Exponential mixing for randomly forced PDE's: method of coupling

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Abstract

The paper is devoted to the description of a coupling method that enables one to study ergodic properties of random dynamical systems associated with stochastic PDE's. This approach was developed in recent years by several authors. We first establish a general criterion for uniqueness of stationary measure and an exponential mixing property. We next illustrate the method on the example of a complex Ginzburg–Landau equation.

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Contents

0 Introduction

The method of coupling was introduced in the famous work of Doeblin [Doe40] to study ergodic properties of Markov chains. To make the main idea of this paper more transparent, let us briefly describe the Doeblin approach in the simplest situation.

Let X be a compact metric space and let $(u_k, \mathbb{P}_u)$ be a family of Markov chains in X parametrised by the initial point $u \in X$. We shall denote by $P_k(u, \Gamma)$ the transition function associated with the Markov family, that is,

$$P_k(u, \Gamma) = \mathbb{P}_u\{u_k \in \Gamma\} \quad \text{for } k \geq 0,\ \Gamma \in \mathcal{B}_X,$$

where $\mathcal{B}_X$ stands for the Borel σ-algebra on X. Recall that a probability measure μ on the space $(X, \mathcal{B}_X)$ is said to be *stationary for* $(u_k, \mathbb{P}_u)$ if

$$\mu(\Gamma) = \int_X P_1(u, \Gamma)\mu(du) \quad \text{for any } \Gamma \in \mathcal{B}_X. \tag{0.1}$$

Suppose there is a constant $\gamma < 1$ such that

$$\|P_1(u, \cdot) - P_1(u', \cdot)\|_{\text{var}} \leq \gamma \tag{0.2}$$

for any $u, u' \in X$, where $\|\cdot\|_{\text{var}}$ denotes the total variation distance. In this case, one can use the following argument to prove that the family $(u_k, \mathbb{P}_u)$ has a unique stationary measure.[1]

Let $(\mathcal{R}(u, u', \cdot), \mathcal{R}'(u, u', \cdot))$ be a pair of random variables depending on $u, u' \in X$ such that the laws of $\mathcal{R}$ and $\mathcal{R}'$ coincide with $P_1(u, \cdot)$ and $P_1(u', \cdot)$, respectively, and

$$\mathbb{P}\bigl\{\mathcal{R}(u, u') \neq \mathcal{R}'(u, u')\bigr\} = \|P_1(u, \cdot) - P_1(u', \cdot)\|_{\text{var}} \quad \text{for all } u, u' \in X. \tag{0.3}$$

[1] It would be easier to observe that the right-hand side of (0.1) defines a contraction in the space of probability measures on X (endowed with the total variation distance) and therefore has a unique fixed point. However, we use a longer coupling argument whose development is applied in the paper.

It can be shown that such random variables exist (see [Lin92]). Let us denote by Ω the direct product of countably many independent copies of the probability space on which $\mathcal{R}$ and $\mathcal{R}'$ are defined and consider a family of Markov chains $\{U_k\}$ in $\boldsymbol{X} = X \times X$ given by the rule

$$U_0(\omega) = U, \qquad U_k(\omega) = (\mathcal{R}(U_{k-1}, \omega_k), \mathcal{R}'(U_{k-1}, \omega_k)) \quad \text{for } k \geq 1, \tag{0.4}$$

where $\omega = (\omega_j, j \geq 1) \in \Omega$ denotes the random parameter and $U \in \boldsymbol{X}$ is an initial point. Writing $U = (u, u')$ and $U_k = (u_k, u'_k)$, we derive from (0.2) and (0.3) that

$$\mathbb{P}_U\{u_{k+1} \neq u'_{k+1} \,|\, \mathcal{F}_k\} \leq \gamma \quad \text{for any } U \in \boldsymbol{X},\, k \geq 0, \tag{0.5}$$

where $\mathcal{F}_k$ denotes the σ-algebra generated by $U_1, \dots, U_k$ and the subscript U indicates that we consider the trajectory starting from U. Iterating inequality (0.5), we obtain

$$\mathbb{P}_U\{u_k \neq u_k\} \leq \gamma^k \quad \text{for any } U \in \boldsymbol{X},\, k \geq 0. \tag{0.6}$$

This estimate implies that

$$\|P_k(u, \cdot) - P_k(u', \cdot)\|_{\text{var}} \leq \gamma^k. \tag{0.7}$$

Combining (0.7) with (0.1) and the Kolmogorov–Chapman relation, we can easily show that there is at most one stationary measure. Moreover, it follows from (0.7) that the sequence $\{P_k(u, \cdot)\}$ converges to a limiting measure μ, which is stationary for $(u_k, \mathbb{P}_u)$.

The Doeblin argument can be used to prove uniqueness of stationary measure for stochastic differential equations (SDE) with non-degenerate diffusion on a compact manifold. At the same time, application of the above scheme to SDE's in $\mathbb{R}^n$ encounters an obstacle related to the fact that the phase space of the problem is not compact, and inequality (0.2) cannot be satisfied uniformly in u and u', unless some restrictive conditions are imposed on the drift. However, one can overcome this difficulty with the help of the following modification of the Doeblin approach.

Let X be a separable Banach space with a norm $\|\cdot\|$ and let $(u_k, \mathbb{P}_u)$ be a family of Markov chains in X. Retaining the notation used above, suppose we can find a closed subset $B \subset X$ for which the two properties below are satisfied:

(i) Inequality (0.2) holds for any $u, u' \in B$ and a constant $\gamma < 1$.

(ii) The first hitting time τ_B of the set B is almost surely finite for any initial point $u \in X$, and there is $\delta > 0$ such that

$$\mathbb{E}_u \exp\big(\delta \tau_B\big) < \infty \quad \text{for all } u \in X. \tag{0.8}$$

Let $(\mathcal{R}, \mathcal{R}')$ be the family of random variables in $\boldsymbol{X}$ defined above and let $\{U_k\}$ be the family of Markov chains given by (0.4). Denote by ρ_n the n-th instant when the trajectory U_k enters the set $\boldsymbol{B} := B \times B$. Then, using (0.2), (0.3), and the strong Markov property (SMP), it can be shown that (cf. (0.5))

$$\mathbb{P}\{u_{\rho_n+1} \neq u'_{\rho_n+1} \,|\, \mathcal{F}_{\rho_n}\} \leq \gamma \quad \text{for any } U \in \boldsymbol{X},\ n \geq 1, \tag{0.9}$$

where $\mathcal{F}_{\rho_n}$ denotes the σ-algebra associated with the Markov time ρ_n. Iteration of (0.9) results in (cf. (0.6))

$$\mathbb{P}_U\{u_{\rho_n+1} \neq u'_{\rho_n+1}\} \leq \gamma^n \quad \text{for any } U \in \boldsymbol{X},\ n \geq 1.$$

Combining this with (0.8), one can prove inequality (0.7) with a larger constant $\gamma < 1$. Thus, the Doeblin method applies also in the case of unbounded phase space, provided that inequality (0.2) is satisfied on a subset that can be reached exponentially fast from any initial point. However, it should be noted that inequality (0.2) is rather restrictive for Markov chains in an infinite-dimensional space. For instance, in the case of stochastic partial differential equations (SPDE), it is satisfied only if the diffusion is "very rough." The aim of this paper is to establish a general criterion for uniqueness of stationary measure and exponential mixing and to show how to apply it to a complex Ginzburg–Landau (CGL) equation. Without going into details, let us describe our scheme in the case of discrete time.

As before, we consider a Markov family $(u_k, \mathbb{P}_u)$ in a separable Banach space X and denote by $P_k(u, \Gamma)$ its transition function. Suppose we can construct a family of Markov chains $(U_k, \mathbb{P}_U)$, $U_k = (u_k, u'_k)$, in the product space $\boldsymbol{X}$ such that the laws of u_k and u'_k under $\mathbb{P}_U$, $U = (u, u')$, coincide with $P_k(u, \cdot)$ and $P_k(u', \cdot)$, respectively, and the following two properties hold (cf. properties (i) and (ii) above):

(i') Let $\sigma = \min\{k \geq 1 : \|u_k - u'_k\| > \gamma^k\}$, where $\gamma < 1$ is a positive constant and the minimum over an empty set is $+\infty$. Then there is a subset $\boldsymbol{B} \subset \boldsymbol{X}$ and positive constants C and $\alpha < 1$ such that

$$\mathbb{P}_U\{\sigma = +\infty\} \geq \tfrac{1}{2}, \quad \mathbb{P}_U\{\sigma = k\} \leq C\alpha^k \quad \text{for } U = (u, u') \in \boldsymbol{B}.$$

(ii') Let $\tau_{\boldsymbol{B}} = \min\{k \geq 0 : U_k \in \boldsymbol{B}\}$. Then there is $\delta > 0$ such that

$$\mathbb{E}_U \exp\bigl(\delta \tau_{\boldsymbol{B}}\bigr) < \infty \quad \text{for any } U \in \boldsymbol{X}.$$

In this case, the difference $P_k(u, \cdot) - P_k(u', \cdot)$, regarded as a signed measure in X, goes to zero in the dual Lipschitz norm $\|\cdot\|^*_{\mathcal{L}}$ exponentially fast. (See Notation for the definition of $\|\cdot\|^*_{\mathcal{L}}$.) Indeed, it follows from (i') that, each time the process is in $\boldsymbol{B}$, with probability $\geq \frac{1}{2}$ we have $\sigma = +\infty$, which means that the difference $\Delta_k = \|u_k - u'_k\|$ goes to zero exponentially fast. Let us consider a sequence of stopping times ρ_k defined by the following rule. Denote by ρ_0 the first hitting time of $\boldsymbol{B}$ (i.e., $\rho_0 = \tau_{\boldsymbol{B}}$). With probability $\geq \frac{1}{2}$, we have $\sigma = +\infty$ for the chain starting from U_{ρ_0}, and in this case we set $\rho_k = +\infty$ for $k \geq 2$. Otherwise we denote by ρ the first instant after σ when U_{ρ_0+k} hits $\boldsymbol{B}$

and define ρ_1 by the formula $\rho_1 = \rho_0 + \rho$. In general, if ρ_k is already defined, then $\rho_{k+1} = \rho_k + \rho$, where ρ is the first instant after σ when the chain starting from U_{ρ_k} hits $\boldsymbol{B}$. As in the case of ρ_0, with probability $\geq \frac{1}{2}$ we have $\rho_l = +\infty$ for $l \geq k+1$.

The above construction implies that, if $\rho_k < +\infty$ and $\rho_{k+1} = +\infty$, then $\Delta_{\rho_k+m} \leq \gamma^m$ for all $m \geq 0$. Using the strong Markov property and assertions (i') and (ii'), it can be shown that $\mathbb{P}_U\{\rho_k < +\infty\} \leq 2^{-k}$. What has been said implies that, with probability $\geq 1 - 2^{-k-1}$, we have

$$\|u_k - u'_k\| \leq \gamma^{k-\rho_k} \quad \text{for all } k \geq \rho_k. \tag{0.10}$$

Moreover, further analysis enables one to show that

$$\mathbb{P}_U\big\{k/2 \leq \rho_k < \infty\big\} \leq C\beta^k, \tag{0.11}$$

where C and $\beta < 1$ are positive constants. Combining (0.10) and (0.11), we see that

$$\mathbb{P}_U\big\{\|u_k - u'_k\| > \gamma^{k/2}\big\} \leq 2^{-k-1} + C\beta^k \quad \text{for } k \geq 1.$$

Thus, the difference $\|u_k - u'_k\|$ converges to zero in probability exponentially fast. This property implies the uniqueness of stationary measure.

Let us mention that the problem of ergodicity for randomly forced equations of mathematical physics was in the focus of attention of many researchers during the last ten–fifteen years, and first results in this direction were obtained in the papers [Sin91, FM95, KS00, EMS01, BKL02]. We refer the reader to the review papers [ES00, Kuk02, Bri02, Shi05b] and to the book [Kuk06] for a detailed account of the results obtained so far. The coupling technique described above is a modified version of the one used in [KS01, KS02, Shi04]. Related approaches were also developed in [Mat02, MY02, Hai02, Oda06].

The paper is organised as follows. In Section 1, we give a description of random dynamical systems (RDS) studied in this work and introduce the concept of an extension for RDS. A general criterion (in terms of extension) for uniqueness of stationary measure and exponential mixing is presented in Section 2. In the third section, we give some simple sufficient conditions under which one of the hypotheses of our criterion is satisfied. The fourth section is devoted to application of these results to complex Ginzburg–Landau equation with random perturbation. We also formulate an open problem. Finally, in Appendix, we present two auxiliary results used in the main text.

