Exponential dichotomy and time-bounded solutions for first-order hyperbolic systems*

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Abstract

The paper is devoted to studying a class of strongly hyperbolic systems of the first order. We show that if the characteristic roots of the full symbol are outside an open strip containing the real axis, then the homogeneous system possesses an exponential dichotomy and the inhomogeneous system is solvable in the space of time-bounded and almost periodic functions. We also discuss some results on the behavior of solutions for nonlinear equations in the neighborhood of a stationary point.

Key words: first-order hyperbolic systems, bounded and almost periodic solutions, exponential dichotomy.

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Contents

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0 Introduction

We consider a first-order hyperbolic system of the form

$$\mathcal{P}(t,x,D)u := D_t u - P(t,x,D_x)u = f(t,x), \tag{0.1}$$

where $D=(D_t,D_x)$, $D_t=-i\partial_t$, $D_x=-i\partial_x=-i(\partial_1,\dots,\partial_n)$, $u=u(t,x)$ is a vector function consisting of m components, and P is a matrix differential operator with smooth coefficients. We will study qualitative properties of solutions for system (0.1), namely, the problem of asymptotic stability and exponential dichotomy (ED) for homogeneous equations and the existence of bounded and almost periodic solutions for inhomogeneous equations. These problems were investigated earlier for different classes of ordinary and partial differential equations, and we will now discuss some results that are related to the subject of the present paper.

1) The problem of ED for linear ordinary differential equations (ODE's) with constant coefficients can easily be described in terms of the characteristic roots of the symbol. Indeed, consider the system

$$D_t u = Pu, \tag{0.2}$$

where P is an $m\times m$ matrix. We assume that the imaginary parts of the eigenvalues of P are nonzero. Then any solution of (0.2) is uniquely representable in the form

$$u(t) = u^+(t) + u^-(t), \tag{0.3}$$

where u^+ and u^- are also solutions of (0.2), $u^+(t)$ decays (grows) exponentially as $t \to +\infty$ ($t \to -\infty$), and vice versa for $u^-(t)$. The converse assertion is also true: if (0.2) possesses an ED (i. e., decomposition (0.3) holds), then P has no real eigenvalues. In the case of general systems with variable coefficients the condition of absence of real eigenvalues does not guarantee existence of an ED. Indeed, the spectral theory of one-dimensional Hill's operator (for instance, see [3, Section 8.10]) implies that if a smooth periodic potential $q(t) \geq 0$ is such that the resolvent set of the operator $D_t^2 - q(t)$ contains a point[1] $\lambda \geq 0$, then the equation

$$(D_t^2 - q(t) - \lambda)u = 0$$

has a nonzero solution $u_0(t)$ satisfying the inequality

$$|u_0(t)| \leq C\, e^{\nu t}, \quad t \in \mathbb{R},$$

where C and ν are positive constants. It follows that the bounded function $u = e^{-\nu t} u_0$ is a nontrivial solution of the equation

$$\big((D_t - i\nu)^2 - q(t) - \lambda\big)u = 0, \tag{0.4}$$

and therefore (0.4) does not possess an ED. On the other hand, the characteristic roots of (0.4) lie on the line $\operatorname{Im} \tau = \nu > 0$ (see also [23, Section 6.3]).

Thus, to ensure that the property of ED holds, some additional conditions must be imposed. The simplest condition of this kind is that the coefficients of the system in question are close to being constant. It can also be shown that the existence of an ED is equivalent to the invertibility of the differential expression in the space of time-bounded functions, so that the above conditions guarantee the existence and uniqueness of a bounded solution for inhomogeneous systems. As a simple consequence, one constructs periodic and almost periodic solutions in the case when the coefficients and right-hand side of the system in question are (almost) periodic (for further details, see [6, 10, 21] and references therein).

All these results can be extended to small nonlinear perturbations of system (0.2) (see [6, 10]) and also to equations with infinite-dimensional phase space on condition that the operators entering the equation under study are bounded (see [1]).

2) In the case of partial differential equations (PDE's) the problem of ED is much more delicate and was investigated by many authors (for instance, see [14, 7, 22, 19] and references therein). Without formulating exact results, let us clarify informally the main difficulty arising in the study of ED for strictly hyperbolic scalar equations. In the case of constant coefficients, the Fourier transform with respect to space variables reduces the problem in question to an ODE whose coefficients depend on a parameter. The solutions of this equation can be written down explicitly as a sum of path integrals whence it can be concluded that the original equation has an ED dichotomy if and only if the characteristic roots of its symbol are separated from the real line (see [23, 26]).

