. We consider the Cauchy problem for equation (0.1),

$$u(0) = u_0 \in \mathbb{R}^m. \tag{0.2}$$

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Let

$$\mathcal{U}_\varepsilon(t,\cdot)\colon \mathbb{R}^m \to \mathbb{R}^m, \quad u_0 \mapsto u(t),$$

denote the resolving operator for problem (0.1), (0.2). For $\varepsilon = 0$, the operator $\mathcal{U}_\varepsilon(t,\cdot)$ is linear, and we denote it by $\mathcal{U}_0(t)$. The Grobman–Hartman theorem asserts that if the real parts of the eigenvalues of the matrix P are non-zero, then there is a homeomorphism $\Phi_\varepsilon\colon \mathbb{R}^m \to \mathbb{R}^m$, $|\varepsilon| \ll 1$, such that the relation

$$\mathcal{U}_0(t)\Phi_\varepsilon(u_0) = \Phi_\varepsilon(\mathcal{U}_\varepsilon(t,u_0))$$

(see Fig. 1, where $u = \mathcal{U}_\varepsilon(t,u_0)$) holds for all $t \in \mathbb{R}$ and $u_0 \in \mathbb{R}^m$.

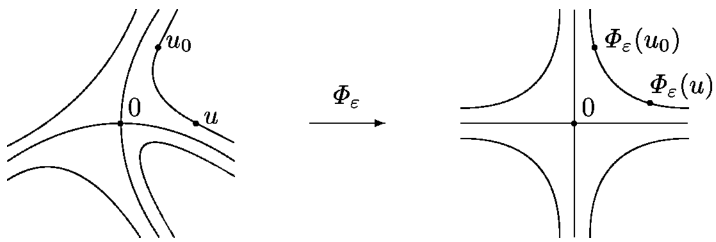

Figure 1. Linearization of the phase diagram

In the general case of a matrix P having roots on the imaginary axis, linearization may not be possible, but, as follows from the results in [29] (also see [21] and [33]), there is a smooth m_c-dimensional manifold $\mathcal{M}$ (where m_c is the number of pure imaginary eigenvalues of P) such that

(i) any solution uniformly bounded for $t \in \mathbb{R}$ lies on $\mathcal{M}$, and the dynamics on $\mathcal{M}$ is described by an equation of order m_c,

(ii) any solution uniformly bounded for $t \geqslant 0$ $(t \leqslant 0)$ is exponentially attracted to $\mathcal{M}$.

The above results are part of the general theory describing the behaviour of ODE solutions in the neighbourhood of a stationary point. For a detailed presentation of the corresponding part of ODE theory, see [2], [6], [9], [10], and [33], where, in particular, questions related to asymptotic stability, exponential dichotomy, and existence and smoothness of unstable, stable, and centre manifolds are studied. All these results can be extended in some form to equations of the form (0.1) with infinite-dimensional phase space $\mathbb{E}$ provided that P is a bounded linear operator and $Q(u)$ is a smooth function on $\mathbb{E}$ (see [5] and [28]). The condition that P is bounded prohibits applying the resulting abstract theory to partial evolution equations since the operator P arising in the reduction of these equations to systems of the form (0.1) is a differential operator with respect to the spatial variables and therefore is unbounded. Thus, the "natural" conditions on the operator P and the non-linear function $Q(u)$ in studying equation (0.1) in an infinite-dimensional phase space are specified by concrete examples of partial differential equations. These difficulties have been studied by many authors (see [1], [8], [11], [14]–[18], [22]–[27], [31], and [34] and the references therein).

In this paper, we study semilinear hyperbolic equations of the form

$$P(\partial)u(t,x) + \varepsilon Q(\varepsilon,t,x,\partial^m u(t,x)) = 0, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{0.3}$$

where $\varepsilon \in \mathbb{R}$ is a small parameter, $\partial = (\partial_t, \partial_{x_1}, \dots, \partial_{x_n})$, $\partial^m = \{\partial^\alpha \colon |\alpha| \leqslant m\}$, and $\partial^\alpha = \partial_t^{\alpha_0} \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$. It is assumed that the operator

$$P(\partial) = P(\partial_t, \partial_x) = \sum_{|\alpha| \leqslant m} p_\alpha \partial^\alpha, \quad p_\alpha \in \mathbb{R}, \tag{0.4}$$

is strictly hyperbolic and the non-linear term Q has the form

$$Q(\varepsilon, t, x, \partial^m u) = \sum_{|\alpha| \leqslant m} q_\alpha(\varepsilon, t, x) \partial^\alpha u + q(\varepsilon, t, x, \partial^{m-1} u), \tag{0.5}$$

where $q_\alpha(\varepsilon, t, x)$ and $q(\varepsilon, t, x, z)$ $(z = \{z_\beta \colon |\beta| \leqslant m-1\} \in \mathbb{R}^d)$ are smooth real-valued functions, and that $q(\varepsilon, t, x, 0) \equiv 0$. Along with the condition of strong hyperbolicity, one of the following conditions is imposed on $P(\partial)$.

Condition (H). *There is a $\delta > 0$ such that*

$$P(\tau, \xi) \neq 0 \quad \textit{for} \quad |\operatorname{Re} \tau| < \delta, \quad (\operatorname{Im} \tau, i\xi) \in \mathbb{R}^{n+1}, \tag{0.6}$$

where $P(\tau, \xi)$ is the full symbol of the operator $P(\partial_t, \partial_x)$.

Condition (H_c). *There are δ and ν, $\delta > \nu \geqslant 0$, such that*

$$P(\tau, \xi) \neq 0 \quad \textit{for} \quad \nu < |\operatorname{Re} \tau| < \delta, \quad (\operatorname{Im} \tau, i\xi) \in \mathbb{R}^{n+1}. \tag{0.7}$$

In other words, it is assumed in Condition (H) that the roots of the polynomial $P(\tau, \xi)$ are separated from the imaginary axis uniformly with respect to $i\xi \in \mathbb{R}^n$, whereas, under Condition (H_c), the symbol $P(\tau, \xi)$ can have some roots in the neighbourhood of the imaginary axis, but it is required that they should be separated from the other roots uniformly with respect to ξ. We shall prove that if Condition (H) holds, then the phase diagram of equation (0.3) can be linearized in an arbitrary finite neighbourhood of zero (for sufficiently small values of ε) using a time-preserving family of homeomorphisms. Moreover, we shall show that if Q is an operator of order $m-1$, then the linearizing homeomorphisms and their inverses are Hölder continuous. For exact statements of these assertions, see §1.2.

In the case when Condition (H_c) holds, we shall prove that an arbitrary finite neighbourhood of zero in the phase space of equation (0.3) contains a smooth infinite-dimensional manifold $\mathcal{M}$ possessing properties similar to (i) and (ii) (see above). Furthermore, it will be established that $\mathcal{M}$ is the graph of a smooth non-linear operator that depends on m_c functions (where m_c[1] is the number of roots lying in the strip $|\operatorname{Re} \tau| \leqslant \nu$) and is a small non-linear perturbation of a pseudo-differential matrix operator whose symbol can be expressed in terms of that of the original equation. The exact statements of these assertions are given in §5.1.

Let us briefly describe the structure of this paper, which is in two parts. The first part is devoted to proving Grobman–Hartman type theorems. In § 1, the main results in the first part are stated (see Theorems 1.3 and 1.4) and some examples are considered. In § 2, which is devoted to studying linear hyperbolic equations,

[1]Everywhere below, we shall assume that $1 \leqslant m_c \leqslant m-1$.

it is proved that the phase diagram of an equation with constant coefficients and those of its small perturbations can be transformed into each other using a time-preserving family of homeomorphisms. In § 3, Theorems 1.3 and 1.4 are proved. In the Appendix (see § 4), some auxiliary assertions used in the main body of the text are collected.

The second part is devoted to constructing the centre manifold for equation (0.3). In § 5, the statements and sketches of the proofs of the main results are given (see Theorems 5.1–5.3). The initial-value problem with growth conditions at infinity for an equation with truncated non-linearity is studied in § 6, and § 7 is devoted to proving Theorems 5.1–5.3. In the Appendix (see § 8), some auxiliary assertions are collected.

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Notation. Let $0 < \gamma \leqslant 1$, let $J \subset \mathbb{R}$ be an interval, let D be a domain in $\mathbb{R}^d$, let X and Y be Banach spaces, and let Ω be an open set in X. We shall use the following function spaces.

The space $C_b^\infty(D)$ of infinitely continuously differentiable functions on D, which, together with all their derivatives, are bounded.

The space $C(J,X)$ of continuous functions on J with range in X.

The space $C^{l,\gamma}(\Omega,Y)$ of l-times continuously (Frechét) differentiable functions $f\colon \Omega \to Y$ whose lth derivatives satisfy Hölder's condition with exponent γ and whose seminorms

$$|f|_{C^{l,\gamma}} := \sup_{u\in\Omega} \sum_{j=1}^{l} \|D^j f(u)\|_{\mathcal{L}^j(X,Y)} + \sup_{\substack{u,v\in\Omega\\ u\neq v}} \frac{\|D^l f(u) - D^l f(v)\|_{\mathcal{L}^l(X,Y)}}{\|u-v\|_X^\gamma}$$

are finite, where $\mathcal{L}^j(X,Y)$ denotes the space of bounded j-linear forms from X to Y (see [11], § 1.2.5). We note that if the domain Ω is bounded, then the norm

$$\|f\|_{C^{l,\gamma}} := \sup_{u\in\Omega} \|f(u)\|_Y + |f|_{C^{l,\gamma}}$$

is finite. In the case $l = 0$, we shall write $C^\gamma(\Omega,Y)$.

The symbols C_i, $i = 1,2,\dots$, will be used to denote all insignificant positive constants.

PART I. THE GROBMAN–HARTMAN THEOREM

§ 1. Statement of linearization theorems and examples

In this section, we recall the theorem on the well-posedness of the Cauchy problem for non-linear hyperbolic equations (Proposition 1.1), state the main results of the first part (Theorems 1.3 and 1.4), and consider some examples.

1.1. The Cauchy problem for equation (0.3)**.** We shall assume that the operators P and Q in equation (0.3) satisfy the following conditions.

Condition (P). The operator $P(\partial)$ has the form (0.4) and is strictly hyperbolic, that is, it is solved with respect to the highest derivative with respect to t, and the roots of the leading symbol

$$P^0(\tau,\xi)=\sum_{|\alpha|=m} p_\alpha \tau^{\alpha_0}\xi^{\alpha'}, \quad \alpha=(\alpha_0,\alpha'),$$

with respect to τ are pure imaginary and pairwise distinct for $i\xi\in\mathbb{R}^n\setminus\{0\}$.

Condition (Q). The operator Q has the form (0.5), where q_α and q are real-valued functions satisfying the relations $q_\alpha\in C_b^\infty([-1,1]\times\mathbb{R}^{n+1}_{t,x})$ and $q\in C_b^\infty([-1,1]\times\mathbb{R}^{n+1}_{t,x}\times B_\rho)$ for any ball $B_\rho=\{z\in\mathbb{R}^d\colon |z|\leqslant\rho\}$, and the condition $q(\varepsilon,t,x,0)\equiv 0$ holds.

We consider the Cauchy problem for equation (0.3):

$$\partial_t^j u(\theta,x)=u_j(x)\in H^{(m-1+k-j)}, \quad j=0,\dots,m-1, \tag{1.1}$$

where $k\geqslant 0$ is an integer and $H^{(s)}=H^{(s)}(\mathbb{R}^n)$ is the Sobolev space of order s with the standard norm $\|\cdot\|_s$. The phase space of equation (0.3) is defined using the formula

$$\mathbb{E}_{m-1,k}=\prod_{j=0}^{m-1} H^{(m-1+k-j)}$$

and is equipped with the norm

$$\|U\|_{m-1,k}=\biggl(\sum_{j=0}^{m-1}\|u_j\|^2_{(m-1+k-j)}\biggr)^{1/2}, \quad U=[u_0,\dots,u_{m-1}].$$

Let $\mathbb{B}_{m-1,k}(\rho)$ denote the open ball in $\mathbb{E}_{m-1,k}$ of radius $\rho>0$ and centre zero.

Proposition 1.1. *Suppose that Conditions* (P) *and* (Q) *hold. Then there is an* $\varepsilon_0>0$ *such that the following assertions are true for* $|\varepsilon|\leqslant\varepsilon_0$.

(i) *For any* $\rho>0$ *and an arbitrary integer* $k>n/2$, *there is a* $T>0$ *such that problem* (0.3), (1.1) *with Cauchy data* $[u_0,\dots,u_{m-1}]\in\mathbb{B}_{m-1,k}(\rho)$ *has a* (*unique*) *solution* $u(t,x)$ *satisfying the relations*

$$\partial_t^j u\in C(J,H^{(m-1+k-j)}), \quad j=0,\dots,m-1, \tag{1.2}$$

where $J=[\theta-T,\theta+T]$.

(ii) *If* $u_i(t,x)$, $i=1,2$, *are two solutions of problem* (0.3), (1.1), (1.2) *with* $J=J_i$, *where* $J_i\subset\mathbb{R}$, $i=1,2$, *are closed intervals containing the point* θ, *then* $u_1(t,x)=u_2(t,x)$ *for* $t\in J_1\cap J_2$ *and* $x\in\mathbb{R}^n$.

Proposition 1.1 is a version of the theorem on the well-posedness of the Cauchy problem for non-linear hyperbolic equations. For a proof, see, for example, [7], Chapter 7.

Remark 1.2. It follows from Proposition 1.1 that for any $\theta \in \mathbb{R}$ and $U_0 = [u_0, \dots, u_{m-1}] \in \mathbb{E}_{m-1,k}$, there are $T_\varepsilon^\pm = T_\varepsilon^\pm(k, \theta, U_0)$ such that $(T_\varepsilon^-, T_\varepsilon^+)$ is the maximum interval on which the solution corresponding to Cauchy data U_0 is defined. In this case, if $T_\varepsilon^+ < +\infty$, then

$$\lim_{t \to T_\varepsilon^+ - 0} \|\mathcal{D}(t)u\|_{m-1,k} = +\infty,$$

where

$$\mathcal{D}(t)u = [u(t,x), \partial_t u(t,x), \dots, \partial_t^{m-1} u(t,x)] \tag{1.3}$$

denotes the *phase trajectory corresponding to* $u(t,x)$. Similarly, if $T_\varepsilon^- > -\infty$, then

$$\lim_{t \to T_\varepsilon^- + 0} \|\mathcal{D}(t)u\|_{m-1,k} = +\infty.$$

We denote by

$$\mathcal{U}_\varepsilon(t, \theta, U_0) \colon \mathbb{E}_{m-1,k} \to \mathbb{E}_{m-1,k}, \quad |\varepsilon| \leqslant \varepsilon_0, \quad t \in (T_\varepsilon^-, T_\varepsilon^+), \tag{1.4}$$

the operator mapping $U_0 = [u_0, \dots, u_{m-1}] \in \mathbb{E}_{m-1,k}$ to the vector function (1.3), where $u(t,x)$ is the solution of problem (0.3), (1.1), (1.2) with $J = (T_\varepsilon^-, T_\varepsilon^+)$. For $\varepsilon = 0$, equation (0.3) is linear, and its coefficients do not depend on (t,x). Therefore, we have $T_0^\pm = \pm\infty$, and (1.4) is an invertible linear operator depending only on the difference $t - \theta$. We denote it by $\mathcal{U}_0(t - \theta)$.

1.2. Statement of results. Recall Condition (H) from the Introduction.

Theorem 1.3. *Let Conditions* (P), (Q), *and* (H) *hold. Then for any* $\rho > 0$ *and an arbitrary integer* $k > n/2$, *there is a constant* $\varepsilon_0 > 0$ *and a family of continuous maps*

$$\Phi_\varepsilon(\theta, U_0) \colon \mathbb{R} \times \mathbb{B}_{m-1,k}(\rho) \to \mathbb{E}_{m-1,k} \tag{1.5}$$

such that $\Phi_\varepsilon(\theta, 0) \equiv 0$ *and the following assertions are true for* $|\varepsilon| \leqslant \varepsilon_0$.

(i) *If* $U_0 = [u_0, \dots, u_{m-1}] \in \mathbb{B}_{m-1,k}(\rho)$ *is a vector function such that*

$$\mathcal{U}_\varepsilon(t, \theta, U_0) \in \mathbb{B}_{m-1,k}(\rho) \quad \text{for} \quad t \in J,$$

where $J \subset \mathbb{R}$ *is an interval containing the point* $\theta \in \mathbb{R}$, *then*

$$\mathcal{U}_0(t - \theta)\Phi_\varepsilon(\theta, U_0) = \Phi_\varepsilon(t, \mathcal{U}_\varepsilon(t, \theta, U_0)). \tag{1.6}$$

(ii) *For an arbitrary fixed* $\theta \in \mathbb{R}$, *the image* $\mathbb{V}_\varepsilon(\rho, \theta)$ *of the open ball* $\mathbb{B}_{m-1,k}(\rho)$ *under the map* $\Phi_\varepsilon(\theta, \cdot)$ *is an open neighbourhood of zero in* $\mathbb{E}_{m-1,k}$, *and* $\Phi_\varepsilon(\theta, \cdot)$ *specifies a homeomorphic map of* $\mathbb{B}_{m-1,k}(\rho)$ *onto* $\mathbb{V}_\varepsilon(\rho, \theta)$.

(iii) *The set* $\mathbb{R}_\theta \times \mathbb{V}_\varepsilon(\rho, \theta) = \{(\theta, U_0) \in \mathbb{R} \times \mathbb{E}_{m-1,k} \colon U_0 \in \mathbb{V}_\varepsilon(\rho, \theta)\}$ *is open in* $\mathbb{R} \times \mathbb{E}_{m-1,k}$, *and the inverse map*

$$\Phi_\varepsilon^{-1}(\theta, U_0) \colon \mathbb{R}_\theta \times \mathbb{V}_\varepsilon(\rho, \theta) \to \mathbb{B}_{m-1,k}(\rho) \tag{1.7}$$

is continuous for any fixed ε.

We now consider the question of equivalence for the operator $\mathcal{U}_\varepsilon(t,\theta,\cdot)$ and the resolving operator $\mathcal{V}_\varepsilon(t,\theta)$ of the Cauchy problem for the linear equation

$$P_\varepsilon(t,x,\partial)u \equiv \sum_{|\alpha|\leqslant m} \bigl(p_\alpha + \varepsilon q_\alpha(\varepsilon,t,x)\bigr)\partial^\alpha u = 0. \tag{1.8}$$

By definition, the linear operator $\mathcal{V}_\varepsilon(t,\theta)$, defined for all $t,\theta \in \mathbb{R}$, transforms $[u_0,\dots,u_{m-1}] \in \mathbb{E}_{m-1,k}$ into the vector function (1.3), where $u(t,x)$ is the solution of problem (1.8), (1.1), (1.2) with $J=\mathbb{R}$.

As is well known, if P is a strictly hyperbolic operator, then the roots $\tau_j(\xi)$, $j=1,\dots,m$, of the full symbol $P(\tau,\xi)$ lie in a strip $|\operatorname{Re}\tau| \leqslant \text{const}$ for $i\xi \in \mathbb{R}^n$ (for example, see [7], Chapter 4, or [32], § 2.2). Let us set

$$\begin{aligned} \sigma_{\max} &= \max_{j=1,\dots,m} \sup_{\xi\in i\mathbb{R}^n} |\operatorname{Re}\tau_j(\xi)|, \\ \sigma_{\min} &= \min_{j=1,\dots,m} \inf_{\xi\in i\mathbb{R}^n} |\operatorname{Re}\tau_j(\xi)|. \end{aligned} \tag{1.9}$$

We note that if the operator $P(\partial)$ satisfies Condition (H), then $\sigma_{\min} \geqslant \delta > 0$.

Theorem 1.4. *Let Conditions* (P), (Q), *and* (H) *be fulfilled. Then for any $\rho>0$ and γ, $0<\gamma<\sigma_{\min}/\sigma_{\max}$, and an arbitrary integer $k>n/2$, there is a constant $\varepsilon_0=\varepsilon_0(k,\gamma,\rho)>0$ and a family of continuous operators*

$$\Psi_\varepsilon(\theta,U_0)\colon \mathbb{R}\times\mathbb{B}_{m-1,k}(\rho) \to \mathbb{E}_{m-1,k}, \quad |\varepsilon|\leqslant\varepsilon_0, \tag{1.10}$$

such that $\Psi_\varepsilon(\theta,0)\equiv 0$, and the following assertions hold for $|\varepsilon|\leqslant\varepsilon_0$.

(i) *If $U_0=[u_0,\dots,u_{m-1}]\in\mathbb{B}_{m-1,k}(\rho)$ is a vector function such that*

$$\mathcal{U}_\varepsilon(t,\theta,U_0)\in\mathbb{B}_{m-1,k}(\rho) \quad \textit{for} \quad t\in J,$$

where $J\subset\mathbb{R}$ is an interval containing the point $\theta\in\mathbb{R}$, then

$$\mathcal{V}_\varepsilon(t,\theta)\Psi_\varepsilon(\theta,U_0) = \Psi_\varepsilon(t,\mathcal{U}_\varepsilon(t,\theta,U_0)) \quad \textit{for} \quad t\in J. \tag{1.11}$$

(ii) *For any fixed $\theta\in\mathbb{R}$, the image $\mathbb{W}_\varepsilon(\rho,\theta)$ of the open ball $\mathbb{B}_{m-1,k}(\rho)$ under the map $\Psi_\varepsilon(\theta,\cdot)$ is an open neighbourhood of zero in $\mathbb{E}_{m-1,k}$, and $\Psi_\varepsilon(\theta,\cdot)$ specifies a one-to-one map of $\mathbb{B}_{m-1,k}(\rho)$ onto $\mathbb{W}_\varepsilon(\rho,\theta)$. Moreover, the operator $\Psi_\varepsilon(\theta,\cdot)$ and its inverse operator $\Psi_\varepsilon^{-1}(\theta,\cdot)$ are Hölder continuous with exponent γ.*

(iii) *The set $\mathbb{R}_\theta\times\mathbb{W}_\varepsilon(\rho,\theta)=\bigl\{(\theta,U_0)\in\mathbb{R}\times\mathbb{E}_{m-1,k}\colon U_0\in\mathbb{W}_\varepsilon(\rho,\theta)\bigr\}$ is open in $\mathbb{R}\times\mathbb{E}_{m-1,k}$, and the inverse map*

$$\Psi_\varepsilon^{-1}(\theta,U_0)\colon \mathbb{R}_\theta\times\mathbb{W}_\varepsilon(\rho,\theta)\to\mathbb{B}_{m-1,k}(\rho)$$

is continuous for any fixed ε.

The proofs of Theorems 1.3 and 1.4 are given in § 3.