Notation

Let X be a separable Banach space endowed with its Borel σ-algebra $\mathcal{B}_X$. Denote by B_R the ball in X of radius R centred at origin, by $\mathcal{P}(X)$ the set of probability measures on $(X, \mathcal{B}_X)$, by $C(X)$ the space of continuous functions $f : X \to \mathbb{R}$, and by $\mathcal{L}(X)$ the space of functions $f \in C(X)$ such that

$$\|f\|_{\mathcal{L}} := \sup_{u \in X} |f(u)| + \sup_{u \neq v} \frac{|f(u) - f(v)|}{\|u - v\|} < \infty,$$

where $\|\cdot\|$ stands for the norm in X. The space $\mathcal{P}(X)$ is endowed with either the total variation distance,

$$\|\mu_1 - \mu_2\|_{\mathrm{var}} := \sup_{\Gamma \in \mathcal{B}_X} |\mu_1(\Gamma) - \mu_2(\Gamma)|,$$

or the dual Lipschitz distance,

$$\|\mu_1 - \mu_2\|_{\mathcal{L}}^* := \sup_{\|f\|_{\mathcal{L}} \le 1} \bigl|(f, \mu_1) - (f, \mu_2)\bigr|,$$

where (f, μ) denotes the integral of the function f with respect to the measure μ. The space $\mathcal{P}(X)$ is complete with respect to both metrics $\|\cdot\|_{\mathrm{var}}$ and $\|\cdot\|_{\mathcal{L}}^*$ (see [Dud89]).

Let $D \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary ∂D and let $T > 0$ be a constant. We shall use the following functional spaces.

$L^2 = L^2(D, \mathbb{C})$ is the space of complex-valued square-integrable functions on D.

$H^1 = H^1(D, \mathbb{C})$ is the Sobolev space of order 1.

$H_0^1 = H_0^1(D, \mathbb{C})$ is the space of functions $u \in H^1$ vanishing on ∂D.

$C^k(0, T; X)$ is the space of continuous functions $u : [0, T] \to X$ that are k times continuously differentiable. In the case $k = 0$, we shall write $C(0, T; X)$.

$L^2(0, T; X)$ is the space of Bochner-measurable square-integrable functions on the interval $[0, T]$ with range in X.

If a and b are real numbers, then $a \vee b$ ($a \wedge b$) stands for their maximum (minimum). For a random variable ξ, we denote by $\mathcal{D}(\xi)$ its distribution. If A is a subset in a given space, then I_A stands for its indicator function and A^c denotes its complement. We denote by $\mathbb{R}_+$ the half-line $[0, \infty)$.

1 Description of the class of problems

1.1 A class of random dynamical systems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $\mathcal{F}_t$, $t \ge 0$, and a semigroup of measure-preserving transformations $\theta_t : \Omega \to \Omega$ such that $\theta_t^{-1}\mathcal{F}_s \subset \mathcal{F}_{t+s}$. We shall always assume that $\mathcal{F}_t$ is augmented with respect to $(\mathcal{F}, \mathbb{P})$, that is, the σ-algebra $\mathcal{F}_t$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$.

We consider a random dynamical system (RDS) whose trajectories form a Markov process. More precisely, let X be a separable Banach space with a norm $\|\cdot\|$, let $\mathcal{B}_X$ be the Borel σ-algebra on X, and let $S_t(u, \omega)$, $t \ge 0$, $\omega \in \Omega$, $u \in X$, be a continuous RDS over θ_t (see Definitions 1.1.1 and 1.1.2 in [Arn98]). We shall *always* assume that the following two properties hold:

- For a.a. $\omega \in \Omega$, the trajectories $S_t(u, \omega)$, $u \in X$, are continuous in $t \ge 0$.

- For any $u \in X$, the random process $S_t(u,\omega)$, $t \geq 0$, is Markov with respect to the filtration $\mathcal{F}_t$, that is, for any $\Gamma \in \mathcal{B}_X$ and any $t, s \geq 0$, we have

$$\mathbb{P}\left(S_{t+s}(u,\cdot) \in \Gamma \,|\, \mathcal{F}_t\right) = P_s(S_t(u,\omega),\Gamma), \tag{1.1}$$

where the equality holds for a.a. $\omega \in \Omega$, and $P_s(u,\Gamma)$ is the transition function defined by the formula

$$P_t(u,\Gamma) = \mathbb{P}\big\{S_t(u,\cdot) \in \Gamma\big\}, \quad u \in X, \quad \Gamma \in \mathcal{B}_X. \tag{1.2}$$

In what follows, random dynamical systems satisfying the above properties (in particular, the continuity condition with respect to time) will be said to be *Markov*. With every Markov RDS, we shall associate a family of Markov processes parametrised by the initial point $u \in X$. To fix notation, let us briefly recall the corresponding construction.

Let us set

$$\Omega' = X \times \Omega, \quad \mathcal{F}' = \mathcal{B}_X \otimes \mathcal{F}, \quad \mathcal{F}'_t = \mathcal{B}_X \otimes \mathcal{F}_t, \quad \mathbb{P}_u = \delta_u \otimes \mathbb{P},$$

where $\delta_u \in \mathcal{P}(X)$ is the Dirac measure concentrated at $u \in X$ and $\otimes$ denotes the direct product of measures and σ-algebras. For $\omega' = (u,\omega) \in \Omega'$, we set

$$S'_t(\omega') = S_t(u,\omega), \quad \theta'_t\omega' = (S_t(u,\omega), \theta_t\omega).$$

We thus obtain a Feller[2] family $(S'_t, \mathbb{P}'_u)$ of homogeneous Markov processes in the phase space X with the transition function (1.2) and the corresponding Markov semigroups

$$\mathfrak{P}_t f(u) = \int_X P_t(u,dv) f(v), \quad \mathfrak{P}^*_t \mu(\Gamma) = \int_X P_t(u,\Gamma)\mu(du), \tag{1.3}$$

where $f \in C_b(X)$ and $\mu \in \mathcal{P}(X)$. In what follows, we shall drop the prime from the notation and write $\omega, \Omega, S_t, \mathcal{F}, \mathcal{F}_t, \theta_t$ instead of $\omega', \Omega', S'_t, \mathcal{F}', \mathcal{F}'_t, \theta'_t$.

In this paper, we consider Markov RDS associated with the randomly forced complex Ginzburg–Landau (CGL) equation

$$\dot{u} - (\nu + i)\Delta u + i|u|^{2p}u = h(x) + \dot{\zeta}(t,x), \quad x \in D, \tag{1.4}$$

$$u\big|_{\partial D} = 0, \tag{1.5}$$

where $u = u(t,x)$ is a complex-valued unknown function, $D \subset \mathbb{R}^n$ is a bounded domain with smooth boundary ∂D, $h \in L^2(D,\mathbb{C})$ stands for a deterministic function, and $\zeta(t,x)$ is a complex-valued coloured Wiener process. We shall show that the problem in question has a unique stationary measure and possesses a property of exponential mixing. We refer the reader to Section 4.2 for an exact formulation of the result.

[2]The Feller property of the transition function follows from the continuity of $S_t(u,\omega)$ with respect to u and the Lebesgue theorem on dominated convergence.

1.2 Extension of random dynamical systems

Let X be a separable Banach space and let $S_t(u,\omega)$ be a Markov RDS in X over a semigroup θ_t. We define the product space $\boldsymbol{X} = X \times X$ endowed with the usual norm and denote by $\mathcal{B}_{\boldsymbol{X}}$ its Borel σ-algebra. Write $\boldsymbol{u} = (u, u')$ and denote by

$$\Pi_X : \boldsymbol{u} \mapsto u, \quad \Pi'_X : \boldsymbol{u} \mapsto u'$$

the natural projections to the components of $\boldsymbol{u}$. Let $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ be a complete probability space endowed with a filtration $\widehat{\mathcal{F}}_t$, $t \geq 0$, which is assumed to be augmented with respect to $(\widehat{\mathcal{F}}, \widehat{\mathbb{P}})$, and let $\hat{\theta}_t : \widehat{\Omega} \to \widehat{\Omega}$ be a semigroup of measure-preserving transformations such that $\theta_t^{-1}\widehat{\mathcal{F}}_s \subset \widehat{\mathcal{F}}_{t+s}$. Consider a Markov RDS $\boldsymbol{S}_t(\boldsymbol{u}, \hat{\omega})$ in $\boldsymbol{X}$ over $\hat{\theta}_t$.

Definition 1.1. A Markov RDS $\boldsymbol{S}_t$ in $\boldsymbol{X}$ defined on the half-line $t \geq 0$ is called an *extension of* S_t if for any $\boldsymbol{u} = (u, u') \in \boldsymbol{X}$ the distributions of the random processes $\Pi_X \boldsymbol{S}_t(\boldsymbol{u}, \hat{\omega})$ and $\Pi'_X \boldsymbol{S}_t(\boldsymbol{u}, \hat{\omega})$ regarded as random variables in $C(\mathbb{R}_+, X)$ coincide with those of $S_t(u, \omega)$ and $S_t(u', \omega)$, respectively.

In what follows, if S_t is an RDS and $\boldsymbol{S}_t$ is its extension, then we shall denote the corresponding stochastic bases by the same symbol $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, \theta_t)$. Moreover, abusing the notation, we shall write $\boldsymbol{S}_t(\boldsymbol{u}, \omega) = (S_t(\boldsymbol{u}, \omega), S'_t(\boldsymbol{u}, \omega))$. Finally, we shall denote by $(\boldsymbol{S}_t, \mathbb{P}_{\boldsymbol{u}})$ the family of Markov processes associated with $\boldsymbol{S}_t$ and parametrised by the initial point $\boldsymbol{u} \in \boldsymbol{X}$.

Let us note that, if $\boldsymbol{S}_t$ is an extension of S_t, then for any $f \in C(X)$ and $\boldsymbol{u} = (u, u') \in \boldsymbol{X}$, we have

$$\mathbb{E}_{\boldsymbol{u}} f(\Pi_X \boldsymbol{S}_t) = \mathfrak{P}_t f(u), \quad \mathbb{E}_{\boldsymbol{u}} f(\Pi'_X \boldsymbol{S}_t) = \mathfrak{P}_t f(u'). \tag{1.6}$$

This observation, which is a simple consequence of the definition of extension, will be important in the next section (see the proof of Theorem 2.3).

We shall also need an auxiliary concept of *extension on a finite time interval*. More precisely, let $\boldsymbol{\mathcal{R}}_t(\boldsymbol{u}, \omega) = (\mathcal{R}_t(\boldsymbol{u}, \omega), \mathcal{R}'_t(\boldsymbol{u}, \omega))$ be a continuous Markov RDS defined for $t \in [0, T]$, where $T > 0$ is a constant not depending on $(\boldsymbol{u}, \omega)$. (In other words, the properties entering the definition of a Markov RDS hold on the interval $[0, T]$; see Definitions 1.1.1 and 1.1.2 in [Arn98].)

Definition 1.2. The RDS $\boldsymbol{\mathcal{R}}_t = (\mathcal{R}_t, \mathcal{R}'_t)$ in $\boldsymbol{X}$ is called an *extension of* S_t *on* $[0, T]$ if for any $\boldsymbol{u} = (u, u') \in \boldsymbol{X}$ the distributions of the random processes $\mathcal{R}_t(\boldsymbol{u}, \cdot)$ and $\mathcal{R}'_t(\boldsymbol{u}, \cdot)$ regarded as random variables in $C(0, T; X)$ coincide with those of $S_t(u, \cdot)$ and $S_t(u', \cdot)$, respectively.

Given an extension $\boldsymbol{\mathcal{R}}_t$ of S_t on an interval $[0, T]$, we can iterate it to construct an extension defined on the half-line $t \geq 0$. To this end, we denote by $(\Omega^k, \mathcal{F}^k, \mathbb{P}^k, \mathcal{F}_t^k, \theta_t^k)$, $k \geq 1$, a countable family of independent copies of the stochastic bases on which $\boldsymbol{\mathcal{R}}_t$ is defined. Let us consider a new stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, \theta_t)$ defined by the following rules:

- The space Ω is the product of Ω^k, $k \geq 1$, and its points are denoted by $\omega = (\omega_1, \omega_2, \dots)$.

- The σ-algebra $\mathcal{F}$ is the direct product of $\mathcal{F}^k$, $k \geq 1$, completed with respect to the product measure $\mathbb{P} = \mathbb{P}^1 \otimes \mathbb{P}^2 \otimes \cdots$.
- If $t = (k-1)T + s$, where $k \geq 1$ is an integer and $0 \leq s < T$, then $\mathcal{F}_t$ is the augmentation (with respect to $(\mathcal{F}, \mathbb{P})$) of the σ-algebra generated by the sets of the form

$$\Gamma = \bigl\{\omega = (\omega_1, \omega_2, \dots) : \omega_m \in \Gamma_m \text{ for } m = 1, \dots, k\bigr\},$$

where $\Gamma_m \in \mathcal{F}_T^m$ for $m = 1, \dots, k-1$ and $\Gamma_k \in \mathcal{F}_s^k$. Furthermore, the shift operator θ_t is given by the formula

$$\theta_t \omega = \theta_t(\omega_1, \omega_2, \dots) = (\theta_s^k \omega_k, \theta_s^{k+1} \omega_{k+1}, \dots)$$

An extension $\boldsymbol{S}_t$ on $t \geq 0$ is now defined by induction. Namely, for $0 \leq t \leq T$ we set

$$\boldsymbol{S}_t(\boldsymbol{u}, \omega) = \boldsymbol{\mathcal{R}}_t(\boldsymbol{u}, \omega_1). \tag{1.7}$$

If $\boldsymbol{S}_t$ is already defined for $0 \leq t \leq kT$, where $k \geq 1$ is an integer, then for $0 \leq s \leq T$ we set

$$\boldsymbol{S}_{kT+s}(\boldsymbol{u}, \omega) = \boldsymbol{\mathcal{R}}_s(\boldsymbol{S}_{kT}(\boldsymbol{u}, \omega), \omega_{k+1}). \tag{1.8}$$

It is a matter of direct verification to show that $\boldsymbol{S}_t(\boldsymbol{u}, \omega)$ is a continuous Markov RDS in $\boldsymbol{X}$ over θ_t and that it is an extension of S_t.

2 Coupling hypothesis

2.1 Markov RDS satisfying a coupling condition

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, \theta_t)$ be a stochastic basis satisfying the conditions formulated in Section 1, let $S_t(u, \omega)$ be a Markov RDS in a separable Banach space X, and let $\mathfrak{P}_t$ and $\mathfrak{P}_t^*$ be the corresponding Markov semigroups (see (1.3)). Recall that $\mu \in \mathcal{P}(X)$ is called a *stationary measure* for $S_t(u, \omega)$ if $\mathfrak{P}_t^* \mu = \mu$ for all $t \geq 0$.

Definition 2.1. We shall say that S_t is *exponentially mixing* if it has a unique stationary measure $\mu \in \mathcal{P}(X)$, and there is a constant $\gamma > 0$ and an increasing function $V : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $u \in X$, we have

$$\|P_t(u, \cdot) - \mu\|_{\mathcal{L}}^* \leq V(\|u\|) e^{-\gamma t}, \quad t \geq 0. \tag{2.1}$$

Let $\boldsymbol{S}_t(\boldsymbol{u}, \omega)$ be an extension of $S_t(u, \omega)$ (see Section 1.2). Let us fix positive constants C, β and a closed subset $\boldsymbol{B} \subset \boldsymbol{X}$ and introduce the stopping times

$$\tau_{\boldsymbol{B}} = \tau_{\boldsymbol{B}}(\boldsymbol{u}, \omega) = \inf\bigl\{t \geq 0 : \boldsymbol{S}_t(\boldsymbol{u}, \omega) \in \boldsymbol{B}\bigr\}, \tag{2.2}$$

$$\sigma = \sigma(\boldsymbol{u}, \omega) = \inf\bigl\{t \geq 0 : \|S_t(\boldsymbol{u}, \omega) - S_t'(\boldsymbol{u}, \omega)\| \geq C\, e^{-\beta t}\bigr\}, \tag{2.3}$$

where $\boldsymbol{u} = (u, u')$, and the infimum over an empty set is $+\infty$. In other words, $\tau_{\boldsymbol{B}}$ is the first hitting time of the closed set $\boldsymbol{B}$ for the trajectory $\boldsymbol{S}_t(\boldsymbol{u}, \omega)$ and σ

is the first instance when the curves $S_t(\boldsymbol{u},\omega)$ and $S_t'(\boldsymbol{u},\omega)$ "stop converging" to each other exponentially fast. In particular, if $\sigma(\boldsymbol{u},\omega)=\infty$, then

$$\|S_t(\boldsymbol{u},\omega)-S_t'(\boldsymbol{u},\omega)\|\le C\,e^{-\beta t}\quad\text{for } t\ge 0. \tag{2.4}$$

Definition 2.2. We shall say that the RDS $S_t(u,\omega)$ satisfies the *coupling hypothesis* if it has an extension $\boldsymbol{S}_t(\boldsymbol{u},\omega)$ possessing the following properties:

(i) There is a constant $\delta>0$, a closed set $\boldsymbol{B}\subset\boldsymbol{X}$, and an increasing function $g(r)\ge 1$ of the variable $r\ge 0$ such that

$$\mathbb{E}_{\boldsymbol{u}}\exp\bigl(\delta\tau_{\boldsymbol{B}}\bigr)\le G(\boldsymbol{u})\quad\text{for all } \boldsymbol{u}=(u,u')\in\boldsymbol{X}, \tag{2.5}$$

where we set $G(\boldsymbol{u})=g(\|u\|)+g(\|u'\|)$.

(ii) There are positive constants δ_1, δ_2, c, K, and $q>1$ such that

$$\mathbb{P}_{\boldsymbol{u}}\bigl\{\sigma=\infty\bigr\}\ge\delta_1, \tag{2.6}$$
$$\mathbb{E}_{\boldsymbol{u}}\bigl\{I_{\{\sigma<\infty\}}\exp\bigl(\delta_2\sigma\bigr)\bigr\}\le c, \tag{2.7}$$
$$\mathbb{E}_{\boldsymbol{u}}\bigl\{I_{\{\sigma<\infty\}}G(\boldsymbol{S}_\sigma)^q\bigr\}\le K \tag{2.8}$$

for any $\boldsymbol{u}\in\boldsymbol{B}$.

Any extension of S_t satisfying properties (i) and (ii) will be called a *mixing extension*.

Before formulating the main result of this section, we wish to make some comments on the above definition. Let us take an arbitrary initial point $\boldsymbol{u}\in\boldsymbol{B}$. Then, in view of (2.6), with probability $\ge\delta_1$, we have $\sigma=\infty$, and therefore, with the same probability, the trajectories $S_t(\boldsymbol{u},\omega)$ and $S_t'(\boldsymbol{u},\omega)$ converge to each other exponentially fast (see (2.4)). On the other hand, if they do not, inequality (2.7) says that the first instant $\sigma(\boldsymbol{u},\omega)$ when the trajectories "stop converging" to each other is not very large. Moreover, by (2.8), we have some control over $\boldsymbol{S}_t(\boldsymbol{u},\omega)$ at the instant $t=\sigma(\boldsymbol{u},\omega)$. If the initial point $\boldsymbol{u}\in\boldsymbol{X}$ does not belong to $\boldsymbol{B}$, we cannot claim that the above properties hold. However, we know that, with probability 1, any trajectory hits the set $\boldsymbol{B}$, and by (2.5), the first hitting time $\tau_{\boldsymbol{B}}$ has a finite exponential moment.

These observations make it plausible that, for any initial point $\boldsymbol{u}\in\boldsymbol{X}$, the trajectories $S_t(\boldsymbol{u},\omega)$ and $S_t'(\boldsymbol{u},\omega)$ converge to each other exponentially fast. In fact, we have the following result, whose proof is given in the next subsection.

Theorem 2.3. *Let $S_t(u,\omega)$ be a continuous Markov RDS satisfying the coupling hypothesis and let $\boldsymbol{S}_t(\boldsymbol{u},\omega)$ be a mixing extension for S_t. Then there is a random time $\ell=\ell(\boldsymbol{u},\omega)$ such that*

$$\|S_t(\boldsymbol{u},\omega)-S_t'(\boldsymbol{u},\omega)\|\le C_1e^{-\beta(t-\ell(\boldsymbol{u},\omega))}\quad\textit{for } t\ge\ell(\boldsymbol{u},\omega), \tag{2.9}$$
$$\mathbb{E}_{\boldsymbol{u}}e^{\alpha\ell}\le C_1\bigl(g(\|u\|)+g(\|u'\|)\bigr), \tag{2.10}$$

where $\boldsymbol{u} \in \boldsymbol{X}$ is an arbitrary initial point, $g(r)$ is the function in Definition 2.2, and C_1, α, and β are positive constants not depending on $\boldsymbol{u}$ and t. If, in addition, there is an increasing function $\tilde{g}(r) \geq 1$, $r \geq 0$, such that

$$\mathbb{E}_u\, g\big(\|S_t\|\big) \leq \tilde{g}(\|u\|) \quad \textit{for } u \in X,\ t \geq 0, \tag{2.11}$$

then $S_t(u,\omega)$ is exponentially mixing, and inequality (2.1) holds with

$$V(r) = 3C_1\big(g(r) + \tilde{g}(0)\big). \tag{2.12}$$

2.2 Proof of Theorem 2.3

We first note that inequalities (2.9), (2.10), and (2.11) imply that $S_t(u,\omega)$ is exponentially mixing. Indeed, to prove this, let us show that, for any $u, u' \in X$, we have

$$\big\|P_t(u,\cdot) - P_t(u',\cdot)\big\|_{\mathcal{L}}^* \leq 3C_1\big(g(\|u\|) + g(\|u'\|)\big)\, e^{-\gamma t}, \quad t \geq 0. \tag{2.13}$$

To this end, we fix an arbitrary functional $f \in \mathcal{L}(X)$ with $\|f\|_{\mathcal{L}} \leq 1$ and note that, in view of (1.6),

$$\begin{aligned} \big|\big(f, P_t(u,\cdot) - P_t(u',\cdot)\big)\big| &= \big|\mathbb{E}_{\boldsymbol{u}}\big(f(S_t) - f(S_t')\big)\big| \leq \mathbb{E}_{\boldsymbol{u}}\big|f(S_t) - f(S_t')\big| \\ &\leq 2\mathbb{P}_{\boldsymbol{u}}\big\{\ell > \tfrac{t}{2}\big\} + \mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\ell \leq \frac{t}{2}\}}\big|f(S_t) - f(S_t')\big|\big\}. \end{aligned} \tag{2.14}$$

In view of (2.10) and the Chebyshev inequality, we have

$$\mathbb{P}_{\boldsymbol{u}}\big\{\ell > \tfrac{t}{2}\big\} \leq C_1\big(g(\|u\|) + g(\|u'\|)\big)\, e^{-\frac{\alpha t}{2}}. \tag{2.15}$$

Furthermore, it follows from the condition $\|f\|_{\mathcal{L}} \leq 1$ and inequality (2.9) that the second term on the right-hand side of (2.14) does not exceed

$$\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\ell \leq \frac{t}{2}\}}\|S_t - S_t'\|\big\} \leq C_1 e^{-\frac{\beta t}{2}}. \tag{2.16}$$

Substituting (2.15) and (2.16) into (2.14), we obtain

$$\big|\big(f, P_t(u,\cdot) - P_t(u',\cdot)\big)\big| \leq 2C_1\big(g(\|u\|) + g(\|u'\|)\big)\, e^{-\frac{\alpha t}{2}} + C_1 e^{-\frac{\beta t}{2}},$$

which implies the required inequality (2.13) with $\gamma = \frac{1}{2}(\alpha \wedge \beta)$.

We now use (2.13) to show that S_t is exponentially mixing. Let us fix arbitrary points $u, u' \in X$ and a functional $f \in \mathcal{L}(X)$ such that $\|f\|_{\mathcal{L}} \leq 1$. By the Kolmogorov–Chapman relation and inequality (2.13), for $t \leq s$ we have

$$\begin{aligned} \big|\big(f, P_t(u,\cdot) - P_s(u',\cdot)\big)\big| &= \left|\int_X P_{s-t}(u',dz)\int_X \big(P_t(u,dv) - P_t(z,dv)\big) f(v)\right| \\ &\leq 3C_1 e^{-\gamma t}\int_X P_{s-t}(u',dz)\big[g(\|u\|) + g(\|z\|)\big] \\ &= 3C_1 e^{-\gamma t}\big[g(\|u\|) + \mathbb{E}_{u'}\, g(\|S_{s-t}\|)\big]. \end{aligned}$$

Taking into account (2.11), we conclude that

$$\big\|P_t(u,\cdot)-P_s(u',\cdot)\big\|_{\mathcal{L}}^* \le 3C_1\big(g(\|u\|)+\tilde g(\|u'\|)\big)\,e^{-\gamma t}. \tag{2.17}$$

By the Prokhorov theorem (see [Dud89, Corollary 11.5.5]), $\mathcal{P}(X)$ is a complete metric space with respect to the norm $\|\cdot\|_{\mathcal{L}}^*$. Hence, we conclude that $P_t(u,\cdot)$ converges, as $t\to+\infty$, to a measure $\mu\in\mathcal{P}(X)$, which does not depend on u and is stationary. Setting $u'=0$ in (2.17) and passing to the limit as $s\to+\infty$, we obtain inequality (2.1) with V given by (2.12).

Thus, we need to establish inequalities (2.9) and (2.10). Their proof is divided into four steps.

Step 1. We introduce the stopping time

$$\rho=\sigma+\tau_{\boldsymbol{B}}\circ\theta_\sigma=\sigma(\boldsymbol{u},\omega)+\tau_{\boldsymbol{B}}\big(\boldsymbol{S}_{\sigma(\boldsymbol{u},\omega)}(\boldsymbol{u},\omega),\theta_{\sigma(\boldsymbol{u},\omega)}\omega\big)\big). \tag{2.18}$$

In other words, we wait until the first instant σ when the trajectories S_t and S_t' "stop converging" to each other and denote by ρ the first hitting time of $\boldsymbol{B}$ after σ. Let δ, δ_1 and δ_2 be the constants in (2.5), (2.6), and (2.7). We claim that, for any $\boldsymbol{u}\in\boldsymbol{B}$,

$$\mathbb{P}_{\boldsymbol{u}}\{\rho=\infty\}\ge\delta_1, \tag{2.19}$$

$$\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\rho<\infty\}}e^{\alpha\rho}\big\}\le a, \tag{2.20}$$

where $\alpha\le\delta_2\wedge\delta$ and $a<1$ are positive constants not depending on $\boldsymbol{u}$. Indeed, the definition of $\rho(\boldsymbol{u},\omega)$ (see (2.18)) implies that $\{\rho=\infty\}=\{\sigma=\infty\}$, and therefore (2.19) is an immediate consequence of (2.6).