Remark 1.5. As will be seen from the proofs of Theorems 1.3 and 1.4, for any $\rho>0$, there are $\rho',\rho''>0$ such that

$$\mathbb{B}_{m-1,k}(\rho')\subset\mathbb{V}_\varepsilon(\rho,\theta), \qquad \mathbb{W}_\varepsilon(\rho,\theta)\subset\mathbb{B}_{m-1,k}(\rho''), \tag{1.12}$$

and $\rho'\to\infty$ as $\rho\to\infty$. The proofs will also imply that if the non-linear operator Q in (0.3) does not depend on t, then the linearizing operators Φ_ε and Ψ_ε do not depend on θ.

1.3. Examples. We present some examples of strictly hyperbolic operators satisfying Condition (H).

Example 1.6. Let $a > 0$, let $b, \sigma \in \mathbb{R} \setminus \{0\}$, and let $c \in \mathbb{R}^n$. We set

$$P_{b,c}(\partial_t, \partial_x) = \partial_t + c\partial_x + b, \qquad P_{a,\sigma}(\partial_t, \partial_x) = \partial_t^2 + \sigma\partial_t + (1 - a^2 \Delta), \tag{1.13}$$

where Δ is the Laplace operator. It is easy to see that (1.13) are strictly hyperbolic operators satisfying Condition (H).

Example 1.7. Let a vector $c \in \mathbb{R}^n$ and real numbers $a_j, \sigma_j,\ \ j = 1, \dots, l$, and b be such that $|c| < a_j$ for all j, $a_j \neq a_k$ for $j \neq k$, and $b, \sigma_j \neq 0$. Then the strictly hyperbolic operators

$$\prod_{j=1}^{l} P_{a_j,\sigma_j}(\partial_t, \partial_x), \qquad P_{b,c}(\partial) \prod_{j=1}^{l} P_{a_j,\sigma_j}(\partial_t, \partial_x)$$

satisfy Condition (H).

Example 1.8. Conditions (P) and (H) are stable with respect to small perturbations of the symbol (see [32], Proposition 3.9). In other words, if an operator $P(\partial)$ of order m satisfies these conditions and $Q(\partial)$ is an arbitrary operator of order m with real coefficients, then $P + \nu Q$ satisfies Conditions (P) and (H) for sufficiently small $\nu \in \mathbb{R}$.

The next example shows that the estimate for the exponent γ in the condition of Hölder continuity is exact in the case $\sigma_{\min} = \sigma_{\max}$.

Example 1.9. Let us consider the linear ODE

$$\dot{u} = -(1 + \varepsilon)u. \tag{1.14}$$

In this case, we have $\sigma_{\min} = \sigma_{\max} = 1$, and the resolving operator of the Cauchy problem for equation (1.14) has the form

$$\mathcal{V}_\varepsilon(t)u_0 = e^{-(1+\varepsilon)t}u_0. \tag{1.15}$$

Let a continuous operator $\Psi\colon [0, \rho] \to \mathbb{R}$ satisfy the condition

$$\mathcal{U}_0(t)\Psi(u_0) = \Psi(\mathcal{U}_\varepsilon(t)u_0), \quad 0 \leqslant u_0 \leqslant \rho, \quad t \geqslant 0, \tag{1.16}$$

for some $\varepsilon > 0$ and $\rho > 0$. Then substituting (1.15) into (1.16) and setting $u_0 = \rho$, we obtain

$$e^{-t}\Psi(\rho) = \Psi(e^{-(1+\varepsilon)t}\rho),$$

whence it follows that

$$\Psi(u) = cu^{\frac{1}{1+\varepsilon}}, \quad u > 0, \quad c = \Psi(\rho)\rho^{-\frac{1}{1+\varepsilon}}.$$

Hence, the operator $\Psi(u)$ is Hölder continuous with exponent $\gamma = 1/(1 + \varepsilon)$, but does not satisfy Lipschitz' condition for any $\varepsilon > 0$.

§ 2. Equivalence of linear equations

We shall show in this section that if Condition (H) holds, then the integral curves of equation (1.8) can be mapped onto those of the equation

$$P(\partial)u(t,x) = 0 \tag{2.1}$$

using a time-preserving family of homeomorphisms. This assertion will be used in the proof of Theorem 1 (see § 3).

2.1. Statement of the result. Recall the operators $\mathcal{U}_0(t-\theta)$ and $\mathcal{V}_\varepsilon(t,\theta)$ from §§ 1.1 and 1.2, respectively.

Theorem 2.1. *Let Conditions* (P), (Q), *and* (H) *hold. Then for any integer* $k \geqslant 0$, *there is a constant* $\varepsilon_0 = \varepsilon_0(k) > 0$ *and a family of continuous maps*

$$L_\varepsilon(\theta, U_0) \colon \mathbb{R} \times \mathbb{E}_{m-1,k} \to \mathbb{E}_{m-1,k}, \quad |\varepsilon| \leqslant \varepsilon_0, \tag{2.2}$$

such that $L_\varepsilon(\theta, 0) \equiv 0$ *and the following assertions are true for* $|\varepsilon| \leqslant \varepsilon_0$.

(i) $\mathcal{U}_0(t-\theta)L_\varepsilon(\theta,U_0) = L_\varepsilon(t, \mathcal{V}_\varepsilon(t,\theta)U_0)$ *for all* $t, \theta \in \mathbb{R}$ *and* $U_0 \in \mathbb{E}_{m-1,k}$.

(ii) *For any fixed* $\theta \in \mathbb{R}$, *the operator* $L_\varepsilon(\theta, U_0)$ *specifies a homeomorphic map of* $\mathbb{E}_{m-1,k}$ *onto itself. Moreover, the inverse operator*

$$L_\varepsilon^{-1}(\theta, U_0) \colon \mathbb{R}_\theta \times \mathbb{E}_{m-1,k} \to \mathbb{E}_{m-1,k}$$

is jointly continuous with respect to the variables (θ, U_0).

Remark 2.2. As will be seen from the proof of the theorem, for any $\rho > 0$, there are $\rho', \rho'' > 0$ such that the inclusion

$$\mathbb{B}_{m-1,k}(\rho') \subset L_\varepsilon(\theta, \mathbb{B}_{m-1,k}(\rho)) \subset \mathbb{B}_{m-1,k}(\rho'') \tag{2.3}$$

holds for the image of the ball $\mathbb{B}_{m-1,k}(\rho)$ under the map L_ε, and $\rho' \to \infty$ as $\rho \to \infty$. The proof will also imply the if the coefficients q_α do not depend on t, then the operators L_ε do not depend on θ.

Theorem 2.1 will be proved in § 2.4. To elucidate the main ideas of the proof, we consider the special case (see § 2.2) in which all roots of the symbol $P(\tau,\xi)$ are stable, that is, lie in the left half-plane. Furthermore, we shall need some results in [3] relating to the property of exponential dichotomy for equation (1.8). They are presented in § 2.3.

2.2. Proof of Theorem 2.1. The case of stable roots. We shall assume in this section that the roots $\tau_j(\xi)$, $j = 1, \dots, m$, of the full symbol $P(\tau,\xi)$ satisfy the condition

$$\operatorname{Re}\tau_j(\xi) \leqslant -\sigma_{\min} \quad \text{for} \quad i\xi \in \mathbb{R}^n, \quad j = 1, \dots, m.$$

(1) As is shown in [32], Theorem 6.1, the inequality

$$\|\mathcal{V}_\varepsilon(t,\theta)U_0\|_{m-1,k} \leqslant C_1 e^{-\mu(t-\theta)} \|U_0\|_{m-1,k}, \quad t \geqslant \theta, \tag{2.4}$$

holds for $|\varepsilon| \leqslant \varepsilon_0 \ll 1$, where $0 < \mu < \sigma_{\min}$ and the constant $C_1 > 0$ does not depend on U_0. For given $\theta \in \mathbb{R}$ and ε, $|\varepsilon| \leqslant \varepsilon_0$, we define the seminorm in $\mathbb{E}_{m-1,k}$ by the formula

$$|U_0|_{(\varepsilon,\theta)} = \biggl(\int_\theta^{+\infty} \|\mathcal{V}_\varepsilon(t,\theta)U_0\|_{m-1,k}^2 \, dt \biggr)^{1/2}, \quad U_0 \in \mathbb{E}_{m-1,k}. \tag{2.5}$$

Let us show that the seminorm (2.5) is equivalent to $\|\cdot\|_{m-1,k}$. More precisely, there is a constant $K > 1$ not depending on ε or θ such that the inequality

$$K^{-1}\|U_0\|_{m-1,k} \leqslant |U_0|_{(\varepsilon,\theta)} \leqslant K\|U_0\|_{m-1,k}, \quad U_0 \in \mathbb{E}_{m-1,k}, \tag{2.6}$$

holds. Indeed, by (2.4), we have

$$|U_0|_{(\varepsilon,\theta)}^2 = \int_\theta^{+\infty} \|\mathcal{V}_\varepsilon(t,\theta)U_0\|_{m-1,k}^2 \, dt \leqslant C_1^2 (2\mu)^{-1} \|U_0\|_{m-1,k}^2.$$

To prove the reverse inequality, we need the following lemma.

Lemma 2.3. *Suppose that Conditions* (P) *and* (Q) *hold. Then there are constants* $\varepsilon_0 > 0$, $\varkappa > 0$, *and* $C > 0$ *such that the inequality*

$$\|\mathcal{V}_\varepsilon(t,\theta)U_0\|_{m-1,k} \leqslant C\, e^{\varkappa|t-\theta|} \|U_0\|_{m-1,k}, \quad t,\theta \in \mathbb{R}, \quad U_0 \in \mathbb{E}_{m-1,k}, \tag{2.7}$$

holds for $|\varepsilon| \leqslant \varepsilon_0$. *More precisely, the operator* $\mathcal{V}_\varepsilon(t,\theta)U_0$ *is continuous in* (t,θ,U_0).

Inequality (2.7) was established in [12], Lemma 23.2.1. The continuity of $\mathcal{V}_\varepsilon$ is proved in the same way as assertion (iii) in Proposition 4.2.

We interchange t and θ in (2.7) and replace U_0 by $\mathcal{V}_\varepsilon(t,\theta)U_0$, which results in

$$\|U_0\|_{m-1,k} \leqslant C_2 \|\mathcal{V}_\varepsilon(t,\theta)U_0\|_{m-1,k}, \quad |t-\theta| \leqslant 1,$$

where the constant $C_2 > 0$ does not depend on ε or U_0. It follows that

$$|U_0|_{(\varepsilon,\theta)}^2 \geqslant \int_\theta^{\theta+1} \|\mathcal{V}_\varepsilon(t,\theta)U_0\|_{m-1,k}^2 \, dt \geqslant C_2^{-2} \|U_0\|_{m-1,k}^2,$$

which completes the proof of (2.6).

(2) Let us denote by $S_\varepsilon(\theta)$ the sphere in $\mathbb{E}_{m-1,k}$ of radius 1 (relative to the norm $|\cdot|_{(\varepsilon,\theta)}$) with centre zero:

$$S_\varepsilon(\theta) = \bigl\{U_0 \in \mathbb{E}_{m-1,k} \colon |U_0|_{(\varepsilon,\theta)} = 1\bigr\}.$$

We claim that for any non-zero solution $u(t,x)$ of problem (1.8), (1.1), there is a unique instant of time T such that

$$U(T) \in S_\varepsilon(T), \tag{2.8}$$

where $U(t) = \mathcal{D}(t)u$ is the phase trajectory corresponding to the solution $u(t,x)$ (see (1.3)). Indeed, by the inequality

$$|U(\theta)|^2_{(\varepsilon,\theta)} = \int_\theta^{+\infty} \|U(t)\|^2_{m-1,k}\,dt > \int_{\theta'}^{+\infty} \|U(t)\|^2_{m-1,k}\,dt = |U(\theta')|^2_{(\varepsilon,\theta')},$$

where $\theta < \theta'$, the function $|U(\theta)|_{(\varepsilon,\theta)}$ is monotone increasing. Moreover, (2.4) and (2.6) imply that

$$\lim_{\theta\to+\infty} |U(\theta)|_{(\varepsilon,\theta)} = 0, \qquad \lim_{\theta\to-\infty} |U(\theta)|_{(\varepsilon,\theta)} = +\infty.$$

Therefore there is a unique $T \in \mathbb{R}$ satisfying the condition $|U(T)|_{(\varepsilon,T)} = 1$, which is equivalent to (2.8).

For given ε, $\theta \in \mathbb{R}$ and $U_0 \in \mathbb{E}_{m-1,k}$, we denote by $T_\varepsilon(\theta, U_0)$ the T satisfying (2.8), where $U(t) = \mathcal{V}_\varepsilon(t,\theta)U_0$. We define the desired map (2.2) by the formula

$$L_\varepsilon(\theta, U_0) = \begin{cases} |\mathcal{V}_\varepsilon(T,\theta)U_0|^{-1}_{(0,T)}\,\mathcal{U}_0(\theta - T)\mathcal{V}_\varepsilon(T,\theta)U_0 & \text{for} \quad U_0 \neq 0, \\ 0 & \text{for} \quad U_0 = 0, \end{cases} \tag{2.9}$$

where $T = T_\varepsilon(\theta, U_0)$. Equivalently, if $U_0 \in \mathbb{E}_{m-1,k}$ lies on $S_\varepsilon(\theta)$, then $L_\varepsilon(\theta, U_0)$ is defined as the intersection point of the ray $\{\lambda U_0,\ \lambda > 0\}$ and the sphere[2] $S_0 = S_0(\theta)$. If the non-zero vector function $U_0 \in \mathbb{E}_{m-1,k}$ does not lie on $S_\varepsilon(\theta)$, then the map $L_\varepsilon(\theta, U_0)$ is defined in the following way: the vector $U(T)$, $T = T_\varepsilon(\theta, U_0)$, which lies on the trajectory passing through U_0 at time θ and satisfies (2.8) is mapped to the sphere S_0 using contraction or stretching, and the resulting function is acted on by the operator $\mathcal{U}_0(\theta - T)$, $= T_\varepsilon(\theta, U_0)$ (see Fig. 2, where $S_\varepsilon = S_\varepsilon(T)$, $A = \mathcal{V}_\varepsilon(T,\theta)U_0$, and $B = |\mathcal{V}_\varepsilon(T,\theta)U_0|^{-1}_{(0,T)}\mathcal{V}_\varepsilon(T,\theta)U_0$).

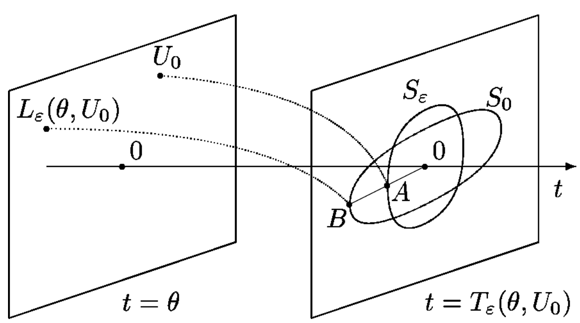

Figure 2. The map $L_\varepsilon(\theta, U_0)$

(3) We claim that the map (2.9) possesses all the required properties. To this end, we need three auxiliary lemmas.

[2]The norm $|\cdot|_{(\varepsilon,\theta)}$ does not depend on θ for $\varepsilon = 0$, and so neither does the sphere $S_0(\theta)$.

Lemma 2.4. (i) *The modulus of the difference* $T_\varepsilon(\theta, U_0) - \theta$ *is uniformly bounded for any* $\rho > 1$ *if* $\theta \in \mathbb{R}$ *and* $\rho^{-1} \leqslant \|U_0\|_{m-1,k} \leqslant \rho$.

(ii) *There are constants* $C > 0$ *and* $\varkappa > 0$ *not depending on* ε *or* θ *such that the inequality*

$$T_\varepsilon(\theta, U_0) \leqslant \theta + \varkappa^{-1} \ln \|U_0\|_{m-1,k} + C \tag{2.10}$$

holds if $T_\varepsilon(\theta, U_0) < \theta$.

Proof. (i) The function $T_\varepsilon(\theta, U_0)$ was defined as the unique solution of the equation

$$F_\varepsilon(T, \theta, U_0) := |\mathcal{V}_\varepsilon(T, \theta) U_0|^2_{(\varepsilon,T)} - 1 = 0. \tag{2.11}$$

Consequently, by (2.6) we have

$$K^{-1} \leqslant \|\mathcal{V}_\varepsilon(T, \theta) U_0\|_{m-1,k} \leqslant K, \quad T = T_\varepsilon(\theta, U_0). \tag{2.12}$$

Note that the norm $\|\mathcal{V}_\varepsilon(t, \theta) U_0\|_{m-1,k}$ tends to zero as $t - \theta \to +\infty$ and to infinity as $t - \theta \to -\infty$, and the convergence is uniform for $\rho^{-1} \leqslant \|U_0\|_{m-1,k} \leqslant \rho$ in both cases. These properties and formula (2.12) imply that the difference $T_\varepsilon(\theta, U_0) - \theta$ is uniformly bounded.

(ii) It follows from (2.7) that

$$\|\mathcal{V}_\varepsilon(t, \theta) U_0\|_{m-1,k} \leqslant C_2 e^{\varkappa(\theta - t)} \|U_0\|_{m-1,k}, \quad t \leqslant \theta.$$

Setting $t = T_\varepsilon(\theta, U_0)$ in this inequality and using (2.12), we obtain

$$-\ln K \leqslant \varkappa(\theta - T_\varepsilon(\theta, U_0)) + \ln C_2 + \ln \|U_0\|_{m-1,k},$$

whence follows the desired inequality (2.10).

Lemma 2.5. *The function* $T_\varepsilon(\theta, U_0)$ *is continuous in* $(\theta, U_0) \in \mathbb{R} \times \mathbb{E}_{m-1,k} \setminus \{0\}$.

Proof. Suppose that sequences $\{\theta_i\} \subset \mathbb{R}$ and $\{U_{i0}\} \subset \mathbb{E}_{m-1,k}$ converge to $\theta \in \mathbb{R}$ and $U_0 \in \mathbb{E}_{m-1,k}$, respectively. We set $T_i = T_\varepsilon(\theta_i, U_{i0})$. By Lemma 2.4, the difference $T_i - \theta_i$ is uniformly bounded with respect to i. Consequently, there is a subsequence $\{T_{i_k}\}$ converging to some limit $T' \in \mathbb{R}$. On setting $T = T_{i_k}$, $\theta = \theta_{i_k}$, and $U_0 = U_{i_k 0}$ in (2.11) and passing to the limit as $k \to \infty$, we conclude that T' is a solution of (2.11). By the uniqueness of this solution, it follows that $T' = T_\varepsilon(\theta, U_0)$ and that the entire sequence $\{T_i\}$ converges to T'.

Lemma 2.6. *For any fixed* ε, $\theta \in \mathbb{R}$ *and* $U_0 \in \mathbb{E}_{m-1,k} \setminus \{0\}$, *the relation*[3]

$$T_\varepsilon(\theta, U_0) = T_0(L_\varepsilon(\theta, U_0)) \tag{2.13}$$

holds

Proof. By definition, $T_0(L_\varepsilon(\theta, U_0))$ is the unique solution of the equation

$$|\mathcal{U}_0(T - \theta) L_\varepsilon(\theta, U_0)|_{(0,T)} = 1. \tag{2.14}$$

[3] The function $T_\varepsilon(\theta, U_0)$ does not depend on θ for $\varepsilon = 0$, and we denote it by $T_0(U_0)$.

By (2.9), we have

$$\mathcal{U}_0(T-\theta)L_\varepsilon(\theta,U_0)=|\mathcal{V}_\varepsilon(T,\theta)U_0|^{-1}_{(0,T)}\mathcal{V}_\varepsilon(T,\theta)U_0$$

for $T=T_\varepsilon(\theta,U_0)$. It remains to note that the right-hand side of this equation satisfies (2.14).

(4) Next, we show that the operator $L_\varepsilon(\theta,U_0)$ defined by formula (2.9) possesses all the required properties.

The continuity of $L_\varepsilon(\theta,U_0)$ for $U_0\neq 0$ is a simple consequence of Lemmas 2.3 and 2.5. Let us prove that $L_\varepsilon(\theta,U_0)$ is continuous at any point of the form $(\theta,0)$. For this, note that if $\|U_0\|_{m-1,k}\ll 1$, then $T=T_\varepsilon(\theta,U_0)<\theta$, and therefore, by (2.4), the inequality

$$\|\mathcal{U}_0(\theta-T)V_0\|_{m-1,k}\leqslant C_1e^{-\mu(\theta-T)}\|V_0\|_{m-1,k} \tag{2.15}$$

holds for any $V_0\in\mathbb{E}_{m-1,k}$. It follows from (2.6) that

$$|\mathcal{V}_\varepsilon(T,\theta)U_0|_{(0,T)}\geqslant K^{-1}\|\mathcal{V}_\varepsilon(T,\theta)U_0\|_{m-1,k}. \tag{2.16}$$

Substituting $V_0=\mathcal{V}_\varepsilon(T,\theta)U_0$ into (2.15) and taking (2.16) into account, we arrive at the estimate

$$\|L_\varepsilon(\theta,U_0)\|_{m-1,k}\leqslant C_1K\,e^{-\mu(\theta-T)}. \tag{2.17}$$

Formulae (2.10) and (2.17) imply that

$$\|L_\varepsilon(\theta,U_0)\|_{m-1,k}\leqslant C_1K\,\exp\bigl(\mu\varkappa^{-1}\ln\|U_0\|_{m-1,k}+\mu C\bigr)\leqslant C_3(\varkappa,\mu)\,\|U_0\|^{\mu/\varkappa}_{m-1,k},$$

and therefore $L_\varepsilon(\theta,U_0)$ is continuous at $(\theta,0)$.

We now complete the proof of Theorem 2.1 for the case of stable roots.

Proof. (i) The desired relation is trivial for $U_0=0$. Suppose that $U_0\neq 0$ and fix some arbitrary $t,\theta\in\mathbb{R}$. Direct verification shows that

$$T_\varepsilon(\theta,U_0)=T_\varepsilon(t,\mathcal{V}_\varepsilon(t,\theta)U_0). \tag{2.18}$$

Setting $T=T_\varepsilon(\theta,U_0)$ and $V_0=\mathcal{V}_\varepsilon(t,\theta)U_0$ and using the group property of the operators $\mathcal{V}_\varepsilon$ and relation (2.18), we find

$$\begin{aligned}\mathcal{U}_0(t-\theta)L_\varepsilon(\theta,U_0)&=|\mathcal{V}_\varepsilon(T,\theta)U_0|^{-1}_{(0,T)}\mathcal{U}_0(t-T)\mathcal{V}_\varepsilon(T,\theta)U_0\\&=|\mathcal{V}_\varepsilon(T,t)V_0|^{-1}_{(0,T)}\mathcal{U}_0(t-T)\mathcal{V}_\varepsilon(T,t)V_0\\&=L_\varepsilon(t,V_0).\end{aligned}$$

(ii) We define the operator

$$K_\varepsilon(\theta,U_0)=\begin{cases}|\mathcal{U}_0(T-\theta)U_0|^{-1}_{(\varepsilon,T)}\mathcal{V}_\varepsilon(\theta,T)\mathcal{U}_0(T-\theta)U_0 & \text{for}\quad U_0\neq 0,\\ 0 & \text{for}\quad U_0=0\end{cases}$$

by analogy with (2.9), where $T = T_0(U_0)$, and prove that it is the inverse of $L_\varepsilon(\theta, U_0)$ for any fixed θ.

Let us set $T = T_\varepsilon(\theta, U_0)$ and $T' = T_0(L_\varepsilon(\theta, U_0))$. By Lemma 2.7, we have $T = T'$. Therefore

$$\mathcal{U}_0(T'-\theta)L_\varepsilon(\theta, U_0) = |\mathcal{V}_\varepsilon(T,\theta)U_0|^{-1}_{(0,T)}\mathcal{V}_\varepsilon(T,\theta)U_0,$$
$$|\mathcal{U}_0(T'-\theta)L_\varepsilon(\theta, U_0)|_{(\varepsilon,T)} = |\mathcal{V}_\varepsilon(T,\theta)U_0|^{-1}_{(0,T)},$$

whence it follows that $K_\varepsilon(\theta, L_\varepsilon(\theta, U_0)) = U_0$. The relation $L_\varepsilon(\theta, K_\varepsilon(\theta, U_0)) = U_0$ is proved in a similar manner.

Thus, the operator $L_\varepsilon(\theta, \cdot)$ specifies a one-to-one map of the space $\mathbb{E}_{m-1,k}$ onto itself. Repeating literally the above argument, it is possible to prove that $L_\varepsilon^{-1}(\theta, U_0) = K_\varepsilon(\theta, U_0)$ is jointly continuous with respect to the variables (θ, U_0).

2.3. The property of exponential dichotomy. For $\mu, \theta \in \mathbb{R}$ and integers $k, l \geqslant 0$, we define the space

$$\mathbb{F}_{l,k,[\mu]}(\mathbb{R}_\pm(\theta)) = \bigl\{u(t,x)\colon e^{\mu t}\partial_t^j u \in C_b(\mathbb{R}_\pm(\theta),\, H^{(l+k-j)}),\ j = 0,\dots,l\bigr\},$$

where $\mathbb{R}_\pm(\theta) = [\theta, \pm\infty)$, and endow it with the norm

$$E_{l,k,[\mu]}(u, \mathbb{R}_\pm(\theta)) = \sup_{\pm(t-\theta)\geqslant 0} e^{\mu(t-\theta)} E_{l,k}(u,t),$$

where

$$E^2_{l,k}(u,t) = \sum_{j=0}^{l} \|\partial_t^j u(t,\cdot)\|^2_{(k+l-j)}.$$

Consider equation (1.8) with a strictly hyperbolic operator $P(\partial)$ satisfying Condition (H). We set $\mathbb{C}_\pm(\delta) := \bigl\{\tau \in \mathbb{C}\colon\ \pm\operatorname{Re}\tau \geqslant \delta\bigr\}$ and denote the numbers of the roots of the symbol $P(\tau,\xi)$ in the half-planes $\mathbb{C}_-(\sigma_{\min})$ and $\mathbb{C}_+(\sigma_{\min})$ by m_s and m_u, respectively.

Proposition 2.7. *Suppose that Conditions* (P), (Q), *and* (H) *are satisfied. Then for any* μ, $0 \leqslant \mu < \sigma_{\min}$, *and an arbitrary integer* $k \geqslant 0$, *there are positive constants* $\varepsilon_0 = \varepsilon_0(k,\mu)$ *and* $C = C(k,\mu)$ *such that the following assertions are true for* $|\varepsilon| \leqslant \varepsilon_0$.

(i) *For any vector function* $U_s = [u_0, \dots, u_{m_s-1}] \in \mathbb{E}_{m_s-1,k+m_u}$, *equation* (1.8) *has a unique solution* $u \in \mathbb{F}_{m-1,k,[\mu]}(\mathbb{R}_+(\theta))$ *satisfying the initial conditions*

$$\partial_t^j u(\theta,x) = u_j(x) \in H^{(m-1+k-j)}, \quad j = 0,\dots,m_s-1. \tag{2.19}$$

This solution satisfies the inequality

$$E_{m-1,k,[\mu]}(u, \mathbb{R}_+(\theta)) \leqslant C \sum_{j=0}^{m_s-1} \|u_j\|_{(m-1+k-j)}. \tag{2.20}$$

Similarly, for an arbitrary vector function $U_u = [u_0, \dots, u_{m_u-1}] \in \mathbb{E}_{m_u-1,k+m_s}$, *equation* (1.8) *has a unique solution* $u \in \mathbb{F}_{m-1,k,[-\mu]}(\mathbb{R}_-(\theta))$ *satisfying the initial conditions*

$$\partial_t^j u(\theta, x) = u_j(x) \in H^{(m-1+k-j)}, \quad j = 0, \dots, m_u - 1. \tag{2.21}$$

This solution satisfies the inequality

$$E_{m-1,k,[-\mu]}(u, \mathbb{R}_-(\theta)) \leqslant C \sum_{j=0}^{m_u-1} \|u_j\|_{(m-1+k-j)}. \tag{2.22}$$

(ii) *The operator*

$$\mathcal{V}_\varepsilon^s(t, \theta) \colon \mathbb{E}_{m_s-1,k+m_u} \to \mathbb{E}_{m-1,k} \tag{2.23}$$

transforming U_s *into the vector function* (1.3), *where* $u \in \mathbb{F}_{m-1,k,[\mu]}(\mathbb{R}_+(\theta))$ *is the solution of problem* (1.8), (2.19), *is continuous with respect to* (t, θ, U_s). *Similarly, the operator*

$$\mathcal{V}_\varepsilon^u(t, \theta) \colon \mathbb{E}_{m_u-1,k+m_s} \to \mathbb{E}_{m-1,k} \tag{2.24}$$

transforming U_u *into the vector function* (1.3), *where* $u \in \mathbb{F}_{m-1,k,[-\mu]}(\mathbb{R}_-(\theta))$ *is the solution of problem* (1.8), (2.21), *is continuous in* (t, θ, U_u).

Assertion (i) of Proposition 2.7 was established in [3], Theorem 4.1. The continuity of (2.23) and (2.24) is proved in the same way as assertion (iii) in Proposition 4.2 (see § 4).

For an arbitrary $\theta \in \mathbb{R}$, we denote by $\mathbb{E}^s_{m-1,k}(\theta)$ and $\mathbb{E}^u_{m-1,k}(\theta)$ the corresponding stable and unstable subspaces of $\mathbb{E}_{m-1,k}$. By definition, they consist of the vector functions $[u_0, \dots, u_{m-1}]$ satisfying the respective relations

$$[u_0, \dots, u_{m-1}] = \mathcal{V}_\varepsilon^s(\theta, \theta)[u_0, \dots, u_{m_s-1}],$$
$$[u_0, \dots, u_{m-1}] = \mathcal{V}_\varepsilon^u(\theta, \theta)[u_0, \dots, u_{m_u-1}].$$

The assertion below was established in [3], Theorem 5.3.

Proposition 2.8. *Under the hypotheses of Proposition* 2.7, *the direct decomposition*

$$\mathbb{E}_{m-1,k} = \mathbb{E}^s_{m-1,k}(\varepsilon, \theta) \dotplus \mathbb{E}^u_{m-1,k}(\varepsilon, \theta) \tag{2.25}$$

holds for any $\theta \in \mathbb{R}$ *and* ε, $|\varepsilon| \leqslant \varepsilon_0$. *In this case, the projections* $\mathcal{P}^s_\varepsilon(\theta)$ *and* $\mathcal{P}^u_\varepsilon(\theta)$ *corresponding to* (2.25) *are continuous with respect to* θ *in the strong operator topology.*[4] *and their norms are uniformly bounded with respect to* (θ, ε).

[4]Recall that a sequence of linear maps $L_k \colon X \to Y$ (where X and Y are Banach spaces) converges to zero in the strong operator topology if $L_k u \to 0$ in Y for any $u \in X$.

2.4. Proof of Theorem 2.1. The general case. We begin by presenting the scheme of the proof. The operator $\mathcal{V}_\varepsilon^s(t,\theta)$ defined for $t \geqslant \theta$ can be extended to the half-line $t \leqslant \theta$ by setting

$$\mathcal{V}_\varepsilon^s(t,\theta) = \mathcal{V}_\varepsilon(t,\theta)\mathcal{V}_\varepsilon^s(\theta,\theta), \quad t \leqslant \theta. \tag{2.26}$$

The operator $\mathcal{V}_\varepsilon^u(t,\theta)$ can similarly be extended to $t \geqslant \theta$. By the direct decomposition (2.25), the operator $\mathcal{V}_\varepsilon(t,\theta)$ can be represented in the form

$$\mathcal{V}_\varepsilon(t,\theta) = \mathcal{V}_\varepsilon^s(t,\theta)\mathcal{Q}_{m_s}\mathcal{P}_\varepsilon^s(\theta) + \mathcal{V}_\varepsilon^u(t,\theta)\mathcal{Q}_{m_u}\mathcal{P}_\varepsilon^u(\theta), \tag{2.27}$$

where we set

$$\mathcal{Q}_l \colon \mathbb{E}_{m-1,k} \to \mathbb{E}_{l-1,k+m-l}, \quad [u_0,\dots,u_{m-1}] \mapsto [u_0,\dots,u_{l-1}$$

for $l = 1,\dots,m$. The first and second terms on the right-hand side of (2.27) correspond to the dynamics on the families of stable and unstable subspaces $\mathbb{E}_{m-1,s}^s(\theta)$ and $\mathbb{E}_{m-1,s}^u(\theta)$, respectively. Note that

$$\mathcal{V}_\varepsilon^s(t,t)\mathcal{Q}_{m_s}U_0 = U_0 \quad \text{for} \quad U_0 \in \mathbb{E}_{m-1,k}^s(t), \quad t \in \mathbb{R}, \tag{2.28}$$

whence

$$\mathcal{V}_\varepsilon^s(t,\theta)\mathcal{Q}_{m_s} = \mathcal{V}_\varepsilon^s(t,t)\bigl[\mathcal{Q}_{m_s}\mathcal{V}_\varepsilon^s(t,\theta)\bigr]\mathcal{Q}_{m_s}.$$

Hence, the dynamics on the stable and unstable subspaces is specified by the family of operators $\mathcal{Q}_{m_s}\mathcal{V}_\varepsilon^s(t,\theta) \colon \mathbb{E}_{m_s-1,k+m_u} \to \mathbb{E}_{m_s-1,k+m_u}$, which is exponentially asymptotically stable as $t-\theta \to +\infty$,

$$\|\mathcal{Q}_{m_s}\mathcal{V}_\varepsilon^s(t,\theta)U_s\|_{m_s-1,k+m_u} \leqslant C\,e^{-\mu(t-\theta)}\|U_s\|_{m_s-1,k+m_u}, \quad t \geqslant \theta. \tag{2.29}$$

Repeating the arguments in § 2.2, we can construct a family of continuous operators

$$L_\varepsilon^s(\theta,U_s) \colon \mathbb{R}\times\mathbb{E}_{m_s-1,k+m_u} \to \mathbb{E}_{m_s-1,k+m_u} \tag{2.30}$$

satisfying the relation

$$\mathcal{Q}_{m_s}\mathcal{U}_0^s(t-\theta)L_\varepsilon^s(\theta,U_s) = L_\varepsilon^s(t,\mathcal{Q}_{m_s}\mathcal{V}_\varepsilon^s(t,\theta)U_s), \quad t,\theta \in \mathbb{R}, \tag{2.31}$$

where $\mathcal{U}_0^s(t)$ denotes the operator $\mathcal{V}_\varepsilon^s(t,0)|_{\varepsilon=0}$. A similar family

$$L_\varepsilon^u(\theta,U_u) \colon \mathbb{R}\times\mathbb{E}_{m_u-1,k+m_s} \to \mathbb{E}_{m_u-1,k+m_s} \tag{2.32}$$

can be constructed for the operators $\mathcal{Q}_{m_u}\mathcal{V}_\varepsilon^u(t,\theta)$ specifying the dynamics on the unstable subspaces. The desired map (2.2) is defined as the "sum" of the operators L_ε^s and L_ε^u,

$$L_\varepsilon(\theta,U_0) = \mathcal{U}_0^s(0)L_\varepsilon^s(\theta,\mathcal{Q}_{m_s}\mathcal{P}_\varepsilon^s(\theta)U_0) + \mathcal{U}_0^u(0)L_\varepsilon^u(\theta,\mathcal{Q}_{m_u}\mathcal{P}_\varepsilon^u(\theta)U_0), \tag{2.33}$$

where $\mathcal{U}_0^u(t) = \mathcal{V}_\varepsilon^u(t,0)|_{\varepsilon=0}$. All the required properties can easily be verified.

We now proceed to the details of the proof, for which the auxiliary assertion below will be needed.

Proposition 2.9. *Let Conditions* (P), (Q), *and* (H) *be fulfilled. Then for an arbitrary integer* $k \geqslant 0$, *there is a constant* $\varepsilon_0 = \varepsilon_0(k) > 0$ *and a family of continuous maps* L^s_ε (*see* (2.30)) *such that the identity* $L^s_\varepsilon(\theta, 0) \equiv 0$ *holds and the following assertions are true for* $|\varepsilon| \leqslant \varepsilon_0$.

(i) *Relation* (2.31) *holds for any* $t, \theta \in \mathbb{R}$ *and* $U_s \in \mathbb{E}_{m_s-1,k+m_u}$.

(ii) *For an arbitrary fixed* $\theta \in \mathbb{R}$, *the operator* $L^s_\varepsilon(\theta, \cdot)$ *specifies a homeomorphic map of* $\mathbb{E}_{m_s-1,k+m_u}$ *onto itself. Moreover, the inverse operator*

$$K^s_\varepsilon(\theta, U_s) \colon \mathbb{R} \times \mathbb{E}_{m_s-1,k+m_u} \to \mathbb{E}_{m_s-1,k+m_u}$$

is jointly continuous in the variables (θ, U_s).

Remark 2.10. (1) We do not give a proof of Proposition 2.9 since the scheme given in § 2.2 for $\mathcal{V}_\varepsilon$ applies word-for-word to the operators $\mathcal{Q}_{m_s}\mathcal{V}^s_\varepsilon$. Indeed, the proof of Theorem 2.1 in the case of stable roots was based on the continuity of the operator $\mathcal{V}_\varepsilon(t, \theta)U_0$ with respect to (t, θ, U_0) and on inequalities (2.4) and (2.7). The joint continuity of the operator $\mathcal{Q}_{m_s}\mathcal{V}^s_\varepsilon$ with respect to the variables (t, θ, U_s) was established in Proposition 2.7. Inequality (2.29) is an analogue of (2.4), and (2.26) and (2.7) imply that

$$\|\mathcal{Q}_{m_s}\mathcal{V}^s_\varepsilon(t, \theta)U_s\|_{m_s-1,k+m_u} \leqslant C\, e^{\varkappa(\theta-t)}\|U_s\|_{m_s-1,k+m_u}, \quad t \leqslant \theta.$$

(2) A similar assertion is true for $\mathcal{Q}_{m_u}\mathcal{V}^u_\varepsilon(t, \theta)$, and we denote the corresponding operators by L^u_ε and K^u_ε.

Let us define the operator L_ε by formula (2.33). Then the continuity of $L_\varepsilon(\theta, U_0)$ follows from that of the operators on the right-hand side of (2.33), and the relation $L_\varepsilon(\theta, 0) \equiv 0$ is obvious.

Proof. (i) Applying the operator $\mathcal{U}^s_0(0)$ to (2.31) and using (2.28) with $\varepsilon = 0$, we obtain

$$\mathcal{U}^s_0(t-\theta)L^s_\varepsilon(\theta, U_s) = \mathcal{U}^s_0(0)L^s_\varepsilon(t, \mathcal{Q}_{m_s}\mathcal{V}^s_\varepsilon(t, \theta)U_s). \tag{2.34}$$

Furthermore, we note that

$$\mathcal{V}^s_\varepsilon(t, \theta)\mathcal{Q}_{m_s}\mathcal{P}^s_\varepsilon(\theta) = \mathcal{V}_\varepsilon(t, \theta)\mathcal{P}^s_\varepsilon(\theta) = \mathcal{P}^s_\varepsilon(t)\mathcal{V}_\varepsilon(t, \theta).$$

Setting $U_s = \mathcal{Q}_{m_s}\mathcal{P}^s_\varepsilon(\theta)U_0$ in (2.34) and using (2.26), we arrive at the relation

$$\begin{aligned} \mathcal{U}_0(t-\theta)\mathcal{U}^s_0(0)L^s_\varepsilon(\theta, \mathcal{Q}_{m_s}\mathcal{P}^s_\varepsilon(\theta)U_0) &= \mathcal{U}^s_0(t-\theta)L^s_\varepsilon(\theta, \mathcal{Q}_{m_s}\mathcal{P}^s_\varepsilon(\theta)U_0) \\ &= \mathcal{U}^s_0(0)L^s_\varepsilon(\theta, \mathcal{Q}_{m_s}\mathcal{V}^s_\varepsilon(t, \theta)\mathcal{Q}_{m_s}\mathcal{P}^s_\varepsilon(\theta)U_0) \\ &= \mathcal{U}^s_0(0)L^s_\varepsilon(\theta, \mathcal{Q}_{m_s}\mathcal{V}_\varepsilon(t, \theta)\mathcal{P}^s_\varepsilon(\theta)U_0) \\ &= \mathcal{U}^s_0(0)L^s_\varepsilon(\theta, \mathcal{Q}_{m_s}\mathcal{P}^s_\varepsilon(t)\mathcal{V}_\varepsilon(t, \theta)U_0). \end{aligned} \tag{2.35}$$

Similarly,

$$\mathcal{U}_0(t-\theta)\mathcal{U}^u_0(0)L^u_\varepsilon(\theta, \mathcal{Q}_{m_u}\mathcal{P}^u_\varepsilon(\theta)U_0) = \mathcal{U}^u_0(0)L^u_\varepsilon(\theta, \mathcal{Q}_{m_u}\mathcal{P}^u_\varepsilon(t)\mathcal{V}_\varepsilon(t, \theta)U_0). \tag{2.36}$$

Adding together (2.35) and (2.36) and taking (2.33) into account, we derive the desired relation.

(ii) By analogy with (2.33), we set

$$K_\varepsilon(\theta, U_0) = \mathcal{U}_\varepsilon^s(\theta,\theta) K_\varepsilon^s(\theta, \mathcal{Q}_{m_s}\mathcal{P}_0^s U_0) + \mathcal{U}_\varepsilon^u(\theta,\theta) K_\varepsilon^u(\theta, \mathcal{Q}_{m_u}\mathcal{P}_0^u U_0), \tag{2.37}$$

where K_ε^s and K_ε^u are the operators defined in Proposition 2.9 and Remark 2.10 and $\mathcal{P}_0^s$ and $\mathcal{P}_0^u$ denote the projections $\mathcal{P}_\varepsilon^s(\theta)$ and $\mathcal{P}_\varepsilon^u(\theta)$ for $\varepsilon = 0$. It is easy to see that the operator (2.37) from $\mathbb{R} \times \mathbb{E}_{m-1,k}$ to $\mathbb{E}_{m-1,k}$ is jointly continuous with respect to the variables (θ, U_0). Hence, assertion (ii) will be proved if we can show that $L_\varepsilon(\theta,\cdot)$ and $K_\varepsilon(\theta,\cdot)$ are mutually inverse maps for any θ.

For example, let us verify the relation

$$K_\varepsilon(\theta, L_\varepsilon(\theta, U_0)) = U_0, \quad U_0 \in \mathbb{E}_{m-1,k}. \tag{2.38}$$

For this purpose, we note that

$$\mathcal{P}_0^s L_\varepsilon(\theta, U_0) = \mathcal{U}_0^s(0) L_\varepsilon^s(\theta, \mathcal{Q}_{m_s}\mathcal{P}_\varepsilon^s(\theta) U_0).$$

This property together with assertion (ii) in Proposition 2.9 and the relation $\mathcal{Q}_{m_s}\mathcal{U}_0^s(0)U_s = U_s$ implies that

$$K_\varepsilon^s(\theta, \mathcal{Q}_{m_s}\mathcal{P}_0^s L_\varepsilon(\theta, U_0)) = K_\varepsilon^s(\theta, L_\varepsilon^s(\theta, \mathcal{Q}_{m_s}\mathcal{P}_\varepsilon^s(\theta)U_0)) = \mathcal{Q}_{m_s}\mathcal{P}_\varepsilon^s(\theta)U_0,$$

and therefore, by (2.28),

$$\mathcal{V}_\varepsilon^s(\theta,\theta) K_\varepsilon^s(\theta, \mathcal{Q}_{m_s}\mathcal{P}_0^s L_\varepsilon(\theta, U_0)) = \mathcal{P}_\varepsilon^s(\theta)U_0.$$

Similarly,

$$\mathcal{V}_\varepsilon^u(\theta,\theta) K_\varepsilon^u(\theta, \mathcal{Q}_{m_u}\mathcal{P}_0^u L_\varepsilon(\theta, U_0)) = \mathcal{P}_\varepsilon^u(\theta)U_0.$$

Adding together these relations, we obtain (2.38).

The relation

$$L_\varepsilon(\theta, K_\varepsilon(\theta, U_0)) = U_0, \quad U_0 \in \mathbb{E}_{m-1,k},$$

can be proved in just the same simple manner. Theorem 2.1 is proved completely.

§ 3. Proof of the main results

This section is devoted to proving Theorems 1.3 and 1.4. We first present an auxiliary assertion on global linearization for an equation with truncated non-linearity and then show that Theorems 1.3 and 1.4 are a simple consequence of this and Theorem 2.1.

3.1. Reducing Theorems 1.3 and 1.4 to linearizing an equation with truncated non-linearity. We write the function q (see (0.5)) in the form

$$q(\varepsilon, t, x, \partial^{m-1}u) = q(\varepsilon, t, x, \partial_x^{m-1}u, \partial_x^{m-2}\partial_t u, \dots, \partial_t^{m-1}u),$$

where $\partial_x^k = \{\partial_x^\alpha : |\alpha| \leqslant k\}$, and set

$$Q_\rho(\varepsilon,t,U) = \chi\bigl(\rho^2 - \|U\|^2_{m-1,k}\bigr) q(\varepsilon,t,x,\partial_x^{m-1}u_0,\partial_x^{m-2}u_1,\dots,u_{m-1}) \tag{3.1}$$

for given $\rho > 0$ and $k \geqslant 0$, where $U = [u_0,\dots,u_{m-1}] \in \mathbb{E}_{m-1,k}$, $\chi(s) \in C^\infty(\mathbb{R})$, $\chi(s) = 0$ for $s \leqslant -1$, and $\chi(s) = 1$ for $s \geqslant 0$. The properties of the function Q_ρ needed in what follows are listed in Proposition 4.1.