To prove (2.20), we first show that

$$\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\rho<\infty\}}e^{\delta_3\rho}\big\}\le M\quad\text{for any }\boldsymbol{u}\in\boldsymbol{B}, \tag{2.21}$$

where $\delta_3=\frac{(q-1)(\delta_2\wedge\delta)}{q}$ and $M>0$ is a constant not depending on $\boldsymbol{u}$. Indeed, using relation (2.18), the strong Markov property (SMP), and inequality (2.5), we derive

$$\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\rho<\infty\}}e^{\delta_3\rho}\big\}=\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\sigma<\infty\}}e^{\delta_3\sigma}\big(\mathbb{E}_{\boldsymbol{S}_\sigma}\,e^{\delta_3\tau_{\boldsymbol{B}}}\big)\big\}\le\mathbb{E}\big\{I_{\{\sigma<\infty\}}e^{\delta_3\sigma}G(\boldsymbol{S}_\sigma)\big\}.$$

Combining this with (2.7) and (2.8), we conclude that

$$\begin{aligned}\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\rho<\infty\}}e^{\delta_3\rho}\big\}&\le\big(\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\sigma<\infty\}}e^{\delta_2\sigma}\big\}\big)^{\frac{q-1}{q}}\big(\mathbb{E}_{\boldsymbol{u}}\,\big\{I_{\{\sigma<\infty\}}G(\boldsymbol{S}_\sigma)^q\big\}\big)^{\frac1q}\\&\le(c^{q-1}K)^{\frac1q}=:M.\end{aligned}$$

To derive (2.20), let us set $\alpha=\varepsilon\delta_3$ and note that, in view of (2.19) and (2.21), we have

$$\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\rho<\infty\}}e^{\alpha\rho}\big\}\le\big(\mathbb{P}_{\boldsymbol{u}}\{\rho<\infty\}\big)^{1-\varepsilon}\big(\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\rho<\infty\}}e^{\delta_3\rho}\big\}\big)^{\varepsilon}\le(1-\delta_1)^{1-\varepsilon}M^{\varepsilon}.$$

The right-hand side of this inequality is less than 1 if $\varepsilon>0$ is sufficiently small.

Step 2. We now consider the iterations of ρ. Namely, we define a sequence of stopping times $\rho_k = \rho_k(\boldsymbol{u}, \omega)$ by the formulas

$$\rho_0 = \tau_{\boldsymbol{B}}, \quad \rho_k = \rho_{k-1} + \rho \circ \theta_{\rho_{k-1}}, \quad k \geq 1.$$

We claim that

$$\mathbb{E}_{\boldsymbol{u}} \big\{ I_{\{\rho_k < \infty\}} e^{\alpha \rho_k} \big\} \leq a^k G(\boldsymbol{u}) \quad \text{for any } \boldsymbol{u} \in \boldsymbol{X}. \tag{2.22}$$

Indeed, since $\boldsymbol{S}_{\rho_k(\boldsymbol{u},\omega)}(\boldsymbol{u}, \omega) \in \boldsymbol{B}$, inequality (2.20) and the SMP imply that

$$\begin{aligned} \mathbb{E}_{\boldsymbol{u}} \big\{ I_{\{\rho_k < \infty\}} e^{\alpha \rho_k} \big\} &\leq \mathbb{E}_{\boldsymbol{u}} \Big\{ I_{\{\rho_{k-1} < \infty\}} e^{\alpha \rho_{k-1}} \sup_{\boldsymbol{v} \in \boldsymbol{B}} \mathbb{E}_{\boldsymbol{v}} \big(I_{\{\rho < \infty\}} e^{\alpha \rho} \big) \Big\} \\ &\leq a \, \mathbb{E}_{\boldsymbol{u}} \big\{ I_{\{\rho_{k-1} < \infty\}} e^{\alpha \rho_{k-1}} \big\} \leq a^k \mathbb{E}_{\boldsymbol{u}} e^{\alpha \tau_{\boldsymbol{B}}}. \end{aligned}$$

The required inequality (2.22) follows now from (2.5) and the fact that $\alpha \leq \delta$.

Step 3. We now note that, if $\rho_k(\boldsymbol{u}, \omega) < \infty$ and $\rho_{k+1}(\boldsymbol{u}, \omega) = \infty$ for an integer $k \geq 0$, then

$$\| S_t(\boldsymbol{u}, \omega) - S_t'(\boldsymbol{u}, \omega) \| \leq C \, e^{-\beta(t - \rho_k(\boldsymbol{u},\omega))} \quad \text{for } t \geq \rho_k(\boldsymbol{u}, \omega). \tag{2.23}$$

For any $\boldsymbol{u} \in \boldsymbol{X}$, let us set

$$\bar{k} = \bar{k}(\boldsymbol{u}, \omega) = \sup\{k \geq 0 : \rho_k(\boldsymbol{u}, \omega) < \infty\}.$$

We wish to show that

$$\bar{k} < \infty \quad \text{for } \mathbb{P}_{\boldsymbol{u}}\text{-almost every } \omega. \tag{2.24}$$

To this end, note that, in view of (2.19) and the SMP,

$$\mathbb{P}_{\boldsymbol{u}}\{\rho_k < \infty\} \leq (1 - \delta_1) \mathbb{P}_{\boldsymbol{u}}\{\rho_{k-1} < \infty\} \leq (1 - \delta_1)^k \mathbb{P}_{\boldsymbol{u}}\{\rho_0 < \infty\} \leq (1 - \delta_1)^k.$$

Hence, the Borel–Cantelli lemma implies (2.24).

Step 4. Let us set

$$\ell = \ell(\boldsymbol{u}, \omega) = \begin{cases} \rho_{\bar{k}(\boldsymbol{u},\omega)}(\boldsymbol{u}, \omega) & \text{if } \bar{k}(\boldsymbol{u}, \omega) < \infty, \\ +\infty & \text{if } \bar{k}(\boldsymbol{u}, \omega) = \infty. \end{cases}$$

Inequality (2.9) follows immediately from (2.23), the definition of ρ_k, and the fact that $\rho_{\ell+1} = \infty$. To prove (2.10), we write

$$\mathbb{E}_{\boldsymbol{u}} e^{\alpha \ell} = \sum_{k=0}^{\infty} \mathbb{E}_{\boldsymbol{u}} \big\{ I_{\{\bar{k} = k\}} e^{\alpha \rho_k} \big\} \leq \sum_{k=0}^{\infty} \mathbb{E}_{\boldsymbol{u}} \big\{ I_{\{\rho_k < \infty\}} e^{\alpha \rho_k} \big\} \leq (1 - a)^{-1} G(\boldsymbol{u}),$$

where we used inequality (2.22) and the fact that $\ell(\boldsymbol{u}, \omega) < \infty$ for $\mathbb{P}_{\boldsymbol{u}}$-a.a. ω. This completes the proof of Theorem 2.3.

Remark 2.4. Analyzing the proof given above, it is not difficult to see that Theorem 2.3 remains valid if $\sigma(\boldsymbol{u}, \omega)$ is replaced with any other stopping time $\tilde{\sigma} \leq \sigma$. In other word, if inequalities (2.6)–(2.9) hold with σ replaced by $\tilde{\sigma}$, then the conclusion of Theorem 2.3 is true. To see this, it suffices to repeat the arguments above, replacing everywhere σ by $\tilde{\sigma}$.

3 Dissipative RDS and their extensions

In this section, we give sufficient conditions for the existence of an extension satisfying inequality (2.5). These results will be used in the next section to prove exponential mixing for the complex Ginzburg–Landau equation.

3.1 Lyapunov function

Let $S_t(u,\omega)$ be a Markov RDS in a separable Banach space X and let $F(u)\ge 1$ be a continuous functional on X tending to $+\infty$ as $\|u\|\to\infty$. Suppose that S_t satisfies the following condition:

(H$_1$) Lyapunov function. There are positive constants t_*, R_*, C_*, and $a<1$ such that

$$\mathbb{E}_u F\bigl(S_{t_*}\bigr) \le a\,F(u) \qquad \text{for } \|u\|\ge R_*, \tag{3.1}$$
$$\mathbb{E}_u F\bigl(S_t\bigr) \le C_* \qquad \text{for } \|u\|\le R_*,\ t\ge 0, \tag{3.2}$$

In what follows, we shall call F a Lyapunov function for S_t. An important property of a Markov RDS possessing a Lyapunov function is that the first hitting time of sufficiently large balls in the phase space is almost surely finite for any initial condition and has a finite exponential moment. Namely, we have the following result:

Proposition 3.1. *Let $S_t(u,\omega)$ be a Markov RDS satisfying Hypothesis* (H$_1$) *and let $\tau_R(u,\omega)$ be the first hitting time of the ball $B_R=\{u\in X:\|u\|\le R\}$, where $R\ge R_*$. Then*

$$\mathbb{P}_u\{\tau_R<\infty\}=1 \quad \textit{for all } u\in X. \tag{3.3}$$

Moreover, there are positive constants δ and C not depending on R and u such that

$$\mathbb{E}_u \exp(\delta\tau_R) \le 1 + CK_R^{-1}F(u), \tag{3.4}$$

where we set

$$K_R = \inf_{\|v\|\ge R} F(v). \tag{3.5}$$

Proposition 3.1 can be established by a standard argument (see [MT93]). However, for the sake of completeness, we give its proof.

Proof of Proposition 3.1. Step 1. The result is trivial for $\|u\|\le R$, since in this case $\tau_R(u,\omega)=0$ for $\mathbb{P}_u$-almost every ω. Let us fix an arbitrary $u\in X$ with $\|u\|>R$ and consider an auxiliary stopping time defined by the formula

$$\bar\tau=\bar\tau(u,\omega)=\min\bigl\{t=mt_*:\|S_t\|\le R,\ m\ge 0 \text{ is an integer}\bigr\}.$$

For any integer $k\ge 0$ and any $v\in X$, we set

$$p_k(v)=\mathbb{E}_v\bigl\{I_{\{\bar\tau>kt_*\}}F(S_{kt_*})\bigr\}. \tag{3.6}$$

We claim that

$$p_k(u) \le a^k F(u) \quad \text{for all } k \ge 0. \tag{3.7}$$

Indeed, the Markov property (1.1) and inequality (3.1) imply that

$$\begin{aligned} p_{k+1}(u) &\le \mathbb{E}_u\bigl\{I_{\{\bar\tau>kt_*\}}\mathbb{E}_u\bigl(F(S_{(k+1)t_*})\,|\,\mathcal{F}_{kt_*}\bigr)\bigr\} \\ &= \mathbb{E}_u\bigl\{I_{\{\bar\tau>kt_*\}}\mathbb{E}_{S_{kt_*}}F(S_{t_*})\bigr\} \\ &\le a\,\mathbb{E}_u\bigl\{I_{\{\bar\tau>kt_*\}}F(S_{kt_*})\bigr\} = ap_k(u), \end{aligned} \tag{3.8}$$

where we used the non-negativity of F and the fact that $\|S_{kt_*}\| > R \ge R_*$ on the set $\{\bar\tau > kt_*\}$. Iterating (3.8) and noting that

$$\mathbb{E}_u\{I_{\{\bar\tau>0\}}F(S_0)\} \le F(u),$$

we arrive at (3.7).

Step 2. It follows from (3.6) and (3.7) that

$$\mathbb{P}_u\{\bar\tau > kt_*\} \le K_R^{-1}\mathbb{E}_u\bigl\{I_{\{\bar\tau>kt_*\}}F(S_{kt_*})\bigr\} \le a^k K_R^{-1}F(u). \tag{3.9}$$

Combining this with the Borel–Cantelli lemma, we see that

$$\mathbb{P}_u\{\bar\tau < \infty\} = 1 \quad \text{for any } u \in X. \tag{3.10}$$

Furthermore, if $\delta > 0$ is so small that $b := e^{\delta t_*}a < 1$, then, by (3.9), we have

$$\begin{aligned} \mathbb{E}_u e^{\delta\bar\tau} &\le 1 + \sum_{k=1}^{\infty}\mathbb{E}_u\bigl\{I_{\{\bar\tau=kt_*\}}e^{\delta\bar\tau}\bigr\} \\ &\le 1 + \sum_{k=1}^{\infty} e^{\delta kt_*}\mathbb{P}_u\{\bar\tau > (k-1)t_*\} \\ &\le 1 + K_R^{-1}F(u)\sum_{k=1}^{\infty} e^{\delta kt_*}a^{k-1} = 1 + CK_R^{-1}F(u), \end{aligned} \tag{3.11}$$

where we set $C = e^{\delta t_*}(1-b)^{-1}$. It remains to note that $\bar\tau \ge \tau_R$, and hence (3.10) and (3.11) imply (3.3) and (3.4). ☐

A result similar to Proposition 3.1 is true for any extension of S_t. More precisely, let $\boldsymbol{S}_t(\boldsymbol{u},\omega)$ be an extension of a Markov RDS satisfying Hypothesis (H_1) and let[3]

$$\tau_R = \min\{t \ge 0 : \|S_t(\boldsymbol{u},\omega)\| \vee \|S_t'(\boldsymbol{u},\omega)\| \le R\}. \tag{3.12}$$

Let $R^* > 0$ be the smallest constant such that $K_{R^*} \ge \frac{2C_*}{1-a}$, where a and C_* are the constants in Hypothesis (H_1) and K_R is defined by (3.5). The assertion below can be established by repeating the arguments in the proof of Proposition 3.1.