Let us consider the following equation obtained from (0.3) by replacing the nonlinearity $q(\varepsilon,t,x,\partial^{m-1}u)$ by $Q_\rho(\varepsilon,t,U(t))$, where $U(t) = \mathcal{D}(t)u$:

$$P_\varepsilon(t,x,\partial)u + \varepsilon Q_\rho(\varepsilon,t,U(t)) = 0. \tag{3.2}$$

By Proposition 4.2 stated below, the Cauchy problem for equation (3.2) with $|\varepsilon| \ll 1$ is well posed, that is, for any $\theta \in \mathbb{R}$ and an arbitrary vector function $[u_0,\dots,u_{m-1}] \in \mathbb{E}_{m-1,k}$, there is a unique solution of (3.2) satisfying the relations

$$\partial_t^j u \in C(\mathbb{R}, H^{(m-1+k-j)}), \quad j = 0,\dots,m-1, \tag{3.3}$$

and the initial conditions

$$\partial_t^j u(\theta,x) = u_j(x) \in H^{(m-1+k-j)}, \quad j = 0,\dots,m-1. \tag{3.4}$$

We write the resolving process for problem (3.2)–(3.4) (see Proposition 4.2) in the form

$$\mathcal{U}_\varepsilon^\rho(t,\theta,U_0)\colon \mathbb{R}_t \times \mathbb{R}_\theta \times \mathbb{E}_{m-1,k} \to \mathbb{E}_{m-1,k}.$$

To prove Theorems 1.3 and 1.4, we need the following assertion. Recall the numbers $\sigma_{\min}$ and $\sigma_{\max}$ from (1.9).

Theorem 3.1. *Let Conditions* (P), (Q), *and* (H) *hold. Then for any $\rho > 0$ and γ, $0 < \gamma < \sigma_{\min}/\sigma_{\max}$, and an arbitrary integer $k > n/2$, there is a constant $\varepsilon_0 = \varepsilon_0(k,\gamma,\rho) > 0$ and a family of continuous operators*

$$N_{\varepsilon,\rho}(\theta,U_0)\colon \mathbb{R} \times \mathbb{E}_{m-1,k} \to \mathbb{E}_{m-1,k}, \quad |\varepsilon| \leqslant \varepsilon_0, \tag{3.5}$$

such that $N_{\varepsilon,\rho}(\theta,0) \equiv 0$ and the following assertions hold for $|\varepsilon| \leqslant \varepsilon_0$.

(i) *$\mathcal{V}_\varepsilon(t,\theta)N_{\varepsilon,\rho}(\theta,U_0) = N_{\varepsilon,\rho}(t,\mathcal{U}_\varepsilon^\rho(t,\theta,U_0))$ for all $t,\theta \in \mathbb{R}$ and $U_0 \in \mathbb{E}_{m-1,k}$.*

(ii) *The operator $N_{\varepsilon,\rho}(\theta,\cdot)$ specifies a one-to-one map of $\mathbb{E}_{m-1,k}$ onto itself for any fixed $\theta \in \mathbb{R}$. Moreover, the operator $N_{\varepsilon,\rho}(\theta,\cdot)$ and its inverse $N_{\varepsilon,\rho}^{-1}(\theta,\cdot)$ are Hölder continuous with exponent γ.*

(iii) *The inverse operator $N_{\varepsilon,\rho}^{-1}(\theta,U_0)\colon \mathbb{R}_\theta \times \mathbb{E}_{m-1,k} \to \mathbb{E}_{m-1,k}$ is jointly continuous in the variables (θ,U_0).*

Theorem 3.1 will be proved in §§ 3.2 and 3.3.

Proof of Theorem 1.4. Let us fix some arbitrary $\rho > 0$ and γ, $0 < \gamma < \sigma_{\min}/\sigma_{\max}$, and an arbitrary integer $k > n/2$ and take a sufficiently small constant $\varepsilon_0 > 0$ such that the assertion of Theorem 3.1 holds for $|\varepsilon| \leqslant \varepsilon_0$. We define the operator Ψ_ε as the restriction of $N_{\varepsilon,\rho}$ to the set $\mathbb{R} \times \mathbb{B}_{m-1,k}(\rho)$,

$$\Psi_\varepsilon(\theta,U_0) = N_{\varepsilon,\rho}(\theta,U_0), \quad \theta \in \mathbb{R}, \quad U_0 \in \mathbb{B}_{m-1,k}(\rho).$$

Clearly, we have $\Psi_\varepsilon(\theta,0)\equiv 0$, and the operator $\Psi_\varepsilon(\theta,U_0)$ is jointly continuous with respect to its variables and is invertible for any fixed θ, and the inverse operator Ψ_ε^{-1} coincides with the restriction of $N_{\varepsilon,\rho}^{-1}$ to the image $\mathbb{W}_\varepsilon(\theta,\rho)$ of the ball $\mathbb{B}_{m-1,k}(\rho)$ under the map $N_{\varepsilon,\rho}(\theta,\cdot)$. Consequently, the operator $\Psi_\varepsilon^{-1}(\theta,U_0)$ is also jointly continuous with respect to its variables. Since equations (0.3) and (3.2) coincide in the ball $\mathbb{B}_{m-1,k}(\rho)$, the desired assertion (i) follows from Theorem 3.1 (i). The set $\mathbb{W}_\varepsilon(\theta,\rho)$ is the image of an open ball under a homeomorphism and therefore is also open. Hölder continuity of the operator $\Psi_\varepsilon(\theta,\cdot)$ and its inverse follows from analogous properties of $N_{\varepsilon,\rho}$. It remains to note that the set $\mathbb{R}_\theta\times\mathbb{W}_\varepsilon(\theta,\rho)$ is the inverse image of the open ball $\mathbb{B}_{m-1,k}(\rho)$ under the continuous map $N_{\varepsilon,\rho}^{-1}$ and therefore is open.

Proof of Theorem 1.3. We fix some arbitrary $\rho>0$ and γ, $0<\gamma<\sigma_{\min}/\sigma_{\max}$, and an arbitrary integer $k>n/2$ and choose a sufficiently small constant $\varepsilon_0>0$ such that the assertions of Theorems 2.1 and 3.1 hold for $|\varepsilon|\leqslant\varepsilon_0$. Let us define the operator Φ_ε as the restriction of the composite of L_ε and $N_{\varepsilon,\rho}$ to the set $\mathbb{R}\times\mathbb{B}_{m-1,k}(\rho)$,

$$\Phi_\varepsilon(\theta,U_0)=L_\varepsilon(\theta,N_{\varepsilon,\rho}(\theta,U_0)),\quad \theta\in\mathbb{R},\quad U_0\in\mathbb{B}_{m-1,k}(\rho).$$

As in the proof of Theorem 4.1, all the required properties can be verified easily.

Remark 3.2. The proof of Theorem 3.1 will be used to show that the operator $N_{\varepsilon,\rho}$ satisfies the inequality

$$\|N_{\varepsilon,\rho}(\theta,U_0)-U_0\|_{m-1,k}\leqslant \text{const}\ll 1\quad\text{for all}\quad \theta\in\mathbb{R},\quad U_0\in\mathbb{E}_{m-1,k}. \tag{3.6}$$

An example from the theory of ordinary differential equations shows that the estimate for the Hölder exponent γ is exact in the class of operators satisfying condition (3.6). Namely, let us consider the system

$$\begin{aligned}\dot u_1&=u_1+\varepsilon q(u_2),\\ \dot u_2&=u_3,\\ \dot u_3&=4u_3+4u_2,\end{aligned} \tag{3.7}$$

where $\varepsilon>0$ and

$$q\in C_0^\infty(\mathbb{R}),\quad q\geqslant 0,\quad q(u)=u^2\quad\text{for}\quad |u|\leqslant 1. \tag{3.8}$$

In this case, we have $\sigma_{\min}=1$ and $\sigma_{\max}=2$. Let $\mathcal{U}_\varepsilon(t,u^0)\colon\mathbb{R}^3\to\mathbb{R}^3$, $t\in\mathbb{R}$, $u^0\in\mathbb{R}$, denote the resolving operator in the Cauchy problem for the system (3.7). As shown in [30], § 4, there is a unique homomorphism $N\colon\mathbb{R}^3\to\mathbb{R}^3$ for $|\varepsilon|\ll 1$ that satisfies the conditions

$$\begin{gathered}\mathcal{U}_0(t)N(u^0)=N(\mathcal{U}_\varepsilon(t,u^0)),\quad t\in\mathbb{R},\quad u^0\in\mathbb{R}^3,\\ |N(u^0)-u^0|\leqslant\text{const}\ll 1,\quad u^0\in\mathbb{R}^3,\end{gathered}$$

where $|\cdot|$ denotes the norm in $\mathbb{R}^3$. We claim that N does not satisfy Hölder's condition with exponent $\gamma = 1/2$. Indeed, according to [2], § 34, the operator N has the form

$$N(u^0) = u^0 + \varepsilon \int_0^{+\infty} \mathcal{U}_0(-\tau)[q(u_2(\tau, u^0)), 0, 0]\, d\tau, \tag{3.9}$$

where $u_2(\tau, u^0)$ is the second component of the vector function $\mathcal{U}_\varepsilon(t, u^0)$. In particular, setting $u^0 = (1/2)[0, 0, v]$, $0 < v \ll 1$, we can write

$$N_1(u^0) = \varepsilon \int_0^{+\infty} e^{-\tau} q(\tau e^{2\tau} v)\, d\tau$$

for the first component of (3.9). Suppose that N satisfies Hölder's condition with exponent $\gamma = \frac{1}{2}$. Then

$$|N_1(u^0)| \leqslant \operatorname{const} |v|^{1/2}. \tag{3.10}$$

On the other hand, denoting the unique solution of the equation $\tau e^{2\tau} v = 1$ by $h(v) > 0$, we readily conclude that

$$h(v) + \frac{1}{2} \ln v \to +\infty \quad \text{as} \quad v \to +0. \tag{3.11}$$

Therefore, by (3.8), we have

$$N_1(u^0) \geqslant \varepsilon \int_0^{h(v)} e^{-\tau} (\tau e^{2\tau} v)^2\, d\tau \geqslant \frac{\varepsilon}{6} (v h(v))^{1/2}.$$

The resulting inequality contradicts (3.10).

3.2. Proof of Theorem 3.1. We shall need an auxiliary assertion (see Proposition 3.3 below). To state it, we take equation (3.2) with the function Q_ρ replaced by another function R_ρ of the form (3.1),

$$P_\varepsilon(t, x, \partial) v + \varepsilon R_\rho(\varepsilon, t, V(t)) = 0. \tag{3.12}$$

Here $V(t) = V(t, \cdot) = \mathcal{D}(t) v$ and

$$R_\rho(\varepsilon, t, V) = \chi\big(\rho^2 - \|V\|^2_{m-1,k}\big) r(\varepsilon, t, x, \partial_x^{m-1} v_0, \partial_x^{m-2} v_1, \dots, v_{m-1}), \tag{3.13}$$

where $V = [v_0, \dots, v_{m-1}] \in \mathbb{E}_{m-1,k}$ and $r(\varepsilon, t, x, z)$ is a smooth function. As in the case of equation (3.2), we consider the solutions of equation (3.12) that satisfy the conditions

$$\partial_t^j v \in C(\mathbb{R}, H^{(m-1+k-j)}), \quad j = 0, \dots, m-1. \tag{3.14}$$

For a given $\mu \geqslant 0$ and arbitrary integers $k, l \geqslant 0$, we define the space

$$\mathbb{F}_{l,k,[\mu,-\mu]} = \big\{u(t, x) \colon e^{-\mu|t|} \partial_t^j u \in C_b(\mathbb{R}, H^{(l+k-j)}),\ j = 0, \dots, l\big\} \tag{3.15}$$

equipped with the norm

$$E_{l,k,[\mu,-\mu]}(u) = \sup_{t \in \mathbb{R}} e^{-\mu|t|} E_{l,k}(u, t). \tag{3.16}$$

If $\mu = 0$, then we write $\mathbb{F}_{l,k}$ and $E_{l,k}(u)$ instead of $\mathbb{F}_{l,k,[\mu,-\mu]}$ and $E_{l,k,[\mu,-\mu]}(u)$.

Proposition 3.3. *Let Conditions* (P), (Q), *and* (H) *hold and let the function* $r(\varepsilon, t, x, z)$ *in* (3.13) *satisfy the same conditions as* $q(\varepsilon, t, x, z)$. *Then for any* $\rho > 0$, $\varkappa > \sigma_{\max}$, *and* μ, $0 < \mu < \sigma_{\min}$, *and an arbitrary integer* $k > n/2$, *there are constants* $\varepsilon_0 > 0$ *and* $C > 0$ *such that the following assertions are true for* $|\varepsilon| \leqslant \varepsilon_0$.

(i) *For any* $\theta \in \mathbb{R}$ *and* $U_0 = [u_0, \dots, u_{m-1}] \in \mathbb{E}_{m-1,k}$, *there is a unique solution* $v(t,x) =: \mathcal{F}_{Q_\rho, R_\rho}(\theta, U_0)$ *of* (3.12), (3.14) *such that*

$$v \in \mathbb{F}_{m-1,k,[\varkappa,-\varkappa]}, \quad u - v \in \mathbb{F}_{m-1,k}, \tag{3.17}$$

where $u(t,x)$ *is the solution of* (3.2)–(3.4).

(ii) *Let* $v_i = \mathcal{F}_{Q_\rho, R_\rho}(\theta_i, U_{i0})$, $i = 1, 2$, *where* $\theta_i \in \mathbb{R}$ *and* $U_{i0} \in \mathbb{E}_{m-1,k}$. *Then*

$$E_{m-1,k,[\varkappa,-\varkappa]}(v_1 - v_2) \leqslant C\big(\|U_{10} - U_{20}\|_{m-1,k} + \|U_{10} - U_{20}\|^{\gamma}_{m-1,k}\big), \tag{3.18}$$

where $\gamma = \mu/\varkappa$.

(iii) *Let sequences* $\{\theta_i\} \subset \mathbb{R}$ *and* $\{U_{i0}\} \subset \mathbb{E}_{m-1,k}$ *converge to the respective limits* $\theta \in \mathbb{R}$ *and* $U_0 \in \mathbb{E}_{m-1,k}$ *and let* $v_i = \mathcal{F}_{Q_\rho, R_\rho}(\theta_i, U_{i0})$ *and* $v = \mathcal{F}_{Q_\rho, R_\rho}(\theta, U_0)$. *Then*

$$E_{m-1,k,[\varkappa,-\varkappa]}(v_i - v) \to 0 \quad \textit{for} \quad i \to \infty. \tag{3.19}$$

We assume that Proposition 3.3 is established and complete[5] the proof of Theorem 3.1.

Let us define the desired map (3.5) by the formula

$$N_{\varepsilon,\rho}(\theta, U_0) = \mathcal{D}(\theta)\big(\mathcal{F}_{Q_\rho, R_\rho}(\theta, U_0)|_{R_\rho \equiv 0}\big),$$

where $\mathcal{D}(\theta)$ is the operator defined in (1.3). The relation $N_{\varepsilon,\rho}(\theta, 0) \equiv 0$ is obvious. By assertion (iii) in Proposition 3.3, the operator $N_{\varepsilon,\rho}$ is jointly continuous with respect to the variables (θ, U_0).

Proof of (i). Let us fix some arbitrary θ and $U_0 = [u_0, \dots, u_{m-1}] \in \mathbb{E}_{m-1,k}$ and denote by $u(t,x)$ the solution of problem (3.2)–(3.4). We set $v(t,x) = \mathcal{F}_{Q_\rho, R_\rho}(\theta, U_0)|_{R_\rho \equiv 0}$. Let $U(t)$ and $V(t)$ be the phase trajectories corresponding to the solutions $u(t,x)$ and $v(t,x)$. It follows from assertion (i) of Proposition 3.3 that

$$v = \mathcal{F}_{Q_\rho, R_\rho}(t, U(t))|_{R_\rho \equiv 0}, \quad t \in \mathbb{R},$$

and therefore

$$V(t) = \mathcal{D}(t)\big(\mathcal{F}_{Q_\rho, R_\rho}(\theta, U_0)|_{R_\rho \equiv 0}\big) = \mathcal{D}(t)\big(\mathcal{F}_{Q_\rho, R_\rho}(t, U(t))|_{R_\rho \equiv 0}\big),$$

whence the desired relation follows.

Proof of (ii) *and* (iii). Simple verification shows that

$$N^{-1}_{\varepsilon,\rho}(\theta, U_0) := \mathcal{D}(\theta)\big(\mathcal{F}_{R_\rho, Q_\rho}(\theta, U_0)|_{R_\rho \equiv 0}\big)$$

is the inverse operator of $N_{\varepsilon,\rho}(\theta, \cdot)$ for any fixed $\theta \in \mathbb{R}$. Consequently, Hölder continuity of the operators $N_{\varepsilon,\rho}(\theta, \cdot)$ and $N^{-1}_{\varepsilon,\rho}(\theta, \cdot)$ follows from inequality (3.18), and the joint continuity of $N^{-1}_{\varepsilon,\rho}$ with respect to the variables (θ, U_0) from assertion (iii) of Proposition 3.3.

[5] We repeat the argument used in [28].

3.3. Proof of Proposition 3.3. (i) We seek the function $v(t,x)$ in the form

$$v = u + w, \quad w \in \mathbb{F}_{m-1,k}. \tag{3.20}$$

Substituting (3.20) into (3.12) and taking (3.2) into consideration, we derive the equation

$$P_\varepsilon(t,x,\partial)w = \varepsilon M(U,W), \quad M(U,W) = Q_\rho(\varepsilon,t,U) - R_\rho(\varepsilon,t,U+W) \tag{3.21}$$

for $w(t,x)$, where $U = \mathcal{D}(t)u$ and $W = \mathcal{D}(t)w$. Using the contraction mapping principle, we see that equation (3.21) is uniquely soluble for $|\varepsilon| \ll 1$ in the space $\mathbb{F}_{m-1,k}$.

Let us consider the operator A transforming the function $z \in \mathbb{F}_{m-1,k}$ into the solution $w \in \mathbb{F}_{m-1,k}$ of the equation

$$P_\varepsilon(t,x,\partial)w = \varepsilon M(U,Z), \quad Z = \mathcal{D}(t)z. \tag{3.22}$$

We show that the operator A is well defined and transforms the space $\mathbb{F}_{m-1,k}$ into itself. Indeed, by Proposition 4.1 (i), if $z \in \mathbb{F}_{m-1,k}$, then $M(U,Z) \in \mathbb{F}_{0,k}$. Therefore, according to Proposition 4.3 with $\mu = 0$, equation (3.22) has a unique solution $w \in \mathbb{F}_{m-1,k}$ for $|\varepsilon| \ll 1$.

We now prove that A is a contraction map. Let $w_i = A(z_i)$, $i = 1,2$. Then the function $w = w_1 - w_2 \in \mathbb{F}_{m-1,k}$ satisfies the equation

$$P_\varepsilon(t,x,\partial)w = \varepsilon\big(M(U,Z_2) - M(U,Z_1)\big),$$

where $Z_i = \mathcal{D}(t)z_i$, $i = 1,2$. Consequently, by inequality (4.8) with $\mu = \theta = 0$, we have

$$E_{m-1,k}(w) \leqslant C_1|\varepsilon|\, E_{0,k}(M(U,Z_1) - M(U,Z_2)). \tag{3.23}$$

Application of inequality (4.2) with $\gamma = 1$ to the right-hand side of (3.23) results in

$$E_{0,k}(M(U,Z_1) - M(U,Z_2)) \leqslant C_2 E_{m-1,k}(z_1 - z_2).$$

Comparing this estimate with (3.23), we arrive at the inequality

$$E_{m-1,k}(A(z_1) - A(z_2)) \leqslant C_1 C_2 |\varepsilon|\, E_{m-1,k}(z_1 - z_2),$$

whence it follows that $A(z)$ is a contraction map.

We have thus established the existence and uniqueness of the solution $w \in \mathbb{F}_{m-1,k}$ of equation (3.21), which implies the desired assertion.

(ii) We claim that if $v_i = \mathcal{F}_{Q_\rho,R_\rho}(\theta, U_{i0})$, $i = 1,2$, then

$$E_{m-1,k,[\mu,-\mu]}(S(\theta)(w_1 - w_2)) \leqslant C_3 \|U_{10} - U_{20}\|^\gamma_{m-1,k}, \tag{3.24}$$

where $w_i = v_i - u_i$ and $u_i(t,x)$ is the solution of problem (3.2)–(3.4) with Cauchy data U_{i0}. Indeed, let us set $w = w_1 - w_2$, $U_i = \mathcal{D}(t)u_i$, and $W_i = \mathcal{D}(t)w_i$. Then the function $w(t,x) \in \mathbb{F}_{m-1,k,[\mu,-\mu]}$ is a solution of the equation

$$P_\varepsilon(t,x,\partial)w = \varepsilon h(t,x), \quad h(t,x) = R_\rho(\varepsilon,t,U_2 + W_2) - R_\rho(\varepsilon,t,U_1 + W_1). \tag{3.25}$$

By inequality (4.2),

$$\|h(t,\cdot)\|_k \leqslant C_4\big(\|U_2(t,\cdot)-U_1(t,\cdot)\|^{\gamma}_{m-1,k}+\|W_2(t,\cdot)-W_1(t,\cdot)\|_{m-1,k}\big),$$

whence it follows that $h\in\mathbb{F}_{0,k,[\mu,-\mu]}$ and

$$\begin{aligned}E_{0,k,[\mu,-\mu]}(S(\theta)h) &\leqslant C_4\sup_{t\in\mathbb{R}}\big(e^{-\mu|t-\theta|}\big[\|U_2(t,\cdot)-U_1(t,\cdot)\|^{\gamma}_{m-1,k}\\ &\qquad+\|W_2(t,\cdot)-W_1(t,\cdot)\|_{m-1,k}\big]\big)\\ &\leqslant C_4\Big(E^{\gamma}_{m-1,k,[\varkappa,-\varkappa]}\big(S(\theta)(u_2-u_1)\\ &\qquad+E_{m-1,k,[\mu,-\mu]}(S(\theta)w)\big)\Big).\end{aligned}\tag{3.26}$$

According to (4.6),

$$E_{m-1,k,[\varkappa,-\varkappa]}(S(\theta)(u_2-u_1))\leqslant K_2\|U_{20}-U_{10}\|_{m-1,k}.\tag{3.27}$$

Substituting (3.27) into (3.26), we obtain

$$E_{0,k,[\mu,-\mu]}(S(\theta)h)\leqslant C_5\big(\|U_{20}-U_{10}\|^{\gamma}_{m-1,k}+E_{m-1,k,[\mu,-\mu]}(S(\theta)w)\big).\tag{3.28}$$

By Proposition 4.3, the solution $w\in\mathbb{F}_{m-1,k,[\mu,-\mu]}$ of (3.25) satisfies the inequality

$$E_{m-1,k,[\mu,-\mu]}(S(\theta)w)\leqslant K_3|\varepsilon|E_{0,k,[\mu,-\mu]}(S(\theta)h).\tag{3.29}$$

Comparing (3.28) and (3.29), we arrive at the inequality

$$E_{m-1,k,[\mu,-\mu]}(S(\theta)w)\leqslant K_3C_5|\varepsilon|\,\big(\|U_{20}-U_{10}\|^{\gamma}_{m-1,k}+E_{m-1,k,[\mu,-\mu]}(S(\theta)w)\big),$$

whence follows (3.24) for $|\varepsilon|\ll 1$.