[3]The stopping time (3.12) is different from the one defined in Proposition 3.1 for the original RDS. However, we retained the same notation since they play similar roles for $\boldsymbol{S}_t$ and S_t.

Proposition 3.2. *Let $S_t(u,\omega)$ be a Markov RDS satisfying Condition* (H_1) *and let $\boldsymbol{S}_t(\boldsymbol{u},\omega)$ be its extension. Then there are positive constants δ and C such that, for any $\boldsymbol{u}\in\boldsymbol{X}$ and $R\ge R^*$, we have*

$$\mathbb{P}_{\boldsymbol{u}}\{\tau_R<\infty\}=1, \tag{3.13}$$

$$\mathbb{E}_{\boldsymbol{u}}\exp(\delta\tau_R)\le 1+CK_R^{-1}\bigl(F(u)+F(u')\bigr). \tag{3.14}$$

3.2 Dissipation

Let $S_t(u,\omega)$ be a continuous Markov RDS in a separable Banach space X and let $\boldsymbol{\mathcal{R}}_t(\boldsymbol{u},\omega)$ be its extension on an interval $[0,T]$. Suppose that $\boldsymbol{\mathcal{R}}_t=(\mathcal{R}_t,\mathcal{R}_t')$ satisfies the following condition.

(H_2) Dissipation. For any $R>0$ there is a constant $q\in(0,1)$ and an increasing function $\varepsilon(d)>0$ defined for $d>0$ such that, for any $\boldsymbol{u}=(u,u')\in\boldsymbol{X}$ with $\|u\|\vee\|u'\|\le R$ and any $d>0$, we have

$$\mathbb{P}_{\boldsymbol{u}}\bigl\{\|\mathcal{R}_T(\boldsymbol{u},\cdot)\|\vee\|\mathcal{R}_T'(\boldsymbol{u},\cdot)\|\le\{q(\|u'\|\vee\|u'\|)\}\vee d\bigr\}\ge\varepsilon(d). \tag{3.15}$$

In other words, the dissipation condition (H_2) means that for any $d>0$, with positive probability, any ball in X of radius $R\ge d/q$ centred at zero is pushed into a ball of radius qR by the maps $\mathcal{R}_T$ and $\mathcal{R}_T'$. Therefore, it is reasonable to expect that, if $\boldsymbol{S}_t$ is the extension of S_t constructed by iteration of $\boldsymbol{\mathcal{R}}_t$ (see (1.7) and (1.8)), then for any initial point $\boldsymbol{u}\in\boldsymbol{X}$ the trajectory $\boldsymbol{S}_t(\boldsymbol{u},\omega)$ will hit, in a finite time, any ball of given radius centred at zero. We have in fact the following result, which shows that the existence of a Lyapunov function combined with the dissipation property (H_2) implies that the first hitting time of any ball centered at zero has a finite exponential moment (cf. (2.5)).

Proposition 3.3. *Let $S_t(u,\omega)$ be a Markov RDS possessing a Lyapunov function $F(u)$ in the sense of* (H_1) *and let $\boldsymbol{\mathcal{R}}_t(\boldsymbol{u},\omega)$ be its extension defined on an interval $[0,T]$ and satisfying condition* (H_2). *Then for any $d>0$ there are positive constants C and ν such that, for the extension $\boldsymbol{S}_t$ constructed by iteration of $\boldsymbol{\mathcal{R}}_t$, we have*

$$\mathbb{E}_{\boldsymbol{u}}\exp(\nu\tau_d)\le C\bigl(F(u)+F(u')\bigr),\quad \boldsymbol{u}=(u,u')\in\boldsymbol{X}, \tag{3.16}$$

Proof. We first describe the main idea, which is well known; for instance, see Sections 3.7 and 4.2 in [Has80] or Section 13 in [Ver00]. By Proposition 3.2, the first hitting time of the set

$$\boldsymbol{B}_R=\{\boldsymbol{u}\in\boldsymbol{X}:\|u\|\vee\|u'\|\le R\} \tag{3.17}$$

has a finite exponential moment for $R\ge R^*$, and by the dissipation property (H_2), each time the process $\boldsymbol{S}_t$ is in $\boldsymbol{B}_R$, with positive probability it hits $\boldsymbol{B}_d$ in finite (deterministic) time. Combining these two observations with the Markov property, we can prove the required result. An accurate proof is divided into four steps.

Step 1. Let R^* and q be the constants in Proposition 3.2 and Hypotheses (H_2). We fix an arbitrary $d > 0$ and set $l_d = \min\{l \geq 0 : q^l R^* \leq d\}$. It follows from inequality (3.15) and the Markov property that, for any $\boldsymbol{u} \in \boldsymbol{B}_{R^*}$, we have

$$\mathbb{P}_{\boldsymbol{u}}\big\{\boldsymbol{S}_{l_d T} \in \boldsymbol{B}_d\big\} \geq p_d := \varepsilon(d)^{l_d} > 0. \tag{3.18}$$

Step 2. Let us set $\tau = \tau_{R^*}$ and define two sequences of stopping times by the formulas

$$\rho_1' = \tau, \quad \rho_1 = \tau + l_d T, \quad \rho_m' = \rho_{m-1} + \tau \circ \theta_{\rho_{m-1}}, \quad \rho_m = \rho_m' + l_d T, \quad m \geq 2.$$

Consider the events $\Gamma_m = \big\{\boldsymbol{S}_{\rho_n} \notin \boldsymbol{B}_d \text{ for } n = 1 \dots, m\big\}$. Let us show that, for any $\boldsymbol{u} \in \boldsymbol{X}$, the sequence $P_m(\boldsymbol{u}) = \mathbb{P}_{\boldsymbol{u}}(\Gamma_m)$ satisfies the inequality

$$P_m(\boldsymbol{u}) = (1 - p_d)^m, \quad m \geq 1. \tag{3.19}$$

Indeed, by the SMP, for any $m \geq 1$ we have [4]

$$\mathbb{P}_{\boldsymbol{u}}\big\{\boldsymbol{S}_{\rho_m} \notin \boldsymbol{B}_d \,\big|\, \mathcal{F}_{\rho_m'}\big\} = \mathbb{P}_{\boldsymbol{S}(\rho_m')}\big\{\boldsymbol{S}_{l_d T} \notin \boldsymbol{B}_d\big\} \leq 1 - p_d, \tag{3.20}$$

where we used inequality (3.18) and the fact that $\boldsymbol{S}_{\rho_m'} \in \boldsymbol{B}_{R^*}$. Therefore, using again the SMP, we derive

$$P_m(\boldsymbol{u}) = \mathbb{E}_{\boldsymbol{u}}\Big(I_{\Gamma_{m-1}} \mathbb{P}_{\boldsymbol{u}}\big\{\boldsymbol{S}_{\rho_m} \notin \boldsymbol{B}_d \,\big|\, \mathcal{F}_{\rho_m'}\big\}\Big) \leq (1 - p_d)\, P_{m-1}(\boldsymbol{u}).$$

Iterating this inequality and using (3.20) with $m = 1$, we obtain (3.19).

Step 3. We now show that for any $d > 0$ there is a constant $K \geq 1$ such that

$$\mathbb{E}_{\boldsymbol{u}} e^{\delta \rho_m} \leq K^m \big(F(u) + F(u')\big), \quad m \geq 1, \tag{3.21}$$

where $\delta > 0$ is the constant in (3.14). Indeed, applying the SMP and inequalities (3.14) and (3.2) (with $t = l_d T$), we derive

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{u}} e^{\delta \rho_m'} &= \mathbb{E}_{\boldsymbol{u}}\big\{e^{\delta \rho_{m-1}} \mathbb{E}_{\boldsymbol{S}(\rho_{m-1})}(e^{\delta \tau})\big\} \\
&\leq C_1 \mathbb{E}_{\boldsymbol{u}}\big\{e^{\delta \rho_{m-1}}\big(F(S_{\rho_{m-1}}) + F(S'_{\rho_{m-1}})\big)\big\} \\
&\leq C_1 e^{\delta l_d T} \mathbb{E}_{\boldsymbol{u}}\big\{e^{\delta \rho'_{m-1}} \mathbb{E}_{\boldsymbol{S}(\rho'_{m-1})}\big(F(S_{l_d T}) + F(S'_{l_d T})\big)\big\} \\
&\leq C_2 e^{\delta l_d T} \mathbb{E}_{\boldsymbol{u}} e^{\delta \rho'_{m-1}},
\end{aligned}$$

where we used the fact that $\boldsymbol{S}_{\rho_{m-1}} \in \boldsymbol{B}_{R^*}$. Iterating this inequality and using again (3.14), we obtain (3.21).

Step 4. We can now prove inequality (3.16) with sufficiently small $\nu > 0$. To this end, we define the random integer

$$\hat{n} = \min\{n \geq 1 : \boldsymbol{S}_{\rho_n} \in \boldsymbol{B}_d\}$$

[4] We write $\boldsymbol{S}(\rho_m')$ instead of $\boldsymbol{S}_{\rho_m'}$ to avoid a double subscript.

and note that $\tau_d \le \rho_{\hat{n}}$. Moreover, it follows from (3.19) and the Borel–Cantelli lemma that $\mathbb{P}_{\boldsymbol{u}}\{\hat{n} < \infty\} = 1$ for any $\boldsymbol{u} \in \boldsymbol{X}$. Hence, for any $\nu > 0$ we have

$$\begin{aligned} \mathbb{E}_{\boldsymbol{u}} e^{\nu\tau_d} \le \mathbb{E}_{\boldsymbol{u}} e^{\nu\rho_{\hat{n}}} &= \sum_{n=1}^{\infty} \mathbb{E}_{\boldsymbol{u}} \bigl(I_{\{\hat{n}=n\}} e^{\nu\rho_n}\bigr) \\ &\le \mathbb{E}_{\boldsymbol{u}} e^{\nu\rho_1} + \sum_{n=2}^{\infty} \mathbb{E}_{\boldsymbol{u}} \bigl(I_{\Gamma_{n-1}} e^{\nu\rho_n}\bigr) \\ &\le \mathbb{E}_{\boldsymbol{u}} e^{\nu\rho_1} + \sum_{m=1}^{\infty} P_m(\boldsymbol{u})^{\frac{1}{2}} \bigl(\mathbb{E}_{\boldsymbol{u}} e^{2\nu\rho_{m+1}}\bigr)^{\frac{1}{2}} \\ &\le K\Bigl(1 + \sum_{m=1}^{\infty} (1-p_d)^{\frac{m}{2}} K^{\frac{\nu m}{\delta}}\Bigr)\bigl(F(u) + F(u')\bigr). \end{aligned} \tag{3.22}$$

Comparing this inequality with (3.19) and (3.21), we see that, for a sufficiently small $\nu > 0$, the right-hand side of (3.22) can be estimated by $C(F(u)+F(u'))$. This completes the proof of Proposition 3.3. □

4 Complex Ginzburg–Landau equation

4.1 Cauchy problem and a priori estimates

Let $D \subset \mathbb{R}^n$ ($n = 3$ or 4) be a bounded domain with smooth boundary ∂D and let $L^2 = L^2(D, \mathbb{C})$ be the space of square-integrable complex-valued functions on D. We regard L^2 as a real Hilbert space and endow it with the scalar product

$$(u, v) = \operatorname{Re} \int_D u(x)\bar{v}(x)\,dx$$

and the corresponding norm $\|\cdot\|$. Let $\{e_j\}$ be a complete set of L^2-normalised eigenfunctions of the Dirichlet Laplacian and let $\{\alpha_j\}$ be the corresponding set of eigenvalues indexed in an increasing order.

We consider the problem

$$\dot{u} - (\nu + i)\Delta u + i|u|^{2p} u = h(x) + \eta(t, x), \tag{4.1}$$

$$u\big|_{\partial D} = 0, \tag{4.2}$$

$$u(0, x) = u_0(x), \tag{4.3}$$

where $\nu > 0$ and $p \ge 0$ are some constants, $h \in L^2$ is a deterministic function, and η is an H^1-valued random force. More precisely, we assume that

$$\eta(t, x) = \frac{\partial}{\partial t}\zeta(t, x), \quad \zeta(t, x) = \sum_{j=1}^{\infty} b_j \beta_j(t) e_j(x), \tag{4.4}$$

where $\beta_j(t) = \beta_{j1}(t) + i\beta_{j2}(t)$ are complex-valued independent Brownian motions and $b_j \geq 0$ are some constant satisfying the condition

$$B_1 := \sum_{j=1}^{\infty} \alpha_j b_j^2 < \infty.$$

In what follows, we always assume that $0 \leq p \leq \frac{2}{n}$. For any function $u(t,x)$, let us set

$$\mathcal{E}_u(t) = \|u(t)\|^2 + \nu \int_0^t \|u(s)\|_1^2 ds. \tag{4.5}$$

The theorem below establishes the well-posedness of problem (4.1)–(4.3) in appropriate functional spaces. We refer the reader to the papers [Kry00, MR01, KS04, Shi06] for proofs of similar (and more general) results.

Theorem 4.1. *Suppose that the above-mentioned conditions are fulfilled, and let u_0 be an L^2-valued random variable that is independent of ζ and satisfies the condition $\mathbb{E}\,\|u_0\|^2 < \infty$. Then the following statements hold.*

(i) *There is a random process $u(t) = u(t,x)$, $t \geq 0$, whose almost every trajectory belongs to the space*

$$\mathcal{X} := C(\mathbb{R}_+; L^2) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+; H_0^1)$$

and satisfies Eqs. (4.1) *and* (4.3) *in the sense that*

$$u(t) = u_0 + \int_0^t \big((\nu + i)\Delta u(s) - i|u(s)|^{2p}u(s)\big)\,ds + th + \zeta(t), \quad t \geq 0.$$

Moreover, the random process $u(t,x)$ is adapted to the filtration $\mathcal{F}_t$ generated by u_0 and ζ.

(ii) *The process $u(t)$ constructed in* (i) *is unique in the sense that if $\tilde{u}(t)$ is another random process satisfying* (i)*, then, with probability* 1*, we have $u(t) = \tilde{u}(t)$ for all $t \geq 0$.*

(iii) *We have the a priori estimates*

$$\mathbb{E}\,\|u(t)\|^2 + \nu \int_0^t \mathbb{E}\,\|u(s)\|_1^2 ds \leq \mathbb{E}\,\|u_0\|^2 + Ct \quad \text{for } t \geq 0, \tag{4.6}$$

$$\mathbb{P}\Big\{\sup_{t\geq 0}\big(\mathcal{E}_u(t) - Lt\big) \geq \|u_0\|^2 + \rho\Big\} \leq e^{-\varkappa\rho} \quad \text{for } \rho > 0, \tag{4.7}$$

where C, L, and $\varkappa$ are positive constants not depending on u_0.

4.2 Formulation of the result and an open problem

Let us denote by $S_t(u_0,\omega)$ the solution of (4.1)–(4.3) constructed in Theorem 4.1. Using a standard argument (e.g., see [Kry00, MR01]), it is not difficult to show that $S_t(u_0,\omega)$ can be regarded as a Markov RDS in L^2, and we shall denote by $(u_t,\mathbb{P}_u)$ the corresponding Markov family (cf. Section 1.1). The transition function and Markov operators associated with $(u_t,\mathbb{P}_u)$ will be denoted by $P_t(u,\Gamma)$, $\mathfrak{P}_t$, and $\mathfrak{P}_t^*$. The following theorem is the main result of this section.

Theorem 4.2. *Suppose that the conditions of Theorem 4.1 are satisfied and that*

$$b_j \neq 0 \quad \text{for all } j \geq 1. \tag{4.8}$$

Then for any $\nu > 0$ the Markov RDS associated with (4.1), (4.2) *has a unique stationary measure $\mu \in \mathcal{P}(L^2)$. Moreover, there are positive constants C and γ such that*

$$|\mathfrak{P}_t f(u) - (f,\mu)| \leq C\,\|f\|_{\mathcal{L}}(1+\|u\|^2)\,e^{-\gamma t} \quad \text{for any } t \geq 0,\ u \in L^2, \tag{4.9}$$

where $f \in \mathcal{L}(L^2)$ is an arbitrary functional.

To prove this theorem, we shall construct an extension $\boldsymbol{S}_t$ for S_t that satisfies the coupling hypothesis in the sense of Definition 2.2, and application of Theorem 2.3 will imply the required result. Moreover, using the regularising property for CGL equation and the associated Markov semigroup (see Proposition 4 in [Shi06]), it is not difficult to show that the stationary measure μ is concentrated on the space H^1, and the exponential convergence to μ holds also for continuous functionals on H_0^1. At the same time, the following question remains open.

Open Problem. The CGL equation is well posed in the space H_0^1 for $n = 3$ or 4 and $p \leq \frac{2}{n-2}$. Prove the uniqueness of stationary measure and exponential mixing property for these values of p.

The rest of this section is organised as follows. In Section 4.3, we construct an extension for S_t. Section 4.4 is devoted to verification of Conditions (H_1) and (H_2) (see Section 3). In Section 4.5, we prove inequalities (2.6) and (2.7). The proof of Theorem 4.2 is completed in Section 4.6.

4.3 Construction of an extension

We wish to construct an extension for S_t that satisfies the coupling hypothesis described in Definition 2.2. As was explained in Section 1.2, if we have an extension $\boldsymbol{\mathcal{R}}_t = (\mathcal{R}_t, \mathcal{R}_t')$ on a time interval $[0,T]$, then its iteration results in an extension defined on the half-line $\mathbb{R}_+$. Our construction of $\boldsymbol{\mathcal{R}}_t$ will depend on $T \geq 1$ and an integer $N \geq 1$. Both parameters will be fixed later.

Step 1. Let H_N be the $2N$-dimensional subspace in L^2 spanned by the vectors $e_j, ie_j, 1 \leq j \leq N$, and let $H_N^\perp$ be its orthogonal complement in L^2.

Denote by P_N and Q_N the orthogonal projections in L^2 onto the subspaces H_N and $H_N^\perp$, respectively.

Let us set $v = \mathsf{P}_N u$, $w = \mathsf{Q}_N u$ and rewrite Eq. (1.4) in the form

$$\dot v - (\nu + i)\Delta v + F_N(v + w) = \mathsf{P}_N h + \dot\varphi(t), \tag{4.10}$$

$$\dot w - (\nu + i)\Delta w + G_N(v + w) = \mathsf{Q}_N h + \dot\psi(t), \tag{4.11}$$

where we set

$$\varphi = \mathsf{P}_N\zeta, \quad \psi = \mathsf{Q}_N\zeta, \quad F_N(u) = i\mathsf{P}_N(|u|^{2p}u), \quad G_N(u) = i\mathsf{Q}_N(|u|^{2p}u).$$

Equations (4.10) and (4.11) are supplemented with the initial conditions

$$v(0) = v_0, \tag{4.12}$$

$$w(0) = w_0, \tag{4.13}$$

where $v_0 \in H_N$ and $w_0 \in H_N^\perp$. Using standard arguments, it is not difficult to check that, for any functions

$$w_0 \in H_N^\perp, \quad v \in C(0, T; H_N), \quad \psi \in C(0, T; H_N^\perp \cap H_0^1),$$

problem (4.11), (4.13) has a unique solution

$$w \in \mathcal{X}_N(T) := C(0, T; H_N^\perp) \cap L^2(0, T; H_N^\perp \cap H_0^1).$$

We shall denote by

$$\mathcal{W} : H_N^\perp \times C(0, T; H_N) \times C(0, T; H_N^\perp \cap H_0^1) \to \mathcal{X}_N(T), \quad (w_0, v, \psi) \mapsto w,$$

the resolving operator for problem (4.11), (4.13) and by $\mathcal{W}_t$ its restriction to the time t. The operators $\mathcal{W}$ and $\mathcal{W}_t$ are uniformly Lipschitz with respect to (w_0, v, ψ) on bounded subsets, and it is easy to see that $\mathcal{W}_t(w_0, v, \psi)$ depends only on the restriction of v and ψ to the interval $[0, t]$.

Step 2. We now fix an arbitrary function $\chi \in C^\infty(\mathbb{R})$ such that

$$0 \le \chi \le 1, \qquad \chi(t) = 1 \text{ for } t \le 0, \qquad \chi(t) = 0 \text{ for } t \ge 1.$$

Let us take any initial points $u_0, u_0' \in L^2$ and set $f_N(u_0, u_0') = \mathsf{P}_N(u_0' - u_0)$. Denote by $\lambda_T(u_0, u_0')$ and $\lambda_T'(u_0, u_0')$ the laws of the processes

$$\left\{\begin{pmatrix}\mathsf{P}_N u(t)\\ \mathsf{Q}_N\zeta(t)\end{pmatrix}, t \in [0, T]\right\}, \quad \left\{\begin{pmatrix}\mathsf{P}_N u'(t) - f_N(u_0, u_0')\chi(t)\\ \mathsf{Q}_N\zeta(t)\end{pmatrix}, t \in [0, T]\right\}, \tag{4.14}$$

respectively, where $u(t) = S_t(u_0, \omega)$ and $u(t) = S_t(u_0', \omega)$. Thus, $\lambda_T(u_0, u_0')$ and $\lambda_T'(u_0, u_0')$ are probability measures on the separable Banach space $C(0, T; L^2)$. Let $(U(u_0, u_0'), U'(u_0, u_0'))$ be a maximal coupling for $(\lambda_T(u_0, u_0'), \lambda_T'(u_0, u_0'))$. [5]

[5] See Section 5.2 for a definition of maximal coupling.

By Proposition 5.2, such a pair of random variables exists and is a measurable function of its arguments. Now let

$$\mathcal{R}_t(u_0,u_0') = \mathsf{P}_N U_t + \mathcal{W}_t(\mathsf{Q}_N u_0, \mathsf{P}_N U, \mathsf{Q}_N U), \tag{4.15}$$

$$\begin{aligned}\mathcal{R}_t'(u_0,u_0') &= \mathsf{P}_N U_t + f_N(u_0,u_0')\chi(t)\\ &\quad + \mathcal{W}_t(\mathsf{Q}_N u_0', \mathsf{P}_N U' + f_N(u_0,u_0')\chi, \mathsf{Q}_N U'),\end{aligned} \tag{4.16}$$

where U_t stands for the restriction of $U(u_0,u_0')$ to the time t, and U_t' is defined in a similar way. We claim that $\boldsymbol{\mathcal{R}}_t = (\mathcal{R}_t, \mathcal{R}_t')$ is an extension of S_t on the interval $[0,T]$.

Indeed, we need to show that the laws of the processes $\{\mathcal{R}_t(u_0,u_0')\}$ and $\{\mathcal{R}_t'(u_0,u_0')\}$ coincide with those of $\{S_t(u_0,\omega)\}$ and $\{S_t(u_0',\omega)\}$, respectively. To this end, let us set

$$\mathcal{X}(T) = C(0,T;L^2) \cap L^2(0,T;H_0^1)$$

and introduce an operator

$$\varUpsilon : H_N^\perp \times C(0,T;H_N) \times C(0,T;H_N^\perp \cap H_0^1) \to \mathcal{X}(T)$$

defined by the relation

$$\varUpsilon(w_0,v,\psi) = v + \mathcal{W}(w_0,v,\psi). \tag{4.17}$$

The definition of $\mathcal{W}$ implies that

$$\{S_t(u_0,\omega), t\in[0,T]\} = \varUpsilon\big(\mathsf{Q}_N u_0, \mathsf{P}_N S.(u_0,\omega), \mathsf{Q}_N\zeta(\cdot)\big). \tag{4.18}$$

Thus, the law of $\{S_t, t\in[0,T]\}$ coincides with the image of the law of the first process in (4.14) under the mapping $\varUpsilon(\mathsf{Q}_N u_0,\cdot,\cdot)$. Furthermore, it follows from (4.15) that the distribution $\mathcal{D}(\mathcal{R}.(u_0,u_0'))$ is the image of $\lambda_T(u_0,u_0')$ under $\varUpsilon(\mathsf{Q}_N u_0,\cdot,\cdot)$. By construction, the law of the first process in (4.14) coincides with $\lambda_T(u_0,u_0')$, and we conclude that

$$\mathcal{D}(\mathcal{R}.(u_0,u_0')) = \mathcal{D}(S.(u_0,\cdot)).$$

A similar argument proves that $\mathcal{D}(\mathcal{R}'.(u_0,u_0')) = \mathcal{D}(S.(u_0',\cdot))$.

Our next goal is to check that Hypotheses (H_1) and (H_2) are satisfied for S_t and $\boldsymbol{\mathcal{R}}_t$. In view of Propositions 3.2 and 3.3, this will imply that property (i) of Definition 2.2 is true for the extension $\boldsymbol{S}_t$.

4.4 Lyapunov function and dissipation

Let us show that S_t satisfies Hypothesis (H_1) with $F(u) = \|u\|^2$ and any $t_* > 0$. Indeed, it follows from (4.6) and the Gronwall inequality that

$$\mathbb{E}_u F(S_t) \le e^{-\nu t} F(u) + C\nu^{-1}, \quad t \ge 0.$$

In particular, fixing any constant $a \in (e^{-\nu t_*}, 1)$, we see that (3.1) and (3.2) hold with

$$R_* = \left(\frac{C}{\nu(a - e^{-\nu t_*})}\right)^{1/2}, \quad C_* = R_*^2 + C\nu^{-1}.$$

We now show that the extension $\boldsymbol{\mathcal{R}}_t$ satisfies Hypothesis (H$_2$) for sufficiently large N and T. Note that, in view of (4.8), the distribution of $\{\zeta(t), 0 \le t \le T\}$ is a non-degenerate Gaussian measure on $C(0,T;H_0^1)$. Combining this with the obvious property of approximate controllability of the CGL equation (1.4) with a control force $\zeta \in C^1(0,T;H_0^1)$, for any $R > 0$, $q \in (0,1)$, and $d > 0$ we can find $\alpha(R,q,d) > 0$ such that (e.g., see [FM95, Shi05a])

$$\mathbb{P}_u\{\|S_T(u,\cdot)\| \le (q\|u\|) \vee d\} \ge \alpha(R,q,d) \quad \text{for any } u \in L^2,\ \|u\| \le R. \tag{4.19}$$

Moreover, using the existence of a Lyapunov function for S_t, the constant $\alpha(R,q,d)$ can be made independent of $T \ge 1$. Since $\boldsymbol{\mathcal{R}}_t$ is an extension for S_t, we conclude from (4.19) that

$$\begin{aligned} \mathbb{P}_{\boldsymbol{u}}\{\|\mathcal{R}_T(u,u')\| \le (q\|u\|) \vee d\} \ge \alpha(R,q,d), \\ \mathbb{P}_{\boldsymbol{u}}\{\|\mathcal{R}'_T(u,u')\| \le (q\|u'\|) \vee d\} \ge \alpha(R,q,d) \end{aligned} \tag{4.20}$$

for any $(u,u') \in L^2 \times L^2$ with $\|u\| \vee \|u'\| \le R$. Inequalities (4.20) would imply (3.15) with $\varepsilon(d) = \alpha(R,q,d)^2$ and any $T \ge 1$, if the processes $\mathcal{R}_t$ and $\mathcal{R}'_t$ were independent. However, this is not the case, and we have to proceed differently.