Next, we prove (3.18). By the representation (3.20) for the functions v_i, $i=1,2$,

$$\begin{aligned}&E_{m-1,k,[\varkappa,-\varkappa]}(S(\theta)(v_1-v_2))\\ &\qquad\leqslant E_{m-1,k,[\varkappa,-\varkappa]}(S(\theta)(u_1-u_2))+E_{m-1,k,[\mu,-\mu]}(S(\theta)(w_1-w_2)).\end{aligned}$$

Therefore the desired estimate is a consequence of (3.24) and (3.27).

(iii) We denote the solution of problem (3.2)–(3.4) with Cauchy data U_0 by $u(t,x)$ and the solution of the same problem with the initial point θ_i and Cauchy data U_{i0} by $u_i(t,x)$. In view of Proposition 4.2, it follows that

$$U_i(\theta)\to U(\theta)=U_0\quad\text{in}\quad\mathbb{E}_{m-1,k}\quad\text{for}\quad i\to\infty,\tag{3.30}$$

where $U_i(t)=\mathcal{D}(t)u_i$ and $U(t)=\mathcal{D}(t)u$. Since the solution $v(t,x)$ of (3.12), (3.14) that satisfies conditions (3.17) is unique, we have

$$v_i=\mathcal{F}_{Q_\rho,R_\rho}(\theta,U_i(\theta))\quad\text{for all}\quad i.$$

Therefore, by (3.18),

$$E_{m-1,k,[\varkappa,-\varkappa]}(S(\theta)(v_i-v))\leqslant C\big(\|U_i(\theta)-U_0\|_{m-1,k}+\|U_i(\theta)-U_0\|^{\gamma}_{m-1,k}\big).$$

Comparing this inequality with (3.30), we obtain (3.19). The proof of Proposition 3.3 is complete.

§ 4. Appendix

4.1. Truncated non-linearity. Recall the function $Q_\rho(\varepsilon, t, U)$ from formula (3.1).

Proposition 4.1. *The assertions below hold for any $\varepsilon \in [-1, 1]$ and $\rho > 0$ and an arbitrary integer $k > n/2$.*

(i) *The operator $Q_\rho(\varepsilon, t, U)\colon \mathbb{R}_t \times \mathbb{E}_{m-1,k} \to H^{(k)}$ is infinitely Frechét differentiable, and all the derivatives are uniformly bounded with respect to $(\varepsilon, t, U) \in [-1, 1] \times \mathbb{R}_t \times \mathbb{E}_{m-1,k}$ for any fixed $\rho > 0$.*

(ii) *For any γ, $0 < \gamma \leqslant 1$, there is a $K_1 = K_1(\rho, k, \gamma) > 0$ such that*

$$\|Q_\rho(\varepsilon, t, U)\|_{(k)} \leqslant K_1 \min\bigl\{\rho, \|U\|_{m-1,k}\bigr\}, \tag{4.1}$$

$$\|Q_\rho(\varepsilon, t, U) - Q_\rho(\varepsilon, t, V)\|_{(k)} \leqslant K_1\bigl(\|U_1 - V_1\|_{m-1,k} + \|U_2 - V_2\|^\gamma_{m-1,k}\bigr), \tag{4.2}$$

where $U, V \in \mathbb{E}_{m-1,k}$, $U = U_1 + U_2$, and $V = V_1 + V_2$.

Proof. We confine ourselves to proving (4.2) since assertions of type (i) are well known in theory of partial differential equations (for example, see [7], Chapter 7, § 3). As to inequality (4.1), it follows readily from the property that the function Q_ρ is compactly supported and its derivative is uniformly bounded.

For $\gamma = 1$ (4.2) is a simple consequence of the mean value formula. It is not obvious that any number in the interval $(0, 1)$ can serve as γ.

(4.2) is trivial for $\min\bigl\{\|U\|_{m-1,k}, \|V\|_{m-1,k}\bigr\} \geqslant \rho$. Therefore it can be assumed that $\|U\|_{m-1,k} \leqslant \rho$. We first suppose that $\|V\|_{m-1,k} \leqslant 3\rho$. Let Q'_ρ denote the derivative of Q_ρ with respect to U. Then, by the mean value theorem and the uniform boundedness of Q'_ρ, we have

$$\|Q_\rho(\varepsilon, t, U) - Q_\rho(\varepsilon, t, V)\|_{(k)} \leqslant C_1 \|U - V\|_{m-1,k}. \tag{4.3}$$

We note that if $\|U_2 - V_2\|_{m-1,k} \leqslant 1$, then

$$\begin{aligned} \|U - V\|_{m-1,k} &\leqslant \|U_1 - V_1\|_{m-1,k} + \|U_2 - V_2\|_{m-1,k} \\ &\leqslant \|U_1 - V_1\|_{m-1,k} + \|U_2 - V_2\|^\gamma_{m-1,k}, \end{aligned} \tag{4.4}$$

and if $\|U_2 - V_2\|_{m-1,k} \geqslant 1$, then

$$\|U - V\|_{m-1,k} \leqslant 4\rho \leqslant 4\rho\bigl(\|U_1 - V_1\|_{m-1,k} + \|U_2 - V_2\|^\gamma_{m-1,k}\bigr). \tag{4.5}$$

Comparing (4.3)–(4.5), we obtain (4.2).

We now suppose that $\|V\|_{m-1,k} \geqslant 3\rho$. Then $\|U - V\|_{m-1,k} \geqslant 2\rho$, and therefore[6]

$$\|U_1 - V_1\|_{m-1,k} + \|U_2 - V_2\|^\gamma_{m-1,k} \geqslant \rho^\gamma.$$

This together with (4.1) implies that

$$\begin{aligned} \|Q_\rho(\varepsilon, t, U) - Q_\rho(\varepsilon, t, V)\|_{(k)} &= \|Q_\rho(\varepsilon, t, U)\|_{(k)} \leqslant C_2 \rho \\ &\leqslant C_2 \rho^{1-\gamma}\bigl(\|U_1 - V_1\|_{m-1,k} + \|U_2 - V_2\|^\gamma_{m-1,k}\bigr). \end{aligned}$$

The proposition is proved.

[6] Without loss of generality, we assume that $\rho \geqslant 1$.

4.2. The Cauchy problem with truncated non-linearity. Recall the number $\sigma_{\max}$ from (1.9).

Proposition 4.2. *Let Conditions* (P) *and* (Q) *be fulfilled. Then for any* $\rho > 0$ *and* $\varkappa > \sigma_{\max}$ *and an arbitrary integer* $k > n/2$, *there are constants* $\varepsilon_0 > 0$ *and* $K_2 > 0$ *such that the following assertions are true for* $|\varepsilon| \leqslant \varepsilon_0$.

(i) *For any* $\theta \in \mathbb{R}$ *and an arbitrary set of Cauchy data* $[u_0, \dots, u_{m-1}] \in \mathbb{E}_{m-1,k}$, *problem* (3.2)–(3.4) *has a unique solutions* $u(t, x)$. *If* $u_i(t, x)$, $i = 1, 2$, *are two solutions of this problem that correspond to Cauchy data* $U_{i0} \in \mathbb{E}_{m-1,k}$, *then*

$$E_{m-1,k}(u_1 - u_2, t) \leqslant K_2\, e^{\varkappa|t-\theta|} \|U_{10} - U_{20}\|_{m-1,k}. \tag{4.6}$$

(ii) *The operator* $\mathcal{U}_\varepsilon^\rho(t, \theta, U_0) \colon \mathbb{R}_t \times \mathbb{R}_\theta \times \mathbb{E}_{m-1,k} \to \mathbb{E}_{m-1,k}$ *transforming* $U_0 = [u_0, \dots, u_{m-1}]$ *into the vector function* (1.3) *is infinitely Frechét differentiable with respect to* U_0 *for any fixed* t *and* θ.

(iii) *The operator* $\mathcal{U}_\varepsilon^\rho$ *is continuous in* $(t, \theta, U_0) \in \mathbb{R}_t \times \mathbb{R}_\theta \times \mathbb{E}_{m-1,k}$.

Proof. We shall prove only assertion (iii) since (i) and (ii) are consequences of general results on ordinary differential equations in a Banach space.

Let sequences $\{t_i\}, \{\theta_i\} \subset \mathbb{R}$ and $\{U_{i0}\} \subset \mathbb{E}_{m-1,k}$ converge to the respective limits $t, \theta \subset \mathbb{R}$ and $U_0 \subset \mathbb{E}_{m-1,k}$. We denote the solution of problem (3.2)–(3.4) by $u(t, x)$ and the corresponding phase trajectory by $U(t)$ (see (1.3)). Then, by the group property of the operators $\mathcal{U}_\varepsilon^\rho(t, \theta, U_0)$, we have

$$\begin{aligned}\mathcal{U}_\varepsilon^\rho(t_i, \theta_i, U_{i0}) - \mathcal{U}_\varepsilon^\rho(t, \theta, U_0) = \bigl(&\mathcal{U}_\varepsilon^\rho(t_i, \theta_i, U_{i0}) - \mathcal{U}_\varepsilon^\rho(t_i, \theta_i, U_0)\bigr) \\ &+ \bigl(\mathcal{U}_\varepsilon^\rho(t_i, \theta_i, U_0) - \mathcal{U}_\varepsilon^\rho(t_i, \theta_i, U(\theta_i))\bigr) + \bigl(U(t_i) - U(t)\bigr).\end{aligned}$$

Application of (4.6) yields the inequality

$$\begin{aligned}&\|\mathcal{U}_\varepsilon^\rho(t_i, \theta_i, U_{i0}) - \mathcal{U}_\varepsilon^\rho(t, \theta, U_0)\|_{m-1,k} \\ &\quad\leqslant \mathrm{const}\bigl(\|U_{i0} - U_0\|_{m-1,k} + \|U(\theta) - U(\theta_i)\|_{m-1,k}\bigr) + \|U(t_i) - U(t)\|_{m-1,k},\end{aligned}$$

whose right-hand side tends to zero as $i \to \infty$.

4.3. Hyperbolic operators in function spaces with exponential weight with respect to t. We consider the non-homogeneous equation

$$P_\varepsilon(t, x, \partial)u \equiv \sum_{|\alpha| \leqslant m} \bigl(p_\alpha + \varepsilon q_\alpha(\varepsilon, t, x)\bigr)\partial^\alpha u = f(t, x). \tag{4.7}$$

Recall the space $\mathbb{F}_{l,k,[\mu,-\mu]}$ and the norm $E_{l,k,[\mu,-\mu]}(u)$ from (3.15) and (3.16). The following assertion was proved in [3], Theorem 2.7.

Proposition 4.3. *Suppose that Conditions* (P), (Q), *and* (H) *hold. Then, for any* μ, $0 \leqslant \mu < \sigma_{\min}$, *and an arbitrary integer* $k \geqslant 0$, *there are positive constants* $\varepsilon_0 = \varepsilon_0(k, \mu)$ *and* $K_3 = K_3(k, \mu)$ *such that equation* (4.7) *possesses a unique solution* $u \in \mathbb{F}_{m-1,k,[\mu,-\mu]}$ *for an arbitrary right-hand side* $f \in \mathbb{F}_{0,k,[\mu,-\mu]}$ *if* $|\varepsilon| \leqslant \varepsilon_0$. *This solution satisfies the inequality*

$$E_{m-1,k,[\mu,-\mu]}(S(\theta)u) \leqslant K_3 E_{0,k,[\mu,-\mu]}(S(\theta)f), \quad \theta \in \mathbb{R}, \tag{4.8}$$

where $S(\theta)$ *is the shift operator with respect to* t (*that is,* $S(\theta)w(t, x) = w(t+\theta, x)$).

PART II. THE CENTRE MANIFOLD THEOREM

§ 5. Main results and scheme of proofs

5.1. Statement of results. We consider equation (0.3) with operators P and Q satisfying Conditions (P) and (Q) in § 1.1. Recall that the phase space $\mathbb{E}_{m-1,k}$ of (0.3) is defined as the direct product of the Sobolev spaces $H^{(m-1+k-j)}$, $j = 0, \dots, m-1$, with the norm

$$\|U\|_{m-1,k} = \biggl(\sum_{j=0}^{m-1} \|u_j\|^2_{(m-1+k-j)}\biggr)^{1/2}, \quad U = [u_0, \dots, u_{m-1}],$$

and $\mathbb{B}_{m-1,k}(\rho)$ denotes an open ball of radius $\rho > 0$ with centre zero in $\mathbb{E}_{m-1,k}$. Also recall the resolving operator of the Cauchy problem for equation (0.3) from § 1.1 (see (1.4)).

For an operator $P(\partial)$ satisfying Condition (H_c) (see Introduction), the number of roots of the full symbol $P(\tau, \xi)$ that lie in the strip $|\operatorname{Re}\tau| \leqslant \nu$ is denoted by m_c. We set $m_h = m - m_c$. The following theorem is the main result in the second part of this paper.

Theorem 5.1. *Let Conditions* (P), (Q), *and* (H_c) *hold. Let* $\gamma \in \mathbb{R}$ *and an integer* $l \geqslant 1$ *satisfy the inequalities*

$$0 < \gamma < 1, \quad l\nu + \gamma < \delta. \tag{5.1}$$

Then, for an arbitrary integer $k > n/2$ *and any* $\rho > 0$ *and* $\mu \in (\nu, \delta/l)$, *there are constants* $\varepsilon_0 > 0$ *and* $C > 0$ *and a family of continuous operators*[7]

$$\mathcal{R}_j(\varepsilon; \theta, u_0, \dots, u_{m_c-1}) \colon \mathbb{R}_\theta \times \mathbb{B}_{m_c-1,k+m_h}(\rho) \to H^{(m-1+k-j)}, \quad j = m_c, \dots, m-1,$$

such that $\mathcal{R}_j(\varepsilon; \theta, 0) = 0$ *and the following assertions are true for* $|\varepsilon| \leqslant \varepsilon_0$.

(i) *Local invariance. The family of manifolds*

$$\begin{aligned} \mathcal{M}(\theta, \rho) = \bigl\{[u_0, \dots, u_{m-1}] \in \mathbb{B}_{m-1,k}(\rho) \colon u_j \\ = \mathcal{R}_j(\varepsilon; \theta, u_0, \dots, u_{m_c-1}),\ j \geqslant m_c\bigr\} \end{aligned} \tag{5.2}$$

is compatible with the action of the resolving operator $\mathcal{U}_\varepsilon(t, \theta, \cdot)$. *In other words, if* $U_0 \in \mathcal{M}(\theta, \rho)$ *and* $\mathcal{U}_\varepsilon(t, \theta, U_0) \in \mathbb{B}_{m-1,k}(\rho)$ *for* $t \in J$ *and an interval* $J \subset \mathbb{R}$ *containing the point* θ, *then* $\mathcal{U}_\varepsilon(t, \theta, U_0) \in \mathcal{M}(t, \rho)$ *for* $t \in J$.

(ii) *Attraction property. Let the initial point* θ *and Cauchy data* $U_0 \in \mathbb{B}_{m-1,k}(\rho)$ *satisfy the condition* $\mathcal{U}_\varepsilon(t, \theta, U_0) \in \mathbb{B}_{m-1,k}(\rho_1)$ *for* $t \geqslant \theta$ *and some* $\rho_1 < \rho$. *Then there are* $T \geqslant \theta$ *and* $V_0 \in \mathcal{M}(T, \rho)$ *such that*

$$\|\mathcal{U}_\varepsilon(t, \theta, U_0) - \mathcal{U}_\varepsilon(t, T, V_0)\|_{m-1,k} \leqslant C e^{-\mu(t-\theta)}, \quad t \geqslant T. \tag{5.3}$$

A similar assertion is true if the phase trajectory $\mathcal{U}_\varepsilon(t, \theta, U_0)$ *belongs to* $\mathbb{B}_{m-1,k}(\rho_1)$ *on the semi-axis* $t \leqslant \theta$. *Moreover, if the phase trajectory* $\mathcal{D}(t)u$ *of a solution*

[7] The number ε is regarded as a parameter.

defined throughout the time axis is entirely contained in the ball $\mathbb{B}_{m-1,k}(\rho)$, *then* $\mathcal{D}(t)u \in \mathcal{M}(t,\rho)$ *for all* $t \in \mathbb{R}$.

(iii) *Smoothness. For any fixed* ε *and* θ, *the operator* $\mathcal{R}_j(\varepsilon;\theta,\cdot)$ *belongs to the class* $C^{l,\gamma}(\mathbb{B}_{m_c-1,k+m_h}(\rho), H^{(m-1+k-j)})$ *and the norm* $\|\mathcal{R}_j\|_{C^{l,\gamma}}$ *is uniformly bounded with respect to* (ε,θ).

We now proceed to describe the operators $\mathcal{R}_j(\varepsilon;\theta,\cdot)$. For this we need a class of symbols of pseudo-differential operators. Such symbols appear in the factorization of strictly hyperbolic polynomials satisfying Condition (H_c).

Let S^j denote the set of functions $p(\varepsilon,y,\xi)$ that are defined and infinitely differentiable for $(\varepsilon,y,\xi) \in [-1,1] \times \mathbb{R}_y^{n+1} \times i\mathbb{R}^n$ and satisfy the following conditions:

(i)

$$[p]_{j,\alpha,\beta} := \sup_{(\varepsilon,y,\xi)} |\partial_\xi^\alpha \partial_z^\beta p(\varepsilon,y,\xi)| \langle\xi\rangle^{|\alpha|-j} < \infty$$

for any multi-indices α and β, where $\langle\xi\rangle = \bigl(1+|\xi_1|^2+\cdots+|\xi_n|^2\bigr)^{1/2}$;

(ii) there is a function $p^0(\varepsilon,y,\xi) \in C^\infty([-1,1]\times\mathbb{R}_y^{n+1}\times(i\mathbb{R}^n\setminus\{0\}))$ positively homogeneous of order j with respect to ξ such that

$$[p-\chi p^0]_{j-1,\alpha,\beta} < \infty \quad \text{for any multi-indices} \quad \alpha,\beta,$$

where $\chi(\xi) \in C^\infty(i\mathbb{R}^n)$, $\chi(\xi)=0$ for $|\xi|<1$, and $\chi(\xi)=1$ for $|\xi|>2$.

With every symbol $p(\varepsilon,y,\xi)$ we associate the pseudo-differential operator

$$p(\varepsilon,t,x,\partial_x)u(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\zeta} p(\varepsilon,t,x,i\zeta)\hat{u}(\zeta)\,d\zeta,$$

where $\hat{u}(\zeta)$ denotes the Fourier transform

$$\hat{u}(\zeta) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\zeta}u(x)\,dx$$

of the function $u(x)$.

Consider the symbol

$$P_\varepsilon(t,x,\tau,\xi) = \sum_{|\alpha|\leqslant m}(p_\alpha + \varepsilon q_\alpha(\varepsilon,t,x))\,\eta^\alpha, \quad \eta=(\tau,\xi). \tag{5.4}$$

According to [32], Theorem 3.10, the symbol (5.4) can be factorized for $|\varepsilon| \ll 1$,

$$P_\varepsilon(y,\tau,\xi) = P_c(\varepsilon,y,\tau,\xi)\,P_c'(\varepsilon,y,\tau,\xi),$$

where P_c and P_c' have the form

$$P_c(\varepsilon,y,\tau,\xi) = \sum_{j=0}^{m_c}(p_{cj}(\xi)+\varepsilon q_{cj}(\varepsilon,y,\xi))\tau^{m_c-j}, \quad p_{cj},q_{cj}\in S^j,$$

$$P_c'(\varepsilon,y,\tau,\xi) = \sum_{j=0}^{m_h}(p_{cj}'(\xi)+\varepsilon q_{cj}'(\varepsilon,y,\xi))\tau^{m_h-j}, \quad p_{cj}',q_{cj}'\in S^j.$$

For $\varepsilon = 0$, the roots of P_c lie in the strip $|\operatorname{Re}\tau| \leqslant \nu$ and those of P_c' outside the strip $|\operatorname{Re}\tau| < \delta$. In what follows, we shall assume that the coefficient of P_c in τ^{m_c} is identically equal to unity. The symbols P_c and P_c' are uniquely defined by this assumption.

We denote by

$$R_j(\varepsilon, t, x, \tau, \xi) = \sum_{i=0}^{m_c-1} r_{ij}(\varepsilon, t, x, \xi)\tau^i, \quad j = m_c, \dots, m-1,$$

the remainder on dividing the polynomial τ^j by $P_c(\varepsilon, t, x, \tau, \xi)$. It is easy to show that

$$r_{ij}(\varepsilon, y, \xi) \in S^{j-i}, \quad i = 0, \dots, m_c - 1,$$

and therefore the corresponding pseudo-differential operator $r_{ij}(\varepsilon, t, x, \partial_x)$ defines a continuous map of $H^{(m-1+k-i)}$ into $H^{(m-1+k-j)}$.

Theorem 5.2. *Let the assumptions of Theorem* 5.1 *hold. Then for each* ε *there are continuous linear operators*

$$\mathcal{B}_j(\varepsilon; \theta, u_0, \dots, u_{m_c-1})\colon \mathbb{R}_\theta \times \mathbb{B}_{m_c-1,k+m_h}(\rho) \to H^{(m-1+k-j)}, \ j = m_c, \dots, m-1,$$

such that $\mathcal{B}_j(\varepsilon; \theta, 0) = 0$ *and the representation*

$$\mathcal{R}_j(\varepsilon; \theta, u_0, \dots, u_{m_c-1}) = \\ = \sum_{i=0}^{m_c-1} r_{ij}(\varepsilon, \theta, x, \partial_x)u_i(x) + \varepsilon\mathcal{B}_j(\varepsilon; \theta, u_0, \dots, u_{m_c-1}) \tag{5.5}$$

holds for $|\varepsilon| \leqslant \varepsilon_0$. *Furthermore, the following assertions are true.*

(i) *The operators* $\mathcal{B}_j$ *satisfy a Lipschitz condition with respect to* $[u_0, \dots, u_{m_c-1}]$ *for any fixed* $\theta \in \mathbb{R}$, *with Lipschitz constant uniformly bounded with respect to* (ε, θ).