Step 1. To prove (3.15), it suffices to show that for any $\delta > 0$ there is $c_\delta > 0$ such that

$$P_\delta := \mathbb{P}_{\boldsymbol{u}}\{\|\mathcal{R}_T(u,u')\| \vee \|\mathcal{R}'_T(u,u')\| \le q_1(\|u\| \vee \|u'\|) + \delta\} \ge c_\delta \tag{4.21}$$

for $u, u' \in B_R$, where $q_1 \in (0,1)$ is a constant and B_R denotes the ball in L^2 of radius R centred at origin. Indeed, suppose that (4.21) is already proved and fix any $d > 0$. Setting $\delta = \frac{1-q_1}{1+q_1} d$ and $q = \frac{1+q_1}{2}$, we derive

$$q_1\|v\| + \delta = (q\|v\|) \vee d \quad \text{for any } v \in L^2.$$

It follows that the probability on the left-hand side of (3.15) is bounded below by P_δ. Since δ depends only on d and q_1, this proves (3.15).

Step 2. We now prove (4.21). In view of the existence of a Lyapunov function for S_t, we can assume that $u, u' \in B_{R_*}$ for some $R_* > 0$. Introduce the events

$$\begin{aligned} G_\delta &= \{\|\mathcal{R}_T(u,u')\| \le q_1(\|u\| \vee \|u'\|) + \delta\}, \\ G'_\delta &= \{\|\mathcal{R}'_T(u,u')\| \le q_1(\|u\| \vee \|u'\|) + \delta\}, \\ E_\rho &= \{\mathcal{E}_{\mathcal{R}}(t) + \mathcal{E}_{\mathcal{R}'}(t) \le 2(R_*^2 + Lt) + \rho \text{ for all } t \ge 0\}, \end{aligned}$$

where $\mathcal{E}_u$ is defined by (4.5). We need to estimate from below the expression $\mathbb{P}_{\boldsymbol{u}}(G_\delta G'_\delta)$. It follows from (4.19) that

$$\mathbb{P}_{\boldsymbol{u}}(G_\delta) \ge \varkappa_\delta, \quad \mathbb{P}_{\boldsymbol{u}}(G'_\delta) \ge \varkappa_\delta \quad \text{for any } u, u' \in B_{R_*}, \tag{4.22}$$

where $\varkappa_\delta > 0$ is a constant not depending on u, u', and T. Moreover, inequality (4.7) implies that

$$\mathbb{P}_{\boldsymbol{u}}(E_\rho) \ge 1 - \beta_\rho \quad \text{for any } u, u' \in B_{R_*}, \tag{4.23}$$

where $\beta_\rho \to 0$ as $\rho \to \infty$. Now recall that (see (4.15) and (4.16))

$$\mathcal{R}_t(u,u') = \Upsilon_t(\mathsf{Q}_N u, U), \quad \mathcal{R}'_t(u,u') = \Upsilon_t(\mathsf{Q}_N u, U' + \tilde f_N(u,u')\chi), \tag{4.24}$$

where (U,U') is a maximal coupling for the pair $(\lambda_T(u,u'), \lambda'_T(u,u'))$, the operator Υ is defined in (4.17), Υ_t stands for its restriction to the time t, and $\tilde f_N(u,u') = \binom{f_N(u,u')}{0}$. Without loss of generality, we can assume that

$$\mathbb{P}_{\boldsymbol{u}}(G'_{\delta/2}\mathcal{N}^c) \le \mathbb{P}_{\boldsymbol{u}}(G_{\delta/2}\mathcal{N}^c), \tag{4.25}$$

where $\mathcal{N} = \{U(u,u') \ne U'(u,u')\}$ and $\mathcal{N}^c$ denotes the complement of $\mathcal{N}$. The case in which the opposite inequality is satisfied can be treated by a similar argument.

Suppose we have shown that

$$G_{\delta/2}E_\rho\mathcal{N}^c \subset G_\delta G'_\delta \quad \text{for any } \rho > 0 \text{ and } T \ge T_\rho, \tag{4.26}$$

where $T_\rho \ge 1$ depends only on ρ. In this case, we can write

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{u}}(G_\delta G'_\delta) &= \mathbb{P}_{\boldsymbol{u}}(G_\delta G'_\delta\mathcal{N}^c) + \mathbb{P}_{\boldsymbol{u}}(G_\delta G'_\delta\mathcal{N}) \\
&\ge \mathbb{P}_{\boldsymbol{u}}(G_\delta G'_\delta E_\rho\mathcal{N}^c) + \mathbb{P}_{\boldsymbol{u}}(G_\delta \,|\, \mathcal{N})\mathbb{P}_{\boldsymbol{u}}(G'_\delta \,|\, \mathcal{N})\mathbb{P}_{\boldsymbol{u}}(\mathcal{N}) \\
&\ge \mathbb{P}_{\boldsymbol{u}}(G_{\delta/2}E_\rho\mathcal{N}^c) + \mathbb{P}_{\boldsymbol{u}}(G_\delta\mathcal{N})\mathbb{P}_{\boldsymbol{u}}(G'_\delta\mathcal{N}),
\end{aligned}$$

where we used inclusion (4.26) and the independence of U and U' conditioned on $\mathcal{N}$. Combining this inequality with (4.23), we derive

$$\mathbb{P}_{\boldsymbol{u}}(G_\delta G'_\delta) \ge \mathbb{P}_{\boldsymbol{u}}(G_{\delta/2}\mathcal{N}^c) + \mathbb{P}_{\boldsymbol{u}}(G_\delta\mathcal{N})\mathbb{P}_{\boldsymbol{u}}(G'_\delta\mathcal{N}) - \beta_\rho. \tag{4.27}$$

We claim that if $\rho > 0$ is so large that $\beta_\rho \le \frac18\varkappa^2_{\delta/2}$, then (4.21) holds with $c_\delta = \frac18\varkappa^2_{\delta/2}$. Indeed, if $\mathbb{P}_{\boldsymbol{u}}(G_{\delta/2}\mathcal{N}^c) \ge \frac14\varkappa^2_{\delta/2}$, then (4.21) follows immediately from (4.27). In the opposite case, inequalities (4.22) and (4.25) imply that

$$\varkappa^2_{\delta/2} \le \mathbb{P}_{\boldsymbol{u}}(G_{\delta/2})\mathbb{P}_{\boldsymbol{u}}(G'_{\delta/2}) \le \mathbb{P}_{\boldsymbol{u}}(G_{\delta/2}\mathcal{N})\mathbb{P}_{\boldsymbol{u}}(G'_{\delta/2}\mathcal{N}) + \frac34\varkappa^2_{\delta/2},$$

whence it follows that

$$\mathbb{P}_{\boldsymbol{u}}(G_\delta\mathcal{N})\mathbb{P}_{\boldsymbol{u}}(G'_\delta\mathcal{N}) \ge \mathbb{P}_{\boldsymbol{u}}(G_{\delta/2}\mathcal{N})\mathbb{P}_{\boldsymbol{u}}(G'_{\delta/2}\mathcal{N}) \ge \frac14\varkappa^2_{\delta/2}.$$

Combining this with (4.27), we obtain (4.21) with $c_\delta = \frac18\varkappa^2_{\delta/2}$.

Step 3. It remains to prove (4.26). The construction implies that if $\omega \in \mathcal{N}^c$, then the processes $\mathcal{R}_t(u,u')$ and $\mathcal{R}'_t(u,u')$ belong to the space $\mathcal{X}(T)$ and satisfy

Eq. (1.4) with some right-hand sides $\zeta, \zeta' \in C(0,T;H_0^1)$. Moreover, we have the relations (cf. (5.1), (5.2))

$$\mathsf{P}_N \mathcal{R}_t(u,u') = \mathsf{P}_N \mathcal{R}'_t(u,u') - f_N(u,u')\chi(t), \tag{4.28}$$

$$\mathsf{Q}_N \zeta(t) = \mathsf{Q}_N \zeta'(t) \tag{4.29}$$

for $0 \le t \le T$. Furthermore, if $\omega \in G_{\delta/2}E_\rho$, then

$$\int_0^t \bigl(\|\mathcal{R}_s(u,u')\|^2 + \|\mathcal{R}'_s(u,u')\|^2\bigr)\,ds \le 2(R^2 + Lt) + \rho \quad \text{for } 0 \le t \le T, \tag{4.30}$$

$$\|\mathcal{R}_T(u,u')\| \le \delta/2 + q_1\bigl(\|u\| \vee \|u'\|\bigr). \tag{4.31}$$

Applying Proposition 5.3 and using (4.28) and (4.30), we see that

$$\begin{aligned}\|\mathcal{R}_t(u,u') - \mathcal{R}'_t(u,u')\| &= \|\mathsf{Q}_N(\mathcal{R}_t(u,u') - \mathcal{R}'_t(u,u'))\| \\ &\le C_1 \exp\bigl\{-\nu\alpha_{N+1}(t-1) + C_1 t + 2R_*^2 + \rho\bigr\}\|u - u'\|,\end{aligned}$$

where $C_1 > 0$ is a constant not depending on u, u', and N. It follows that if N is sufficiently large, then for any $\rho > 0$ we can choose $T_\rho \ge 1$ such that

$$\|\mathcal{R}_T(u,u') - \mathcal{R}'_T(u,u')\| \le \tfrac{\delta}{2} \quad \text{for } u, u' \in B_{R_*},\ T \ge T_\rho. \tag{4.32}$$

Combining this with (4.31), we obtain the inequality

$$\|\mathcal{R}_T(u,u')\| \vee \|\mathcal{R}'_T(u,u')\| \le q_1\bigl(\|u\| \vee \|u'\|\bigr) + \delta,$$

which shows that $G_{\delta/2}E_\rho\mathcal{N}^c \subset G_\delta G'_\delta$. This completes the verification of Hypothesis (H_2).

4.5 Squeezing: verification of (2.6) and (2.7)

Let us recall that the extension $\boldsymbol{S}_t = (S_t, S'_t)$ is obtained by the iteration of $\boldsymbol{\mathcal{R}}_t = (\mathcal{R}_t, \mathcal{R}'_t)$ and that the random processes $S_t(\boldsymbol{u},\omega)$ and $S'_t(\boldsymbol{u},\omega)$ satisfy Eq. (1.4) with some right-hand sides $\zeta = \zeta(t,u,u')$ and $\zeta = \zeta(t,u,u')$, respectively. Introduce the Markov times

$$\begin{aligned}\sigma_1(\boldsymbol{u},\omega) &= \inf\{t \ge 0 : \mathsf{P}_N S_t \ne \mathsf{P}_N S'_t - f_N(u,u')\chi(t) \text{ or } \mathsf{Q}_N\zeta(t) \ne \mathsf{Q}_N\zeta'(t)\} \\ \sigma_2(\boldsymbol{u},\omega) &= \inf\{t \ge 0 : \mathcal{E}_{S_\cdot}(t) + \mathcal{E}_{S'_\cdot}(t) \ge \|\boldsymbol{u}\|^2 + 2(L+M)t + 2\rho\},\end{aligned}$$

where M and ρ are positive parameters that will be chosen later. Let us set

$$\tilde\sigma(\boldsymbol{u},\omega) = \sigma_1(\boldsymbol{u},\omega) \wedge \sigma_2(\boldsymbol{u},\omega).$$

The Foiaş–Prodi estimate (5.3) implies that if $N \gg 1$ and $u, u' \in B_1$, then (cf. the derivation of (4.32))

$$\|S_t(\boldsymbol{u},\omega) - S'_t(\boldsymbol{u},\omega)\| \le C\,e^{-t} \quad \text{for } 0 \le t \le \tilde\sigma(\boldsymbol{u},\omega), \tag{4.33}$$

where $C > 0$ does not depend on u and u'. It follows that $\tilde{\sigma} \leq \sigma$, where σ is defined by relation (2.3) with $\beta = 1$. We shall show that if $N \gg 1$, $\rho \gg 1$, and $\boldsymbol{B} = B_d \times B_d$ with $d \ll 1$, then $\tilde{\sigma}$ satisfies (2.6) and (2.7).