(ii) *If the non-linear term* $q(\varepsilon, y, \partial^{m-1}u)$ (*see* (0.5)) *does not depend on the derivatives* $\partial^\alpha u$ *for* $|\alpha| = m - 1$, *then* $\mathcal{B}_j$ *are continuous operators from* $\mathbb{R}_\theta \times \mathbb{B}_{m_c-1,k+m_h}(\rho)$ *to* $H^{(m+k-j)}$, *and assertion* (i) *with* $H^{(m-1+k-j)}$ *replaced by* $H^{(m+k-j)}$ *holds for them.*

Theorems 5.1 and 5.2 will be proved in § 7. Here we give only sketches of the proofs (see § 5.2).

Definition. The manifold

$$\mathcal{M}_\rho = \bigl\{[\theta, u_0, \dots, u_{m-1}] \in \mathbb{R} \times \mathbb{B}_{m-1,k}(\rho)\colon [u_0, \dots, u_{m-1}] \in \mathcal{M}(\theta, \rho)\bigr\} \tag{5.6}$$

embedded in the extended phase space $\mathbb{R} \times \mathbb{E}_{m-1,k}$ is called the *centre manifold* of equation (0.3).

By the property of local invariance (see Theorem 5.1), the neighborhood of any point of the centre manifold $\mathcal{M}_\rho$ consists of integral curves of problem (0.3), (1.2). In this connection, the problem of describing the dynamics on $\mathcal{M}_\rho$ arises. The *reduction principle* below provides a partial solution of this problem.

Theorem 5.3. *The following assertions are true under the assumptions of Theorem* 5.1 *for sufficiently small values of* ε.

(i) *Let the solution* $u(t,x)$ *of problem* (0.3), (1.2) *satisfy the condition* $\mathcal{D}(t)u \in \mathcal{M}(t,\rho)$ *for* $t \in J$. *Then* $u(t,x)$ *satisfies the equation*

$$P_c(\varepsilon,t,x,\partial)u - \varepsilon\mathcal{B}_{m_c}(\varepsilon;t,u,\partial_t u,\dots,\partial_t^{m_c-1}u) = 0. \tag{5.7}$$

(ii) *Suppose that the non-linear term* $q(\varepsilon,t,x,\partial^{m-1}u)$ *does not depend on the derivatives* $\partial^\alpha u$, $|\alpha| = m-1$. *Let a function* $u(t,x)$ *for which*

$$\partial_t^j u \in C(I, H^{(m-1+k-j)}), \quad j = 0,\dots,m_c-1, \tag{5.8}$$

where $I \subset \mathbb{R}$ *is an interval, satisfy the conditions that the vector function*

$$\mathcal{D}_c(t)u := [u(t,\cdot), \partial_t u(t,\cdot), \dots, \partial_t^{m_c-1}u(t,\cdot)]$$

lies entirely in $\mathbb{B}_{m_c-1,k+m_h}(\rho)$ *and equation* (5.7) *holds. Let* $I_1 \subset I$ *be an arbitrary interval on which the curve*

$$[u(t,\cdot),\dots,\partial_t^{m_c-1}u(t,\cdot), u_{m_c}(t,\cdot),\dots,u_{m-1}(t,\cdot)],$$

where $u_j = \mathcal{R}_j(\varepsilon;t,u,\dots,\partial_t^{m_c-1}u)$ *for* $j \geqslant m_c$, *is contained in the ball* $\mathbb{B}_{m-1,k}(\rho)$. *Then* $u(t,x)$ *is the solution of problem* (0.3), (1.2) *with* $J = I_1$, *and* $\mathcal{D}(t)u \in \mathcal{M}(t,\rho)$ *for* $t \in I_1$.

Remark 5.4. By Theorem 5.3, if the non-linear term q in (0.5) does not depend on the derivatives of order $m-1$, then the dynamics on the centre manifold is described by problem (5.7), (5.8). We note that the left-hand side of (5.7) is a small Lipschitzian perturbation of the strictly hyperbolic operator $P_c(\varepsilon,y,\partial)$, and therefore the Cauchy problem for equation (5.7) is well posed (for example, see [19]). The question of an "explicit" description of the dynamics on the centre manifold in the general case remains open. If q depends on the highest derivatives, then the perturbation $\varepsilon\mathcal{B}_{m_c}$ in equation (5.7) is of order $m_c = \operatorname{ord} P_c$ and, in view of the hyperbolicity, is not subordinate to the principal linear part.

5.2. Sketches of the proofs of Theorems 5.1 and 5.2. In this subsection, we present the main ideas used in constructing the centre manifold and studying its properties. For detailed proofs, see §§ 6 and 7.

Passage to an equation with truncated non-linearity. We note that all assertions in Theorems 5.1 and 5.2 relate to solutions whose phase trajectories are contained in the ball $\mathbb{B}_{m-1,k}(\rho)$. Therefore the original equation (0.3) can be replaced by an equation with truncated non-linearity. Namely, repeating the constructions in § 3.1, we define the function $Q_\rho(\varepsilon,t,U)$ for a given $\rho > 0$ and an arbitrary integer $k > n/2$ using formula (3.1). Instead of (0.3), we shall study the equation

$$P_\varepsilon(t,x,\partial)u + \varepsilon Q_\rho(\varepsilon,t,\mathcal{D}(t)u) = 0, \tag{5.9}$$

where the symbol $P_\varepsilon(t,x,\eta)$ and the vector function $\mathcal{D}(t)u$ are defined in (5.4) and (1.3), respectively. We shall show (Theorem 7.1) that the "global versions" of the assertions in Theorems 5.1 and 5.2 are true for equation (5.9). Since equations (0.3) and (5.9) coincide on solutions whose phase trajectories are contained in $\mathbb{B}_{m-1,k}(\rho)$, this will imply Theorems 5.1 and 5.2.

Initial-value problem with growth conditions at infinity. Following the standard scheme for constructing the centre manifold (for example, see [24], [26], [33]), we consider the following initial-value problem for equation (5.9) with a given $\theta\in\mathbb{R}$:

$$\partial_t^j u\in C(\mathbb{R},H^{(m-1+k-j)}),\qquad j=0,\dots,m-1,\tag{5.10}$$

$$\partial_t^j u(\theta,x)=u_j(x)\in H^{(m-1+k-j)},\quad j=0,\dots,m_c-1.\tag{5.11}$$

We have set $m_c<m$ initial conditions for an equation of order m. The missing $m_h=m-m_c$ conditions are replaced by a constraint on the growth rate as $|t|\to\infty$. Namely, it is assumed that the energy $E_{m-1,k}(u,t)$ of the solution grows no faster than $e^{\mu|t|}$, where $\nu<\mu<\delta$. In other words, we seek a solution belonging to the class $\mathbb{F}_{m-1,k,[\mu,-\mu]}$ (see (3.15)). It will be shown (Theorem 6.1) that problem (5.9)–(5.11) has a unique solution $u(y)\in\mathbb{F}_{m-1,k,[\mu,-\mu]}$ for $|\varepsilon|\ll 1$. Let

$$\mathcal{G}(\varepsilon;\theta,\cdot)\colon [u_0,\dots,u_{m_c-1}]\mapsto u(t,x)\quad \bigl(\mathbb{E}_{m_c-1,k+m_h}\to\mathbb{F}_{m-1,k,[\mu,-\mu]}\bigr)\tag{5.12}$$

denote the corresponding resolving operator.

Constructing the family of invariant manifolds. We define the operators

$$\mathcal{R}_j(\varepsilon;\theta,u_0,\dots,u_{m_c-1})=\bigl(\partial_t^j\mathcal{G}(\varepsilon;\theta,u_0,\dots,u_{m_c-1})\bigr)\Big|_{t=\theta}\tag{5.13}$$

and set (compare (5.2))

$$\mathcal{M}(\theta)=\bigl\{[u_0,\dots,u_{m-1}]\in\mathbb{E}_{m-1,k}\colon\\ u_j=\mathcal{R}_j(\varepsilon;\theta,u_0,\dots,u_{m_c-1}),\ j\geqslant m_c\bigr\}.\tag{5.14}$$

Hence, the vector function $U_0=[u_0,\dots,u_{m-1}]$ belongs to the manifold $\mathcal{M}(\theta)$ if and only if it is the set of Cauchy data for a solution $u(y)\in\mathbb{F}_{m-1,k,[\mu,-\mu]}$ of equation (5.9). We denote by

$$\mathcal{U}^\rho_\varepsilon(t,\theta,\cdot)\colon\mathbb{E}_{m-1,k}\to\mathbb{E}_{m-1,k},\quad t,\theta\in\mathbb{R},\quad |\varepsilon|\ll 1,$$

the resolving operator of the Cauchy problem for (5.9), (5.10). It readily follows from (5.14) (see Theorem 7.1) that the family $\mathcal{M}(\theta)$ is compatible with the action of the operator $\mathcal{U}^\rho_\varepsilon$, that is, if $U_0\in\mathcal{M}(\theta)$ for some $\theta\in\mathbb{R}$, then $\mathcal{U}^\rho_\varepsilon(t,\theta,U_0)\in\mathcal{M}(t)$ for any $t\in\mathbb{R}$.

Attraction property. We note that if the energy $E_{m-1,k}(u,t)$ of a solution $u(t,x)$ of problem (5.9), (5.10) is uniformly bounded (or grows no faster than $e^{\mu|t|}$ as $|t|\to\infty$), then $u\in\mathbb{F}_{m-1,k,[\mu,-\mu]}$, and therefore the corresponding phase trajectory $\mathcal{D}(t)u$ belongs to $\mathcal{M}(t)$ for any $t\in\mathbb{R}$. The fact that a solution whose energy is bounded on a semi-axis is attracted to a solution lying on the manifold (compare (5.6))

$$\mathcal{M}=\bigl\{[\theta,u_0,\dots,u_{m-1}]\in\mathbb{R}\times\mathbb{E}_{m-1,k}\colon [u_0,\dots,u_{m-1}]\in\mathcal{M}(\theta)\bigr\}$$

is harder to establish. We outline the proof of this property on the basis of the solubility of the non-homogeneous equation

$$P_\varepsilon(t,x,\partial)u(t,x)=h(t,x)\tag{5.15}$$

in the function space with exponential weight $e^{\mu t}$ or $e^{-\mu t}$.

Let the energy $E_{m-1,k}(u,t)$ of a solution $u(y)$ of problem (5.9), (1.2) with $J=[\theta,+\infty)$ be uniformly bounded. An approximating solution is sought in the form

$$v(t,x)=\zeta(t)u(t,x)+w(t,x), \tag{5.16}$$

where $\zeta(t)\in C^\infty(\mathbb{R})$, $\zeta(t)=0$ for $t\leqslant\theta$, and $\zeta(t)=1$ for $t\geqslant\theta+1$. Substituting (5.16) into (5.9) and taking the fact that $u(y)$ satisfies (5.9) into account, we obtain the equation

$$P_\varepsilon(t,x,\partial)w(t,x)=g(t,x)-\varepsilon\bigl(Q_\rho(\varepsilon,t,V(t))-\zeta(t)Q_\rho(\varepsilon,t,U(t))\bigr) \tag{5.17}$$

for the function $w(t,x)$, where $U(t)=\mathcal{D}(t)u$, $V(t)=\mathcal{D}(t)v$, and

$$g(t,x)=-\sum_{l=1}^{m}\frac{1}{l!}D_t^l\zeta(t)P_\varepsilon^{(l)}(t,x,\partial)u(t,x), \qquad P_\varepsilon^{(l)}(t,x,\eta)=\partial_\tau^l P_\varepsilon(t,x,\eta). \tag{5.18}$$

We shall show (Theorem 7.1) that equation (5.17) has a unique solution $u(y)$ belonging to the space

$$\mathbb{F}_{m-1,k,[\mu]}=\bigl\{f(t,x)\colon e^{\mu t}\partial_t^j f\in C_b(\mathbb{R},H^{(m-1+k-j)}),\ j=0,\dots,m-1\bigr\}.$$

This together with (5.16) will imply that the norm of the difference $U(t)-V(t)$ in $\mathbb{E}_{m-1,k}$ decreases as $e^{-\mu t}$ as $t\to+\infty$ and the norm of $V(t)$ increases no faster than $e^{\mu|t|}$ as $|t|\to\infty$, and therefore $V(t)\in\mathcal{M}(t)$ for all $t\in\mathbb{R}$.

Smoothness. We shall establish the smoothness of $\mathcal{R}_j(\varepsilon;\theta,\cdot)$ as a consequence of similar properties of the resolving operator $\mathcal{G}(\varepsilon;\theta,\cdot)$. The existence of the first derivative and its Hölder property are fairly easy to prove. Namely, for a given solution $u(y)\in\mathbb{F}_{m-1,k,[\mu,-\mu]}$ of problem (5.9), (5.11), we consider the linearized equation

$$P_\varepsilon(t,x,\partial)v+\varepsilon(DQ_\rho)(\varepsilon,t,\mathcal{D}(t)u)\mathcal{D}(t)v=0 \tag{5.19}$$

supplemented with the initial conditions

$$\partial_t^j v(\theta,x)=v_j(x)\in H^{(m-1+k-j)}, \quad j=0,\dots,m_c-1, \tag{5.20}$$

where $(DQ_\rho)(\varepsilon,t,U)W$ is the value of the derivative of the function $Q_\rho(\varepsilon,t,U)$ with respect to U on the vector $W=[w_0,\dots,w_{m-1}]$. We shall show (see § 6.3) that problem (5.19), (5.20) is uniquely soluble in the space $\mathbb{F}_{m-1,k,[\mu,-\mu]}$ and that the solution $v^{(1)}(y)$ coincides with the value of the derivative of the operator $\mathcal{G}(\varepsilon;\theta,\cdot)$ on the vector $[v_0,\dots,v_{m_c-1}]$,

$$v(y)=(D\mathcal{G})(\varepsilon;\theta,u_0,\dots,u_{m_c-1})[v_0,\dots,v_{m_c-1}].$$

The proof of the existence of the derivatives of order $i\geqslant 2$ for the operator $\mathcal{G}(\varepsilon;\theta,\cdot)$ is somewhat more complicated. This comes down to the fact that the ith linearization of equation (5.9) is not generally soluble in the space $\mathbb{F}_{m-1,k,[\mu,-\mu]}$. For example, the second linearization has the form

$$P_\varepsilon(t,x,\partial)v+\varepsilon(DQ_\rho)(\varepsilon,t,U(t))V(t)=-\varepsilon(D^2Q_\rho)(\varepsilon,t,U(t))[V_1(t),V_1(t)], \tag{5.21}$$

where $U(t) = \mathcal{D}(t)u$, $V(t) = \mathcal{D}(t)v$, $V_1(t) = \mathcal{D}(t)v^{(1)}$, and $(D^2Q_\rho)(\varepsilon, t, U)[W, W]$ is the value of the second derivative of the function $Q_\rho(\varepsilon, t, U)$ with respect to U on the vector $W = [w_0, \dots, w_{m-1}]$. We note here that the $H^{(k)}$-norm of the right-hand side of equation (5.21) can grow as $e^{2\mu|t|}$ as $t \to \infty$. Therefore the solution of equation (5.21) does not necessarily belong to $\mathbb{F}_{m-1,k,[\mu,-\mu]}$. However, if μ is not very large (say $\mu < \delta/2$, whence it follows that $\nu < \mu, 2\mu < \delta$), then equation (5.21) has a unique solution $v(y) \in \mathbb{F}_{m-1,k,[2\mu,-2\mu]}$ satisfying the initial conditions[8]

$$\partial_t^j v(\theta, x) = 0, \quad j = 0, \dots, m_c - 1. \tag{5.22}$$

We shall prove that the operator $\mathcal{G}(\varepsilon; \theta, \cdot)$ regarded as a map from $\mathbb{E}_{m_c-1,k+m_h}$ to $\mathbb{F}_{m-1,k,[\mu',-\mu']}$, where $2\mu < \mu' < \delta$, is twice continuously differentiable and that the value of its second derivative on the vector $[v_0, \dots, v_{m_c-1}]$ coincides with the solution $v(y)$ of problem (5.21), (5.22).

Similarly, the ith linearization of equation (5.9) involves terms growing as $e^{i\mu|t|}$ as $t \to \infty$. In this case, if $\mu < \delta/i$ and $\mathcal{G}(\varepsilon; \theta, \cdot)$ is regarded as an operator from $\mathbb{E}_{m_c-1,k+m_h}$ to $\mathbb{F}_{m-1,k,[\mu',-\mu']}$, where $i\mu < \mu' < \delta$, then the ith derivatives exists, and its value on the vector $[v_0, \dots, v_{m_c-1}]$ coincides with the solution of the ith linearization supplemented with the additional initial conditions (5.22) (see [16], [33] and § 6.4).

The structure of the operators $\mathcal{R}_j(\varepsilon; \theta, \cdot)$. Let

$$G(\varepsilon, \theta)\colon \mathbb{E}_{m_c-1,k+m_h} \times \mathbb{F}_{0,k,[\mu,-\mu]} \to \mathbb{F}_{m-1,k,[\mu,-\mu]}$$

denote the operator transforming the vector function $[u_0, \dots, u_{m_c-1}, h]$ into the solution

$$u(y) \in \mathbb{F}_{m-1,k,[\mu,-\mu]}$$

of problem (5.15), (5.11). As was shown in [3], § 5, this operator is well defined and continuous for $|\varepsilon| \ll 1$. Note that

$$\mathcal{G}(\varepsilon; \theta, u_0, \dots, u_{m_c-1}) = G(\varepsilon, \theta)[u_0, \dots, u_{m_c-1}, 0] + \varepsilon G(\varepsilon, \theta)[0, \dots, 0, f], \tag{5.23}$$

where

$$f(t, x) = Q_\rho(\varepsilon, t, \mathcal{D}(t)\mathcal{G}(\varepsilon; \theta, u_0, \dots, u_{m_c-1})). \tag{5.24}$$

Applying the operator ∂_t^j to (5.23) and setting $t = \theta$, we find

$$\begin{aligned}\mathcal{R}_j(\varepsilon; \theta, u_0, \dots, u_{m_c-1}) = \partial_t^j \bigl(G(\varepsilon, \theta)&[u_0, \dots, u_{m_c-1}, 0] \\ &+ \varepsilon G(\varepsilon, \theta)[0, \dots, 0, f]\bigr)\Big|_{t=\theta}.\end{aligned} \tag{5.25}$$

The representation (5.5) follows from (5.25) and the analogue of formula (5.5) in the case of linear equations (see [3], § 5.3).

[8]It is easy to see that if the resolving operator $\mathcal{G}(\varepsilon; \theta, \cdot)$ has a second derivative, then its value on an arbitrary vector function $[v_0, \dots, v_{m_c-1}]$ satisfies zero initial conditions.

§ 6. An initial-value problem with growth conditions at infinity

This section is devoted to investigating problem (5.9)–(5.11). We shall prove that it is uniquely soluble in the space $\mathbb{F}_{m-1,k,[\mu,-\mu]}$ and that the resolving operator is smooth in the corresponding spaces. The results obtained will be used in proving Theorems 5.1 and 5.2 (see § 7).

6.1. Statement of results. Recall that the space $\mathbb{F}_{l,k,[\mu,-\mu]}$ and the corresponding norm $E_{l,k,[\mu,-\mu]}(\cdot)$ were defined in (3.15) and (3.16). We set $S(\theta)w(t,x) = w(t+\theta,x)$ for given $\theta \in \mathbb{R}$ and $w(t,x)$.

Theorem 6.1. *Let Conditions* (P), (Q), *and* (H_c) *hold. Then for any $\rho > 0$ and $\mu \in (\nu, \delta)$ and an arbitrary integer $k > n/2$, there is a constant $\varepsilon_0 > 0$ such that the following assertions are true for $|\varepsilon| \leqslant \varepsilon_0$.*

(i) *For any $\theta \in \mathbb{R}$ and arbitrary initial data $[u_0, \dots, u_{m_c-1}] \in \mathbb{E}_{m_c-1,k+m_h}$, problem* (5.9)–(5.11) *has a unique solution*

$$u(y) \in \mathbb{F}_{m-1,k,[\mu,-\mu]}.$$

(ii) *If $u_1, u_2 \in \mathbb{F}_{m-1,k,[\mu,-\mu]}$ are two solutions corresponding to the initial points $\theta_1, \theta_2 \in \mathbb{R}$, $|\theta_1 - \theta_2| \leqslant 1$, and the initial data $U_1, U_2 \in \mathbb{E}_{m_c-1,k+m_h}$, then*

$$E_{m-1,k,[\mu,-\mu]}\big(S(\theta_1)(u_1-u_2)\big) \leqslant C\|U_1-U_2\|_{m_c-1,k+m_h} + b(\theta_1, U_1; |\theta_1-\theta_2|), \quad (6.1)$$

where the constant $C > 0$ and the continuous function $b(\theta_1, U_1; r) \geqslant 0$ of the variable $r \in [0,1]$ depend on the parameters k, ρ, and μ, and $b(\theta_1, U_1; r)$ tends to zero as $r \to 0$.

Theorem 6.1 will be proved in § 6.2.

Recall the operator $\mathcal{G}(\varepsilon; \theta, u_0, \dots, u_{m_c-1})$ from § 5.2 (see formula (5.12)). In particular, the inequality (6.1) implies that this operator is jointly continuous in the variables $(\theta, u_0, \dots, u_{m_c-1})$.

Theorem 6.2. *Let Conditions* (P), (Q), *and* (H_c) *and inequality* (5.1) *hold, where $l \geqslant 1$ is an integer. Then there is an $\varepsilon_0 > 0$ such that the operator $\mathcal{G}(\varepsilon;\theta,\cdot)$ with $|\varepsilon| \leqslant \varepsilon_0$ belongs to the class $C^{l,\gamma}\big(\mathbb{E}_{m_c-1,k+m_h}, \mathbb{F}_{m-1,k,[\mu',-\mu']}\big)$, where $\mu' = l\mu + \gamma$, and the seminorm $|\mathcal{G}|_{C^{l,\gamma}}$ is uniformly bounded with respect to (ε, θ).*

The proof of Theorem 6.2 is given in §§ 6.3 and 6.4.

6.2. Proof of theorem 6.1. We shall need the following assertion on the solubility of linear equations in $\mathbb{F}_{m-1,k,[\mu,-\mu]}$. See [3], § 4, for a proof.

Proposition 6.3. *Under the assumptions of Theorem* 6.1, *for any $\mu \in (\nu, \delta)$ and an arbitrary integer $k \geqslant 0$, there are constants $\varepsilon_0 > 0$ and $C > 0$ such that, for $|\varepsilon| \leqslant \varepsilon_0$, problem* (5.15), (5.11) *with right-hand side $h \in \mathbb{F}_{0,k,[\mu,-\mu]}$ and initial conditions $[u_0, \dots, u_{m_c-1}] \in \mathbb{E}_{m_c-1,k+m_h}$ has a unique solution $u(y) \in \mathbb{F}_{m-1,k,[\mu,-\mu]}$ satisfying the inequality*

$$E_{m-1,k,[\mu,-\mu]}(S(\theta)u) \leqslant C\left(\sum_{j=0}^{m_c-1} \|u_j\|_{(m-1+k-j)} + E_{0,k,[\mu,-\mu]}(S(\theta)h)\right). \quad (6.2)$$

Proof. (i) To simplify the notation we assume that $\theta = 0$. The existence of a solution is proved by the contraction mapping principle. Recall that $\mathcal{D}(t)w$ denotes the phase trajectory corresponding to the function $w = w(t,x)$ (see (1.3)). Consider the operator A transforming $v(y) \in \mathbb{F}_{m-1,k,[\mu,-\mu]}$ into the solution $u(y) \in \mathbb{F}_{m-1,k,[\mu,-\mu]}$ of the equation

$$P_\varepsilon(t,x,\partial)u(t,x) = -\varepsilon Q_\rho(\varepsilon,t,\mathcal{D}(t)v) \tag{6.3}$$

with initial conditions (5.11). It is clear that the fixed point of the operator A is the desired solution.

By Lemma 8.2, the right-hand side of (6.3) belongs to $\mathbb{F}_{0,k,[\mu,-\mu]}$. Therefore, by Proposition 6.3, problem (6.3), (5.11) has a unique solution $u \in \mathbb{F}_{m-1,k,[\mu,-\mu]}$ for $|\varepsilon| \ll 1$. Consequently, the operator

$$A\colon \mathbb{F}_{m-1,k,[\mu,-\mu]} \to \mathbb{F}_{m-1,k,[\mu,-\mu]}$$

is well defined. Moreover, if $u_i = A(v_i)$, then the difference $u = u_1 - u_2$ satisfies (5.15) with right-hand side

$$h(t,x) = \varepsilon\big(Q_\rho(\varepsilon,t,\mathcal{D}(t)v_2) - Q_\rho(\varepsilon,t,\mathcal{D}(t)v_1)\big)$$

and has zero Cauchy data up to order $m_c - 1$. Therefore, by (6.2),

$$E_{m-1,k,[\mu,-\mu]}(u) \leqslant C\, E_{0,k,[\mu,-\mu]}(h). \tag{6.4}$$

By Lemma 8.2, we have

$$E_{0,k,[\mu,-\mu]}(h) \leqslant C_1\, |\varepsilon|\, E_{m-1,k,[\mu,-\mu]}(v_1 - v_2).$$

Substituting this estimate into (6.4), we find

$$E_{m-1,k,[\mu,-\mu]}(u_1 - u_2) \leqslant C_2\, |\varepsilon|\, E_{m-1,k,[\mu,-\mu]}(v_1 - v_2),$$

whence it follows that A is a contraction operator for $|\varepsilon| \ll 1$ and consequently has a fixed point $u(y)$.

The uniqueness of this solution follows from (6.1), which is established below.

(ii) The difference $u = u_1 - u_2$ satisfies (5.15) with right-hand side

$$h(t,x) = \varepsilon\big(Q_\rho(\varepsilon,t,\mathcal{D}(t)u_2) - Q_\rho(\varepsilon,t,\mathcal{D}(t)u_1)\big). \tag{6.5}$$

Consequently, by (6.2),

$$\begin{aligned} &E_{m-1,k,[\mu,-\mu]}(S(\theta_2)u) \\ &\quad\leqslant C\left(\sum_{j=0}^{m_c-1} \|\partial_t^j u(\theta_2,\cdot)\|_{(m-1+k-j)} + E_{0,k,[\mu,-\mu]}(S(\theta_2)h)\right). \end{aligned} \tag{6.6}$$

Let us estimate the right-hand side of (6.6). We set

$$u_{ij}(x) = \partial_t^j u_i(\theta_i,x), \quad i = 1,2, \quad j = 0,\dots,m_c - 1.$$

Then

$$\partial_t^j u(\theta_2,x) = \big(\partial_t^j u_1(\theta_2,x) - u_{1j}(x)\big) + \big(u_{1j}(x) - u_{2j}(x)\big),$$

which implies that

$$\|\partial_t^j u(\theta_2,\cdot)\|_{(m-1+k-j)} \leqslant b_1(\theta_1,U_1;|\theta_1-\theta_2|) + \|u_{1j}-u_{2j}\|_{(m-1+k-j)}, \tag{6.7}$$

where the function $b_1(\theta_1,U_1;r) \geqslant 0$ continuous in $r \geqslant 0$ tends to zero as $r \to 0$. Furthermore, by Lemma 8.2, (6.5) satisfies the inequality

$$E_{0,k,[\mu,-\mu]}(S(\theta_2)h) \leqslant C_1\,|\varepsilon|\,E_{m-1,k,[\mu,-\mu]}(S(\theta_2)u). \tag{6.8}$$

For $|\varepsilon| \ll 1$, the substitution of (6.7) and (6.8) into (6.6) results in

$$E_{m-1,k,[\mu,-\mu]}(S(\theta_2)u) \leqslant C\Bigg(\sum_{j=0}^{m_c-1}\|u_{1j}-u_{2j}\|_{(m-1+k-j)} + b_1(\theta_1,U_1;|\theta_1-\theta_2|)\Bigg).$$

It remains to note that the norms $E_{m-1,k,[\mu,-\mu]}(S(\theta)u)$ are equivalent for $\theta \in \mathbb{R}$ with bounded constants of equivalence if θ varies in a finite interval.

6.3. Proof of Theorem 6.2 in the case $l=1$. We note that problem (5.19), (5.20) has a unique solution $v(y) \in \mathbb{F}_{m-1,k,[\mu,-\mu]}$ for any function $u(t,x)$ satisfying (5.10) and arbitrary initial data $[v_0,\dots,v_{m_c-1}] \in \mathbb{E}_{m_c-1,k+m_h}$. This assertion can easily be proved using the arguments in § 6.2. Let

$$\mathcal{G}_1(\varepsilon;\theta,u_0,\dots,u_{m_c-1})\colon \mathbb{E}_{m_c-1,k+m_h} \to \mathbb{F}_{m-1,k,[\mu,-\mu]}$$

be a bounded linear operator transforming the vector $[v_0,\dots,v_{m_c-1}]$ into the solution $v \in \mathbb{F}_{m-1,k,[\mu,-\mu]}$ of problem (5.19), (5.20) with $u = \mathcal{G}(\varepsilon;\theta,u_0,\dots,u_{m_c-1})$. For simplicity, in what follows we shall assume that $\theta=0$ and drop the parameter θ in the notation for the operators $\mathcal{G}$ and $\mathcal{G}_1$.

By the converse of Taylor's theorem (see [13] or [11], § 1.2.5), the existence of the derivative of the operator

$$\mathcal{G}(\varepsilon;\cdot)\colon \mathbb{E}_{m_c-1,k+m_h} \to \mathbb{F}_{m-1,k,[\mu',-\mu']}, \quad \mu' = \mu+\gamma,$$

the Hölder property of the derivative, and the uniform boundedness of the seminorm $|\mathcal{G}|_{C^{1,\gamma}}$ will be proved if we can show that

$$\mathcal{G}(\varepsilon;U_0+V_0) = \mathcal{G}(\varepsilon;U_0) + \mathcal{G}_1(\varepsilon;U_0)V_0 + \mathcal{F}_1(\varepsilon;U_0,V_0), \tag{6.9}$$

where $U_0 = [u_0,\dots,u_{m_c-1}]$, $V_0 = [v_0,\dots,v_{m_c-1}]$, and the operator $\mathcal{F}_1$ satisfies the inequality

$$E_{0,k,[\mu',-\mu']}(\mathcal{F}_1(\varepsilon;U_0,V_0)) \leqslant \text{const}\,\|V_0\|^{1+\gamma}_{m_c-1,k+m_h}. \tag{6.10}$$

We set

$$\tilde u(y) = \mathcal{G}(\varepsilon;U_0+V_0), \qquad u(y) = \mathcal{G}(\varepsilon;U_0), \qquad v^{(1)}(y) = \mathcal{G}_1(\varepsilon;U_0)V_0.$$

Direct verification shows that $w := \tilde{u} - u - v^{(1)} = \mathcal{F}_1(\varepsilon; U_0, V_0)$ is the solution of the problem

$$P_\varepsilon(t,x,\partial)w(t,x) = -\varepsilon h(t,x), \tag{6.11}$$

$$\partial_t^j w(0,x) = 0, \quad j = 0,\dots,m_c - 1, \tag{6.12}$$

where

$$\begin{aligned} h(t,x) &= h_1(t,x) + h_2(t,x),\\ h_1(t,x) &= Q_\rho(\varepsilon,t,\mathcal{D}(t)\tilde{u}) - Q_\rho(\varepsilon,t,\mathcal{D}(t)u) - (DQ_\rho)(\varepsilon,t,\mathcal{D}(t)u)\mathcal{D}(t)(\tilde{u}-u),\\ h_2(t,x) &= (DQ_\rho)(\varepsilon,t,\mathcal{D}(t)u)\mathcal{D}(t)w. \end{aligned}$$

By (6.2) with $\theta = 0$ and $\mu = \mu'$, we have

$$E_{m-1,k,[\mu',-\mu']}(w) \leqslant C\,|\varepsilon|\,E_{0,k,[\mu',-\mu']}(h). \tag{6.13}$$

Let us estimate the right-hand side of (6.13). According to Lemma 8.1,

$$\|h_1(t,\cdot)\|_{(k)} \leqslant C_1 \|\mathcal{D}(t)(\tilde{u}-u)\|_{m-1,k}^{1+\gamma}.$$

This inequality and (6.1) with $\theta_1 = \theta_2 = 0$ imply that

$$E_{0,k,[\mu',-\mu']}(h_1) \leqslant C_2 \sum_{j=0}^{m_c-1} \|v_j\|_{(m-1+k-j)}^{1+\gamma}. \tag{6.14}$$

Next, by the uniform boundedness of the derivative of Q_ρ,

$$E_{0,k,[\mu',-\mu']}(h_2) \leqslant C_3\,E_{m-1,k,[\mu',-\mu']}(w). \tag{6.15}$$

Substituting (6.14) and (6.15) into (6.13) with $h = h_1 + h_2$, we obtain

$$E_{m-1,k,[\mu',-\mu']}(w) \leqslant C_4 \sum_{j=0}^{m_c-1} \|v_j\|_{(m-1+k-j)}^{1+\gamma},$$

which is equivalent to (6.10).

6.4. Proof of Theorem 6.2 in the case $l \geqslant 2$. Despite the title of this section, we confine ourselves to the case $l = 2$. The proof for arbitrary $l \geqslant 3$ is technically rather complicated, but does not involve any fundamentally new ideas. For detailed proofs of similar assertions in the case of ordinary differential equations and abstract equations in Banach spaces, see [16] and [33].

As in § 6.3, let us assume that $\theta = 0$. We want to prove that the operator

$$\mathcal{G}(\varepsilon;\cdot)\colon \mathbb{E}_{m_c-1,k+m_h} \to \mathbb{F}_{m-1,k,[\mu',-\mu']}, \quad \mu' = 2\mu + \gamma,$$

is twice continuously differentiable in the sense of Fredchét, the second derivative satisfies Hölder's condition with exponent γ, and the seminorm $|\mathcal{G}|_{C^{2,\gamma}}$ is uniformly bounded.

We note that problem (5.21), (5.22) (with $U(t) = \mathcal{D}(t)u$, $V(t) = \mathcal{D}(t)v$, and $V_1(t) = \mathcal{D}(t)v^{(1)}$) has a unique solution $v(y) \in \mathbb{F}_{m-1,k,[2\mu,-2\mu]}$ (compare 6.3) for any function $u(t,x)$ satisfying (5.10). Let

$$\mathcal{G}_2(\varepsilon; u_0, \dots, u_{m_c-1})\colon \mathbb{E}_{m_c-1,k+m_h} \to \mathbb{F}_{m-1,k,[2\mu,-2\mu]}$$

denote the bounded quadratic operator transforming $[v_0, \dots, v_{m_c-1}]$ into the solution $v \in \mathbb{F}_{m-1,k,[2\mu,-2\mu]}$ of problem (5.21), (5.22) with $u = \mathcal{G}(\varepsilon; u_0, \dots, u_{m_c-1})$.

By the converse of Taylor's theorem, we have to show that

$$\mathcal{G}(\varepsilon; U_0 + V_0) = \mathcal{G}(\varepsilon; U_0) + \mathcal{G}_1(\varepsilon; U_0)V_0 + \frac{1}{2}\mathcal{G}_2(\varepsilon; U_0)[V_0, V_0] + \mathcal{F}_2(\varepsilon; U_0, V_0),$$

where $U_0 = [u_0, \dots, u_{m_c-1}]$, $V_0 = [v_0, \dots, v_{m_c-1}]$, and the operator $\mathcal{F}_2$ satisfies the inequality

$$E_{0,k,[\mu',-\mu']}(\mathcal{F}_2(\varepsilon; U_0, V_0)) \leqslant \text{const}\, \|V_0\|^{2+\gamma}_{m_c-1,k+m_h}. \tag{6.16}$$

We set $v^{(2)}(y) = \mathcal{G}_2(\varepsilon; U_0)[V_0, V_0]$. Then

$$w := \tilde{u} - u - v^{(1)} - \frac{1}{2}v^{(2)} = \mathcal{F}_2(\varepsilon; U_0, V_0)$$

is the solution of problem (6.11), (6.12) with[9] $h = h_1 + h_2 + h_3$,

$$\begin{aligned} h_1(t,x) &= Q_\rho(\varepsilon, t, \tilde{u}) - Q_\rho(\varepsilon, t, u) - (DQ_\rho)(\varepsilon, t, u)(\tilde{u} - u) \\ &\qquad - \frac{1}{2}(D^2Q_\rho)(\varepsilon, t, u)[\tilde{u} - u, \tilde{u} - u], \\ h_2(t,x) &= (DQ_\rho)(\varepsilon, t, u)w, \\ h_3(t,x) &= \frac{1}{2}\big((D^2Q_\rho)(\varepsilon, t, u)[\tilde{u} - u, \tilde{u} - u] - (D^2Q_\rho)(\varepsilon, t, u)[v^{(1)}, v^{(1)}]\big). \end{aligned}$$

Therefore, by (6.2), the function $w(y)$ satisfies inequality (6.13) with $\mu' = 2\mu + \gamma$. Let us estimate the right-had side of (6.13). Using Lemma 8.1 and inequality (6.1), it is easy to prove that (compare the derivation of (6.14))

$$E_{0,k,[\mu',-\mu']}(h_1) \leqslant C_1 \sum_{j=0}^{m_c-1} \|v_j\|^{2+\gamma}_{(m-1+k-j)}. \tag{6.17}$$

The function $h_2(t,x)$ satisfies (6.15) with $\mu' = 2\mu + \gamma$. To estimate the norm of h_3, note that

$$h_3(t,x) = \frac{1}{2}(D^2Q_\rho)(\varepsilon, t, u)[\tilde{u} - u - v^{(1)}, \tilde{u} - u + v^{(1)}].$$

Comparing this relation with (6.9) and (6.10), we conclude that h_3 satisfies an inequality of type (6.17). This fact and formula (6.13) imply the desired inequality (6.16). Other details are left to the reader.

[9]To simplify the formulae, we shall drop the operator $\mathcal{D}(t)$ up to the end of this section. It will always be clear from the context where it must stand.

§ 7. Proof of the main results

This section is devoted to the proof of Theorems 5.1 – 5.3. We first show (see Theorem 7.1) that the global versions of the assertions in Theorems 5.1 and 5.2 are true for the equation with truncated non-linearity (5.9). Theorems 5.1 and 5.2 are then deduced as a simple consequence. Finally, the results obtained and the uniqueness of the solution of the Cauchy problem for strictly hyperbolic equations with Lipschitzian non-linearity are used to prove Theorem 5.3.

7.1. The centre manifold for the equation with truncated non-linearity. Recall that $\mathcal{U}_\varepsilon^\rho(t,\theta,U_0)$ denotes the resolving operator of the Cauchy problem (5.9), (5.10), (1.1). By Proposition 4.2, if $|\varepsilon| \ll 1$, then this operator is defined for all $t,\theta \in \mathbb{R}$.

Theorem 7.1. *Suppose that Conditions* (P), (Q), *and* (H_c) *and inequalities* (5.1), *where $l \geqslant 1$ is an integer, hold. Then for any $\rho > 0$ and $\mu \in (\nu, \delta/l)$ and an arbitrary integer $k > n/2$, there is a constant $\varepsilon_0 > 0$ and a family of continuous operators*

$$\begin{gathered}\mathcal{R}_j(\varepsilon;\theta,u_0,\dots,u_{m_c-1})\colon \mathbb{R}_\theta \times \mathbb{E}_{m_c-1,k+m_h} \to H^{(m-1+k-j)},\\ j = m_c,\dots,m-1,\end{gathered} \tag{7.1}$$

such that $\mathcal{R}_j(\varepsilon;\theta,0) = 0$ and the following assertions are true for $|\varepsilon| \leqslant \varepsilon_0$.

(i) *The family of manifolds $\mathcal{M}(\theta)$ defined by formula* (5.14) *is compatible with the action of the resolving operator $\mathcal{U}_\varepsilon^\rho(t,\theta,\cdot)$ in the sense that if $U_0 \in \mathcal{M}(\theta)$ for some $\theta \in \mathbb{R}$, then $\mathcal{U}_\varepsilon^\rho(t,\theta,U_0) \in \mathcal{M}(t)$ for all $t \in \mathbb{R}$.*

(ii) *If $u(t,x) \in \mathbb{F}_{m-1,k,[\mu,-\mu]}$ satisfies equation* (5.9), *then $\mathcal{D}(t)u \in \mathcal{M}(t)$ for all $t \in \mathbb{R}$.*

(iii) *If the energy $E_{m-1,k}(u,t)$ of a solution $u(t,x)$ of problem* (5.9), (5.10) *grows no faster than $e^{\mu t}$ as $t \to +\infty$, then there is a solution $v(t,x) \in \mathbb{F}_{m-1,k,[\mu,-\mu]}$ of* (5.9) *such that $\mathcal{D}(t)v \in \mathcal{M}(t)$ for all $t \in \mathbb{R}$ and*

$$E_{m-1,k}(u-v,t) \leqslant C\, e^{-\mu(t-\theta)} E_{m-1,k}(u,\theta), \quad t \geqslant \theta, \tag{7.2}$$

where the constant $C > 0$ does not depend on t, θ, or $u(y)$. A similar assertion holds for solutions whose energy grows no faster than $e^{-\mu t}$ as $t \to -\infty$.

(iv) *For any fixed ε and θ, the operator $\mathcal{R}_j(\varepsilon;\theta,\cdot)$ belongs to the class $C^{l,\gamma}(\mathbb{E}_{m_c-1,k+m_h}, H^{(m-1+k-j)})$, and the seminorm $|\mathcal{R}_j|_{C^{l,\gamma}}$ is uniformly bounded with respect to (ε,θ).*

(v) *There are continuous linear operators*

$$\begin{gathered}\mathcal{B}_j(\varepsilon;\theta,u_0,\dots,u_{m_c-1})\colon \mathbb{R}_\theta \times \mathbb{E}_{m_c-1,k+m_h} \to H^{(m-1+k-j)},\\ j = m_c,\dots,m-1,\end{gathered} \tag{7.3}$$

such that $\mathcal{B}_j(\varepsilon;\theta,0) = 0$ and the representation (5.5) *holds. Moreover, the operators $\mathcal{B}_j$ satisfy the global Lipschitz condition*

$$\|\mathcal{B}_j(\varepsilon;\theta,U_0) - \mathcal{B}_j(\varepsilon;\theta,V_0)\|_{(m-1+k-j)} \leqslant \mathrm{const}\, \|U_0 - V_0\|_{m_c-1,k+m_h} \tag{7.4}$$

for a fixed $\theta \in \mathbb{R}$, where $U_0, V_0 \in \mathbb{E}_{m_c-1,k+m_h}$.

(vi) *If the non-linear term* $q(\varepsilon, t, x, \partial^{m-1}u)$ *in* (0.5) *does not depend on the* $(m-1)$*th derivatives, then assertion* (v) *holds with* $H^{(m-1+k-j)}$ (*see* (7.3) *and* (7.4)) *replaced by* $H^{(m+k-j)}$.

Proof. We define $\mathcal{R}_j$ by formula (5.13), where $\mathcal{G}(\varepsilon;\theta,U_0)$ is the resolving operator of problem (5.9)–(5.11) (see (5.12)). Then, by uniqueness, we have $\mathcal{G}(\varepsilon;\theta,0)=0$, and consequently $\mathcal{R}_j(\varepsilon;\theta,0)=0$. Moreover, (6.1) implies that $\mathcal{G}$ is jointly continuous with respect to the variables (θ,U_0), and therefore $\mathcal{R}_j$ also possesses this property. We now prove assertions (i)–(vi).

(i) Suppose that $U_0\in\mathcal{M}(\theta)$ for some $\theta\in\mathbb{R}$ and set $u=\mathcal{G}(\varepsilon;\theta,U_0)$. We have to prove that

$$\partial_t^j u(t,\cdot)=\mathcal{R}_j(\varepsilon;t,\mathcal{D}_c(t)u) \quad \text{for all} \quad t\in\mathbb{R}. \tag{7.5}$$

As is easily seen, (7.5) follows from the relation

$$u(t,x)=\mathcal{G}(\varepsilon;t,\mathcal{D}_c(t)u), \quad t\in\mathbb{R}, \tag{7.6}$$

which is a consequence of the uniqueness of the solution of problem (5.9)–(5.11).

(ii) Let $u\in\mathbb{F}_{m-1,k,[\mu,-\mu]}$ be the solution of equation (5.9). Then (7.6) holds, whence follows the desired assertion.

(iii) Suppose that a solution $u(y)$ of problem (5.9), (5.10) satisfies the inequality

$$E_{m-1,k}(u,t)\leqslant \text{const}\, e^{\mu t}, \quad t\geqslant 0. \tag{7.7}$$

An approximating solution $v(y)$ is sought in the form (5.16). Substitution of $v(y)$ into (5.9) for the unknown function $w(y)$ gives equation (5.17) with $U(t)=\mathcal{D}(t)u$ and $V(t)=\mathcal{D}(t)v$, and the function $g(t,x)$ has the form (5.18). Let us show that this equation is uniquely soluble in $\mathbb{F}_{m-1,k,[\mu]}$ and that the solution $w(y)$ satisfies the inequality

$$E_{m-1,k,[\mu]}(w):=\sup_{t\in\mathbb{R}} e^{\mu t}E_{m-1,k}(w,t)\leqslant \text{const}\, E_{m-1,k}(u,\theta).$$

To this end, we need the following auxiliary assertion, which is a consequence of Theorem 2.6 in [3].