Step 1. Let us set

$$Q_k = \{\tilde{\sigma}(\boldsymbol{u}, \omega) \in I_k\}, \quad I_k = [(k-1)T, kT].$$

Suppose we have shown that

$$\mathbb{P}_{\boldsymbol{u}}(Q_k) \leq 2e^{-2k} \quad \text{for any } k \geq 1,\ \boldsymbol{u} \in \boldsymbol{B}. \tag{4.34}$$

In this case, we can write

$$\mathbb{P}_{\boldsymbol{u}}\{\tilde{\sigma} = \infty\} = 1 - \sum_{k=1}^{\infty} \mathbb{P}_{\boldsymbol{u}}(Q_k) \geq 1 - 2\sum_{k=1}^{\infty} e^{-2k} =: \delta_1 > 0,$$

$$\mathbb{E}_{\boldsymbol{u}}\big(I_{\{\tilde{\sigma}<\infty\}} e^{\delta_2 \tilde{\sigma}}\big) \leq \sum_{k=1}^{\infty} \mathbb{P}_{\boldsymbol{u}}(Q_k) e^{\delta_2 T k} \leq 2\sum_{k=1}^{\infty} e^{-(2-\delta_2 T)k} \leq K,$$

where $\delta_2 < T^{-1}$. Thus, it suffices to prove (4.34).

Step 2. To prove (4.34), we shall need the following result. Recall that the measures $\lambda_T(u, u')$ and $\lambda'_T(u, u')$ are defined in Section 4.3.

Proposition 4.3. *There is an integer $N_0 \geq 1$ such that if $N \geq N_0$, then*

$$\big\|\lambda_T(u, u') - \lambda'_T(u, u')\big\|_{\text{var}} \leq Ce^{-cR^2} + C_N d e^{CR^2} \tag{4.35}$$

for any $u, u' \in B_R$ such that $\|u - u'\| \leq d$. Here C_N, C, and c are positive constants not depending on R and d.[6]

The proof of this result is based on a well-known argument using the Girsanov theorem (see [EMS01, KS02]). The case of the CGL equation is technically more complicated; however, the main ideas remain the same, and therefore we omit the proof. We refer the reader to Proposition 3 in [Shi06] for a weaker version of (4.35).

The proof of (4.34) is by induction on k. Let us denote by A_k the set of $\omega \in \Omega$ for which

$$\mathsf{P}_N S_t = \mathsf{P}_N S'_t - f_N(u, u')\chi(t), \quad \mathsf{Q}_N \zeta(t) = \mathsf{Q}_N \zeta'(t) \quad \text{for } t \in I_k.$$

For $k = 1$, we have

$$Q_1 = \{\sigma_2 \in [0, T]\} \cup A_1^c. \tag{4.36}$$

It follows from (4.7) that

$$\mathbb{P}_{\boldsymbol{u}}\{\sigma_2 \in [0, T]\} \leq 2e^{-\varkappa\rho} \leq e^{-2} \quad \text{for } \rho \geq 4/\varkappa. \tag{4.37}$$

[6] However, they may depend on T.

Furthermore, Proposition 4.3 and the definition of maximal coupling imply that

$$\mathbb{P}_{\boldsymbol{u}}(A_1^c) \le Ce^{-c\,R^2} + C_N d e^{CR^2}. \tag{4.38}$$

The right-hand side of this inequality is smaller then e^{-2} if

$$R \ge c^{-1}(\ln C + 4), \quad d \le (2C_N)^{-1} e^{-CR^2}. \tag{4.39}$$

Combining (4.36)–(4.38), we arrive at (4.34) for $k = 1$.

We now assume that $k = l + 1 \ge 2$ and that inequality (4.34) is established for $1 \le k \le l$. Let us denote by $\bar{A}_l$ the intersection of $A_1, \dots, A_l$. We have

$$Q_{l+1} \subset \{\sigma_2 \in I_{l+1}\} \cup D_{l+1}, \tag{4.40}$$

where $D_{l+1} = \bar{A}_l \cap A_{l+1}^c \cap \{\sigma_2 \ge (l+1)T\}$. Let us estimate the probabilities of the events on the right-hand side of (4.40). Inequality (4.7) implies that

$$\mathbb{P}_{\boldsymbol{u}}\{\sigma_2 \in I_{l+1}\} \le 2e^{-\varkappa(\rho+Ml)} \le e^{-2(l+1)}, \tag{4.41}$$

on condition that

$$M \ge 2/\varkappa, \quad \rho \ge 4/\varkappa. \tag{4.42}$$

Furthermore, using inequality (4.34) for $0 \le k \le l$, we derive

$$\mathbb{P}_{\boldsymbol{u}}\big(\bar{A}_l \cap \{\sigma_2 \ge lT\}\big) \ge \mathbb{P}_{\boldsymbol{u}}\{\tilde{\sigma} \ge lT\} \ge 1 - 2\sum_{k=1}^{l} e^{-2k} \ge 1/2 \tag{4.43}$$

for $\boldsymbol{u} \in \boldsymbol{B}$. The Foiaş–Prodi inequality (5.3) implies that, for any $P > 0$ and sufficiently large N, we have (cf. the derivation of (4.32))

$$\begin{aligned} \|S_{lT}\| \vee \|S'_{lT}\| &\le C_1(\rho + MTl)^{1/2}, \\ \|S_{lT} - S'_{lT}\| &\le C_2 d\, e^{C_2\rho - PTl} \end{aligned}$$

on the set $\bar{A}_l \cap \{\sigma_2 \ge lT\}$, where C_1 and C_2 are positive constants not depending on N, d, and l. Applying now the Markov property and using inequalities (4.35) and (4.43), we obtain

$$\begin{aligned} \mathbb{P}_{\boldsymbol{u}}(D_{l+1}) &\le \mathbb{P}_{\boldsymbol{u}}\big(A_{l+1}^c \,|\, \bar{A}_l \cap \{\sigma_2 \ge lT\}\big)\, \mathbb{P}_{\boldsymbol{u}}\big(\bar{A}_l \cap \{\sigma_2 \ge lT\}\big) \\ &\le Ce^{-c\,C_1^2(\rho+MTl)} + C_N C_2 d \exp\big\{\rho(CC_1^2 + C_2) + (CC_1^2 M - P)Tl\big\}. \end{aligned} \tag{4.44}$$

The right-hand side of this inequality is smaller than $e^{-2(l+1)}$ if

$$\begin{aligned} &M \ge (2c\,C_1^2 T)^{-1}, \qquad \rho \ge \frac{\ln C + 2}{c\,C_1^2}, \\ &P \ge CC_1^2 M + 2, \qquad d \le (C_N C_2)^{-1} e^{-\rho(CC_1^2 + C_2) - 1}. \end{aligned} \tag{4.45}$$

Note that the conditions imposed on the parameters M, ρ, P, and d by inequalities (4.39), (4.42), and (4.45) are compatible. Combining (4.40), (4.41), and (4.44), we arrive at (4.34) for $k = l + 1$. This completes the proof (4.34).

4.6 Completion of the proof of Theorem 4.2

We have thus shown that the RDS associated with the CGL equation (4.2) possesses an extension $\boldsymbol{S}_t = (S_t, S_t')$ that satisfies (2.5)–(2.7) with

$$\sigma = \tilde{\sigma}, \quad \boldsymbol{B} = B_d \times B_d, \quad g(r) = r^2,$$

where $d > 0$ is sufficiently small. If we show that

$$\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\tilde{\sigma}<\infty\}}\|\boldsymbol{S}_{\tilde{\sigma}}\|^{2q}\big\} \le K \quad \text{for any } \boldsymbol{u} \in \boldsymbol{B}, \tag{4.46}$$

where K and q are positive constants not depending on $\boldsymbol{u}$, then application of Theorem 2.3 and Remark 2.4 will prove that problem (4.2), (4.3) possesses a unique stationary measure $\mu \in \mathcal{P}(L^2)$ and inequality (4.9) holds.

To prove (4.46), note that if $\tilde{\sigma} < \infty$, then

$$\|S_{\tilde{\sigma}}\|^2 + \|S'_{\tilde{\sigma}}\|^2 \le 2(d^2 + L\tilde{\sigma}) + \rho \quad \text{for } u, u' \in B_d.$$

It follows that

$$\|\boldsymbol{S}_{\tilde{\sigma}}\|^{2q} \le C_q(\tilde{\sigma}^2 + 1) \quad \text{for any } q > 1,$$

where $C_q > 0$ depends only on L, d, and ρ. Multiplying this inequality by $I_{\{\tilde{\sigma}<\infty\}}$, taking the mean value, and using (2.7), we arrive at (4.46). The proof of Theorem 4.2 is complete.

5 Appendix

5.1 Maximal coupling of measures

Let X be a Polish space and let μ, μ' be two probability Borel measures on X. Recall that a pair (ξ, ξ') of X-valued random variables defined on the same probability space is called a *coupling for* (μ, μ') if

$$\mathcal{D}(\xi) = \mu, \quad \mathcal{D}(\xi') = \mu'.$$

Definition 5.1. A coupling (ξ, ξ') for (μ, μ') is said to be *maximal* if

$$\mathbb{P}\{\xi \ne \xi'\} = \|\mu - \mu'\|_{\mathrm{var}},$$

and the random variables ξ and ξ' conditioned on the event $\mathcal{N} = \{\xi \ne \xi'\}$ are independent, that is,

$$\mathbb{P}\{\xi \in \Gamma, \xi' \in \Gamma' \,|\, \mathcal{N}\} = \mathbb{P}\{\xi \in \Gamma \,|\, \mathcal{N}\}\,\mathbb{P}\{\xi' \in \Gamma' \,|\, \mathcal{N}\}$$

for any $\Gamma, \Gamma' \in \mathcal{B}_X$.

In Section 4.3, we have used the following result on the existence of maximal coupling for measures depending on a parameter. Let Y be a Polish space endowed with its Borel σ-algebra $\mathcal{B}_Y$ and let $\{\mu_y\}_{y\in Y}$ be a family of measures on X. We shall say that μ_y *measurable depends on* $y \in Y$ if the function $y \mapsto \mu_y(\Gamma)$ is $(\mathcal{B}_Y, \mathcal{B}_{\mathbb{R}})$-measurable for any $\Gamma \in \mathcal{B}_X$.

Proposition 5.2. *Let $\{\mu_y\}, \{\mu'_y\} \subset \mathcal{P}(X)$ be two families that measurably depend on $y \in Y$. Then there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two measurable functions*

$$\xi : Y \times \Omega \to X, \quad \xi' : Y \times \Omega \to X$$

such that $(\xi(y,\cdot), \xi'(y,\cdot))$ is a maximal coupling for (μ_y, μ'_y) for any $y \in Y$.

In the case $X = \mathbb{R}^n$, a proof can be found in [KS01]. In the general case, it suffices to use the fact that any Polish space is measurably isomorphic to $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

5.2 Foiaş–Prodi estimate

In this subsection, we present an estimate for the difference between two solutions of problem (1.4), (1.5) in which $\zeta : \mathbb{R}_+ \to H^1$ is a deterministic continuous function. Recall that $\{e_j\} \subset H$ is the complete set of eigenfunctions for the Dirichlet Laplacian in the domain D, H_N is the $2N$-dimensional subspace in L^2 generated by $\{e_j, ie_j, 1 \le j \le N\}$, and $H_N^\perp$ is the orthogonal complement of H_N in L^2. Denote by $\mathsf{P}_N : L^2 \to H_N$ and $\mathsf{Q}_N : L^2 \to H_N^\perp$ the corresponding orthogonal projections.

The following result provides a Foiaş–Prodi type estimate for the difference between two solutions whose projections to H_N coincide (cf. [FP67]). Its proof can be found in [Shi06, Section 4]. [7]

Proposition 5.3. *Let $n = 3$ or 4, let $p \le \frac{2}{n}$, and let*

$$u_1, u_2 \in \mathcal{X}(T) = C(0,T;L^2) \cap L^2(0,T;H_0^1)$$

be two solutions of problem (1.4), (1.5) that correspond to deterministic functions $\zeta_1, \zeta_2 \in C(0,T;H_0^1)$ and $h \in L^2(D,\mathbb{C})$. Suppose that

$$\mathsf{P}_N u_1(t) = \mathsf{P}_N u_2(t) \quad \text{for } t_0 \le t \le T, \tag{5.1}$$

$$\mathsf{Q}_N \zeta_1(t) = \mathsf{Q}_N \zeta_2(t) \quad \text{for } 0 \le t \le T, \tag{5.2}$$

where $t_0 \in [0,T]$ and $N \ge 1$ is an integer. Then there is a constant $C > 0$ not depending on u_1, u_2, t_0, and N such that

$$\begin{aligned} \big\|\mathsf{Q}_N(u_1(t) - u_2(t))\big\|^2 &\le \exp\big\{-\nu\alpha_{N+1}t + q(t)\big\} \bigg(\big\|\mathsf{Q}_N(u_1(0) - u_2(0))\big\|^2 \\ &+ Ce^{\nu\alpha_{N+1}t_0 + q(t_0)} \int_0^t \big(\|u_1(s)\|_1 + \|u_2(s)\|_1\big)^{(4p-2)\vee 0} \big\|\mathsf{P}_N(u_1(s) - u_2(s))\big\|_1^2 ds \bigg) \end{aligned} \tag{5.3}$$

for $0 \le t \le T$, where we set

$$q(t) = C \int_0^t \big(\|u_1(s)\|_1^2 + \|u_2(s)\|_1^2 + 1\big)\, ds.$$

[7] The estimate established in [Shi06] is slightly different. However, a similar argument enables one to prove (5.3).

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