Proposition 7.2. *Under the assumptions of Theorem* 7.1, *for any* $\mu\in(\nu,\delta)$ *and an arbitrary integer* $k\geqslant 0$, *there are constants* $\varepsilon_0>0$ *and* $C>0$ *such that equation* (5.15) *with right-hand side* $h\in\mathbb{F}_{0,k,[\mu]}$ *has a unique solution* $u(y)\in\mathbb{F}_{m-1,k,[\mu]}$ *for* $|\varepsilon|\leqslant\varepsilon_0$. *This solution satisfies the inequality*

$$E_{m-1,k,[\mu]}(S(\theta)u)\leqslant E_{0,k,[\mu]}(S(\theta)h) \quad \textit{for any} \quad \theta\in\mathbb{R}. \tag{7.8}$$

Let us consider the operator A transforming $z\in\mathbb{F}_{m-1,k,[\mu]}$ into the solution $w\in\mathbb{F}_{m-1,k,[\mu]}$ of the equation

$$P_\varepsilon(t,x,\partial)w(t,x)=g(t,x)-\varepsilon g_1(t,x), \tag{7.9}$$

where

$$g_1(t,x)=Q_\rho(\varepsilon,t,\mathcal{D}(t)(\zeta u+z))-\zeta(t)Q_\rho(\varepsilon,t,\mathcal{D}(t)u).$$

We shall show that the operator A is well defined for $|\varepsilon| \ll 1$ and is a contraction operator on the space $\mathbb{F}_{m-1,k,[\mu]}$. This will imply that A has a fixed point $w(y)$, which is precisely the desired solution.

We note that the right-hand side of equation (7.9) belongs to the space $\mathbb{F}_{0,k,[\mu]}$ and satisfies the inequality

$$E_{0,k,[\mu]}(S(\theta)(g-\varepsilon g_1)) \leqslant C_1\big(E_{m-1,k}(u,\theta)+|\varepsilon|\, E_{m-1,k,[\mu]}(S(\theta)z)\big) \tag{7.10}$$

for any function $u(t,x)$ satisfying (5.10), where $C_1>0$ is a constant. Indeed, it is clear that g and g_1 are continuous functions on $\mathbb{R}$ with range in $H^{(k)}$. Therefore it suffices to establish (7.10). By Proposition 4.3,

$$E_{m-1,k}(u,t) \leqslant C_2\, e^{\varkappa|t-\theta|} E_{m-1,k}(u,\theta), \quad t\in\mathbb{R}, \tag{7.11}$$

where C_2 and $\varkappa$ are positive constants. Relations (7.11) and (5.18) imply that the function $g(t,x)$ satisfies the inequality

$$\|g(t,\cdot)\|_{(k)} \leqslant C_3 E_{m-1,k}(u,\theta), \quad \theta\leqslant t\leqslant \theta+1, \tag{7.12}$$

on the closed interval $[\theta,\theta+1]$ and vanishes outside it. To estimate $g_1(t,x)$ we use the mean value theorem and take the uniform boundedness of the derivatives of Q_ρ and inequality (7.11) into consideration. As a result, we obtain

$$E_{0,k,[\mu]}(S(\theta)g_1) \leqslant C_4\big(E_{m-1,k}(u,\theta)+E_{m-1,k,[\mu]}(S(\theta)z)\big). \tag{7.13}$$

Comparing (7.12) and (7.13), we arrive at (7.10).

Thus, the right-hand side of (7.9) belongs to $\mathbb{F}_{0,k,[\mu]}$ for any $z\in\mathbb{F}_{m-1,k,[\mu]}$. Therefore, by Proposition 7.2, for $|\varepsilon|\ll 1$ equation (7.9) has a unique solution $w=A(z)\in\mathbb{F}_{m-1,k,[\mu]}$ for which the estimate

$$\begin{aligned} E_{m-1,k,[\mu]}\big(S(\theta)A(z)\big) &\leqslant C\, E_{0,k,[\mu]}(S(\theta)(g-\varepsilon g_1)) \\ &\leqslant C_5\big(E_{m-1,k}(u,\theta)+|\varepsilon|\, E_{m-1,k,[\mu]}(S(\theta)z)\big) \end{aligned} \tag{7.14}$$

holds.

We now show that A is a contraction operator. Let us take two arbitrary functions $z_1,z_2\in\mathbb{F}_{m-1,k,[\mu]}$, set $w_i=A(z_i)$, and consider the difference $w=w_1-w_2$. It can readily be seen that this difference is a solution of equation (5.15) with right-hand side

$$h(t,x)=\varepsilon\big(Q_\rho(\varepsilon,t,\mathcal{D}(t)(\zeta u-z_2))-Q_\rho(\varepsilon,t,\mathcal{D}(t)(\zeta u-z_1))\big).$$

Using the mean value theorem it is easy to prove the inequality

$$\|h(t,\cdot)\|_{(k)} \leqslant C_6|\varepsilon|\, E_{m-1,k}(z_1-z_2,t),$$

whence it follows that

$$E_{0,k,[\mu]}(S(\theta)h) \leqslant C_6|\varepsilon|\, E_{m-1,k,[\mu]}\big(S(\theta)(z_1-z_2)\big). \tag{7.15}$$

By (7.8), the solution $w \in \mathbb{F}_{m-1,k,[\mu]}$ of (5.15) satisfies the inequality

$$E_{m-1,k,[\mu]}(S(\theta)w) \leqslant C E_{0,k,[\mu]}(S(\theta)h).$$

Comparing this inequality with (7.15), we conclude that A is a contraction operator for $|\varepsilon| \ll 1$.

Let us denote by $w \in \mathbb{F}_{m-1,k,[\mu]}$ the fixed point of A, $A(w) = w$. It follows from (7.14) that

$$E_{m-1,k,[\mu]}(S(\theta)w) \leqslant C_7 E_{m-1,k}(u,\theta). \tag{7.16}$$

We shall show that all the required assertions hold for the function $v(t,x)$ defined by (5.16).

Indeed, by construction, $v(t,x)$ satisfies (5.9). Since $\zeta(t) = 0$ for $t \leqslant \theta$, it follows from (5.16), (7.7), (7.16) that $v \in \mathbb{F}_{m-1,k,[\mu,-\mu]}$. Consequently, according to assertion (ii), $\mathcal{D}(t)v \in \mathcal{M}(t)$ for all $t \in \mathbb{R}$. Furthermore, (5.16) and (7.16) imply that

$$E_{m-1,k}(u-v,t) = E_{m-1,k}(w,t) \leqslant C_7 e^{-\mu(t-\theta)} E_{m-1,k}(u,\theta) \quad \text{for} \quad t \geqslant \theta + 1.$$

Comparing this inequality with (7.11) and (7.16), we arrive at (7.2).

We have thus established that for any $\theta \in \mathbb{R}$, there is a solution $v = v_\theta \in \mathbb{F}_{m-1,k,[\mu,-\mu]}$ satisfying (7.2). It remains to show that $v_{\theta_1} \equiv v_{\theta_2}$ for any $\theta_1, \theta_2 \in \mathbb{R}$.

Direct verification shows that the difference $w := v_{\theta_1} - v_{\theta_2}$ belongs to $\mathbb{F}_{m-1,k,[\mu,-\mu]}$ and is a solution of (5.15) with right-hand side

$$h(t,x) = \varepsilon\big(Q_\rho(\varepsilon,t,\mathcal{D}(t)v_{\theta_1}) - Q_\rho(\varepsilon,t,\mathcal{D}(t)v_{\theta_2})\big).$$

Inequality (7.2) with $v = v_{\theta_i}$ implies that

$$E_{m-1,k}(v_{\theta_1} - v_{\theta_2}, t) \leqslant C_8 e^{-\mu t} E_{m-1,k}(u,\theta).$$

Therefore $w \in \mathbb{F}_{m-1,k,[\mu]}$. Furthermore, it can easily be shown that h belongs to $\mathbb{F}_{0,k,[\mu]}$ and satisfies the inequality

$$E_{0,k,[\mu]}(h) \leqslant C_9 |\varepsilon|\, E_{m-1,k,[\mu]}(w).$$

By (7.8), we have

$$E_{m-1,k,[\mu]}(w) \leqslant C\, E_{0,k,[\mu]}(h) \leqslant C\, C_9 |\varepsilon|\, E_{m-1,k,[\mu]}(w),$$

which implies that $w \equiv 0$ for $|\varepsilon| \ll 1$.

(iv) This assertion is an obvious consequence of the smoothness of the operator $\mathcal{G}(\varepsilon;\theta,\cdot)$ and formula (5.13).

(v) As was shown in § 5, the operator $\mathcal{R}_j$ can be represented in the form (5.25). We now note that the first term in the right-hand side of (5.25) is an analogue of the operator $\mathcal{R}_j$ for the linear equation

$$P_\varepsilon(t,x,\partial)u(t,x) = 0.$$

According to [3], Theorem 6.4, the representation

$$\partial_t^j \big(G(\varepsilon,\theta)U_0\big)\Big|_{t=\theta} = \sum_{i=0}^{m_c-1} \big(r_{ij}(\varepsilon,\theta,x,\partial_x) + \varepsilon d_{ij}(\varepsilon,\theta)\big)u_i(x) \tag{7.17}$$

holds for linear equations, where $U_0 = [u_0,\dots,u_{m_c-1}]$, the operators

$$d_{ij}(\varepsilon,\theta)\colon H^{(m-1+k-i)} \to H^{(m+k-j)} \tag{7.18}$$

depend continuously on θ in the strong operator topology, and their norms are uniformly bounded with respect to (ε,θ). We set

$$\mathcal{B}_j(\varepsilon;\theta,U_0) = \sum_{i=0}^{m_c-1} d_{ij}(\varepsilon,\theta)u_i + \partial_t^j\big(G(\varepsilon,\theta)[0,\dots,0,f]\big)\Big|_{t=\theta}, \tag{7.19}$$

where the function f is defined by (5.24). The representation (5.5) is a trivial consequence of (5.25), (7.17), and (7.19). The continuity of the operator $\mathcal{B}_j$ with respect to $[\theta,U_0]$ and its uniform Lipschitz property with respect to U_0 follow from the corresponding properties of the operators on the right-hand side of (7.19). Other details are left to the reader.

(vi) We shall use formula (7.19). Since $d_{ij}(\varepsilon,\theta)$ are continuous operators from $H^{(m-1+k-i)}$ to $H^{(m+k-j)}$ (see (7.18)), it suffices to consider the second term on the right-hand side of (7.19). We note that if q does not depend on the $(m-1)$th derivatives, then Q_ρ is a smooth operator from $\mathbb{R}_t \times \mathbb{E}_{m-1,k}$ to $H^{(k+1)}$ and all its derivatives are uniformly bounded. This together with Theorem 6.1 implies that

$$[\theta,u_0,\dots,u_{m_c-1}] \mapsto f(t,x) = Q_\rho\big(\varepsilon,t,\mathcal{D}(t)\mathcal{G}(\varepsilon;\theta,u_0,\dots,u_{m_c-1})\big)$$

is a continuous operator from $\mathbb{R}_\theta \times \mathbb{E}_{m-1,k}$ to $\mathbb{F}_{0,k+1,[\mu,-\mu]}$ for $|\varepsilon| \ll 1$ and satisfies a Lipschitz condition with uniformly bounded Lipschitz constant with respect to (ε,θ) for any fixed θ. Furthermore,

$$f \mapsto G(\varepsilon,\theta)[0,\dots,0,f]$$

is a continuous linear operator from $\mathbb{F}_{0,k+1,[\mu,-\mu]}$ to $\mathbb{F}_{m-1,k+1,[\mu,-\mu]}$ for sufficiently small ε. It depends continuously on θ in the strong operator topology and its norm is uniformly bounded with respect to (ε,θ). What has been said implies that the second term on the right-hand side of (7.19) also possesses all the required properties.

7.2. Proof of Theorems 5.1 and 5.2. We fix an arbitrary $\rho > 0$ and define the desired operators $\mathcal{R}_j$ as the restrictions of operators (7.1) constructed in Theorem 7.1 to the cylinder $\mathbb{R} \times \mathbb{B}_{m_c-1,k+m_h}(\rho)$. All assertions of Theorems 5.1 and 5.2 are readily deduced from Theorem 7.1. For instance, let us prove the attraction property.

Suppose that a solution $u(t,x)$ of (0.3), (1.2) is defined throughout the time axis and that its phase trajectory $\mathcal{D}(t)u$ is entirely contained in the ball $\mathbb{B}_{m-1,k}(\rho)$.

Then $u(t,x)$ is the solution of problem (5.9), (5.10) and belongs to the space $\mathbb{F}_{m-1,k,[\mu,-\mu]}$. Therefore, according to assertion (ii) in Theorem 7.1, $\mathcal{D}(t)u \in \mathcal{M}(t)$ for all $t \in \mathbb{R}$. It remains to note that the manifold $\mathcal{M}(t,\rho)$ defined in (5.2) is the intersection of $\mathcal{M}(t)$ (see (5.14)) and the ball $\mathbb{B}_{m-1,k}(\rho)$.

We now suppose that $u(t,x)$ is the solution of (0.3), (1.2) with $J = [\theta, +\infty)$ and that $\mathcal{D}(t)u \in \mathbb{B}_{m-1,k}(\rho_1)$ for $t \geqslant \theta$ and some $\rho_1 < \rho$. Let us extend $u(t,x)$ throughout the time axis $\mathbb{R}$ as the solution of (5.9), (5.10), (1.1) with Cauchy data

$$u_j(x) = \partial_t^j u(\theta + 0, x), \quad j = 0, \dots, m-1.$$

Clearly, the extended function $u(t,x)$ satisfies all conditions in assertion (iii) of Theorem 7.1. Consequently, there is a solution $v(t,x)$ of problem (5.9), (5.10) such that $\mathcal{D}(t)v \in \mathcal{M}(t)$ for all $t \in \mathbb{R}$ and (7.2) holds. We now note that the phase trajectory $\mathcal{D}(t)v$ is contained in the ball $\mathbb{B}_{m-1,k}(\rho)$ for $t \geqslant T \gg 1$ since it is exponentially attracted to $\mathcal{D}(t)u$ as $t \to +\infty$. Hence, $v(t,x)$ also satisfies (0.3) for $t \geqslant T$. Setting $V_0 = \mathcal{D}(T)v$, we derive (5.3) from (7.2).

The proof of the remaining assertions is left to the reader.

7.3. Proof of Theorem 5.3. (i) By (5.2), if $\mathcal{D}(t)u \in \mathcal{M}(t,\rho)$ for $t \in J$, then

$$\partial_t^{m_c} u = \mathcal{R}_{m_c}(\varepsilon; t, u, \partial_t u, \dots, \partial_t^{m_c-1} u), \quad t \in J. \tag{7.20}$$

Note that the remainder $R_{m_c}(\varepsilon, y, \tau, \xi)$ on dividing the polynomial τ^{m_c} by $P_c(\varepsilon, y, \tau, \xi)$ is equal to $P_c(\varepsilon, y, \tau, \xi) - \tau^{m_c}$. Therefore, according to (5.5),

$$\mathcal{R}_{m_c}(\varepsilon; t, u, \partial_t u, \dots, \partial_t^{m_c-1} u) = \partial_t^{m_c} u - P_c(\varepsilon, t, x, \partial) u + \varepsilon \mathcal{B}_{m_c}(\varepsilon; t, u, \dots, \partial_t^{m_c-1} u).$$

Substituting this formula into (7.20), we obtain (5.7).

(ii) We begin by showing that the solution of problem (5.7), (5.8) is uniquely determined by the Cauchy data up to order $m_c - 1$ at an arbitrary point $\theta \in I$.

Indeed, suppose that u_1 and u_2 are two solutions of (5.7), (5.8) and that $\mathcal{D}_c(\theta)u_1 = \mathcal{D}_c(\theta)u_2$. Then the difference $w = u_1 - u_2$ satisfies (5.8), the initial conditions

$$\partial_t^j w(\theta, x) = 0, \quad j = 0, \dots, m_c - 1,$$

and the equation

$$P_c(\varepsilon, t, x, \partial) w(t,x) = h(t,x),$$

where $h(t,x) = \varepsilon\big(\mathcal{B}_{m_c}(\varepsilon; t, u_1, \dots, \partial_t^{m_c-1} u_1) - \mathcal{B}_{m_c}(\varepsilon; t, u_2, \dots, \partial_t^{m_c-1} u_2)\big)$.

By Theorem 5.2 (ii), the right-hand side $h(y)$ belongs to the space[10] $C(I, H^{(k+m_h)})$, and we have

$$\|h(t,\cdot)\|_{(k+m_h)} \leqslant C_1 |\varepsilon| \, E_{m_c-1,k+m_h}(u_1 - u_2, t), \quad t \in I, \tag{7.21}$$

[10]It is here that we use the property that q does not involve derivatives of order $m-1$, whence, by Theorem 5.2 (ii), it follows that the range of the operator $\mathcal{B}_{m_c}$ is $H^{(k+m_h)}$ rather than $H^{(k+m_h-1)}$.

where the constant $C_1 > 0$ does not depend on ε or t. We shall use an estimate for the norm of the solution of the Cauchy problem for strictly hyperbolic equations (see [12], Lemma 23.2.1):

$$E_{m_c-1,k+m_h}(w,t) \leqslant C_2 \int_{[\theta,t]} \|h(r,\cdot)\|_{(k+m_h)} dr, \quad t \in I. \tag{7.22}$$

Comparing (7.21) and (7.22), we conclude that $u_1(t,x) \equiv u_2(t,x)$ for $|\varepsilon| \ll 1$.

We now suppose that the interval $I_1 \subset I$ satisfies the conditions in assertion (ii) of Theorem 5.3. Let us fix an arbitrary $\theta \in I_1$ and consider the solution $v(t,x)$ of problem (0.3), (1.1), (1.2), where

$$u_j(x) = \begin{cases} \partial_t^j u(\theta,x), & j = 0,\dots,m_c-1, \\ \mathcal{R}_j(\varepsilon;\theta,u(\theta,\cdot),\dots,\partial_t^{m_c-1}u(\theta,x)), & j = m_c,\dots,m-1. \end{cases}$$

By construction, the initial vector function $[u_0,\dots,u_{m-1}]$ lies on the manifold $\mathcal{M}(\theta,\rho)$. Consequently, by local invariance (see Theorem 5.1), the phase trajectory $\mathcal{D}(t)v$ belongs to $\mathcal{M}(t,\rho)$ as long as it stays in the ball $\mathbb{B}_{m-1,k}(\rho)$. Therefore, according to assertion (i), the function $v(t,x)$ satisfies (5.7). Moreover, its Cauchy data up to order $m_c - 1$ at the initial instant of time $t = \theta$ coincide with the Cauchy data for the function $u(t,x)$. Since the solution of problem (5.7), (5.8), (5.11) is unique, $u(t,x)$ and $v(t,x)$ coincide on the common domain. It follows that the maximum interval J on which $v(t,x)$ is defined contains I_1 and that $u(t,x) = v(t,x)$ for $t \in I_1$. The theorem is proved.

§ 8. Appendix

In this section we collect some auxiliary assertions used in the main body of the text.

Lemma 8.1. *Let $\rho > 0$, let $0 < \gamma \leqslant 1$, and let $k > n/2$ and $l \geqslant 0$ be integers. Then there is a constant $C = C(\rho,\gamma,k,l) > 0$ such that*

$$\left\| Q_\rho(\varepsilon,t,U+V) - \sum_{j=0}^{l} \frac{1}{j!}(D^j Q_\rho)(\varepsilon,t,U)[V,\dots,V] \right\|_{(k)} \leqslant C\,\|V\|_{m-1,k}^{l+\gamma}, \tag{8.1}$$

where $U, V \in \mathbb{E}_{m-1,k}$ and $D^j Q_\rho$ denotes the jth derivative of Q_ρ with respect to U.

Proof. According to Proposition 4.1, the operator

$$Q_\rho(\varepsilon,t,U)\colon \mathbb{R}_t \times \mathbb{E}_{m-1,k} \to H^{(k)} \tag{8.2}$$

is infinitely Frechét differentiable and all its derivatives are uniformly bounded. Therefore inequality (8.1) with $\gamma = 1$ is an obvious consequence of Taylor's formula. Let us show that any number in the interval $(0,1)$ can serve as γ.

Since $Q_\rho(\varepsilon,t,U) = 0$ for $\|U\|_{m-1,k} \geqslant \rho$, it can be assumed that $\|U\|_{m-1,k} < \rho$. We first suppose that $\|V\|_{m-1,k} \geqslant 2\rho$. Then $\|U+V\|_{m-1,k} \geqslant \rho$, and inequality (8.1) follows from the uniform boundedness of the derivatives $D^j Q_\rho$,

$$\| (D^j Q_\rho)(\varepsilon,t,U)[V,\dots,V] \|_{(k)} \leqslant \text{const}\, \|V\|_{m-1,k}^{j}.$$

Now let $\|V\|_{m-1,k} \leqslant 2\rho$. By Taylor's formula, the left-hand side of (8.1) has const $\|V\|_{m-1,k}^{l+1}$ as an upper bound. It remains to note that

$$\|V\|_{m-1,k}^{l+1} \leqslant (2\rho)^{1-\gamma} \|V\|_{m-1,k}^{l+\gamma} \quad \text{for} \quad \|V\|_{m-1,k} \leqslant 2\rho.$$

Recall that we set $S(\theta)w(t,x) = w(t+\theta, x)$ for the function $w(t,x)$.

Lemma 8.2. *For any $\rho > 0$ and $\mu \geqslant 0$ and an arbitrary integer $k > n/2$, the composite operator*

$$\mathcal{C}\colon u(y) \mapsto Q_\rho(\varepsilon, t, \mathcal{D}(t)u)$$

is a continuous operator from $\mathbb{F}_{m-1,k,[\mu,-\mu]}$ to $\mathbb{F}_{0,k,[\mu,-\mu]}$ and satisfies a global Lipschitz condition. Moreover, if $u, v \in \mathbb{F}_{m-1,k,[\mu,-\mu]}$, then

$$E_{0,k,[\mu,-\mu]}\big(S(\theta)(\mathcal{C}u - \mathcal{C}v)\big) \leqslant C\, E_{m-1,k,[\mu,-\mu]}\big(S(\theta)(u-v)\big) \tag{8.3}$$

for any $\theta \in \mathbb{R}$, where the constant $C = C(k,\rho) > 0$ does not depend on u, v, θ, or μ.

Proof. Since the operator (8.2) is infinitely continuously differentiable and its derivatives are bounded, it follows that if $u \in \mathbb{F}_{m-1,k,[\mu,-\mu]}$, then $\mathcal{C}u \in \mathbb{F}_{0,k,[\mu,-\mu]}$. Moreover, by the mean value formula, we have

$$\big\|Q_\rho(\varepsilon, t, \mathcal{D}(t)u) - Q_\rho(\varepsilon, t, \mathcal{D}(t)v)\big\|_{(k)} \leqslant \text{const}\, E_{m-1,k}(u-v, t),$$

whence follows (8.3).

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