[1] The resolvent set contains infinitely many positive points for potentials of general position (see [3]).

It is a natural conjecture that this condition is sufficient for existence of an ED for small perturbations of an equation with constant coefficients. However, the proof of this assertion encounters an essential difficulty related to the fact that, when solving a hyperbolic equation, one derivative is lost (i. e., a solution of an equation of order m possesses $m-1$ derivatives), and therefore, if the perturbation is of the same order as the original equation, then it is not subordinate to the resolving operator of the unperturbed problem. Nevertheless, the technique of Leray's separating operator enables one to derive some a priori estimates for solutions and, as a consequence, to establish the property of ED. Note that similar results are true for nonlinear equations (see [23, 25, 26] for details).

3) The aim of this paper is to extend the above results to a class of non-strictly hyperbolic systems. Let us mention some earlier achievements in this direction. Asymptotic stability and exponential dichotomy were studied for abstract evolution equations in Banach spaces (for instance, see [14, 7, 22, 19]). The results obtained in the framework of the abstract theory can be applied to hyperbolic systems. However, as a rule, this approach leads to the rather restrictive condition that the principal symbol of the system have constant coefficients. The problem of stability was studied in [18, 2] for symmetric hyperbolic systems and also in [12] for strongly hyperbolic systems on a torus under the condition that the characteristic roots of the symbol lie in the half-plane $\operatorname{Im}\tau \geq \delta > 0$. In the book [16], time-bounded and almost periodic solutions are constructed for symmetric hyperbolic systems on a bounded domain with dissipative boundary conditions.

4) Let us turn to a description of the results in this paper. We consider the system

$$\mathcal{P}_\varepsilon(t,x,D)u := D_t u - P_\varepsilon(t,x,D_x)u = f(t,x). \tag{0.5}$$

Here $\varepsilon \in [-1,1]$ is a parameter and P_ε is a first-order matrix operator whose coefficients are close to being constant, i. e., they have the form

$$p_\varepsilon(t,x) = p_0 + \varepsilon p_1(\varepsilon,t,x), \tag{0.6}$$

where p_0 is a constant and p_1 is a smooth function of its arguments that is uniformly bounded together with all its derivatives. It is assumed that the operator P_ε is strongly hyperbolic. This means that its principal symbol admits uniform diagonalization, the characteristic roots are real, and their multiplicity does not depend on (ε,t,x,ξ), where ξ is the variable dual to x. (See Section 1.1 for the classical definition due to Petrovskii [17, Section 3.1].)

Let $H^s = H^s(\mathbb{R}^n)$ be the Sobolev space of order s with norm $\|\cdot\|_s$ and let $C(\mathbb{R},H^s)$ (accordingly, $C_b(\mathbb{R},H^s)$) be the space of continuous (bounded continuous) functions on $\mathbb{R}$ with range in H^s. We will use the same notation for the spaces of scalar and vector functions. Let us consider the Cauchy problem for Eq. (0.5):

$$u(0,x) = u_0(x). \tag{0.7}$$

It is well known that for any right-hand side $f \in C(\mathbb{R},H^s)$ and any initial function $u_0 \in H^s$ the problem (0.5), (0.7) has a unique solution $u \in C(\mathbb{R},H^s)$, and the corresponding resolving operator $\mathcal{U}_\varepsilon(t) : H^s \to H^s$, which takes u_0 to $u(t,\cdot)$, is bounded and continuously depends on t in the strong operator topology.

We denote by $\tau_j(\varepsilon,t,x,\xi)$, $j=1,\dots,m$, the characteristic roots of the full symbol $P_\varepsilon(t,x,\xi)$. We will assume that, apart from strong hyperbolicity, P_ε satisfies the following condition: there is $\delta>0$ such that

$$\bigl|\operatorname{Im}\tau_j(\varepsilon,t,x,\xi)\bigr| \ge \delta \quad \text{for} \quad (\varepsilon,t,x,\xi)\in[-1,1]\times\mathbb{R}^{n+1}\times\mathbb{R}^n_\xi, \quad j=1\dots,m. \tag{0.8}$$

The theorem below is the main result of this paper.

Main Theorem. *Under the above conditions, for any $s_0>0$ there are positive constants ε_0, C, and μ such that the following statements hold for $|\varepsilon|\le\varepsilon_0$ and $|s|\le s_0$.*

(i) *Let $f\equiv 0$. Then there are closed subspaces $\mathbb{E}^+$ and $\mathbb{E}^-$ such that*

$$H^s = \mathbb{E}^+ \dotplus \mathbb{E}^-, \tag{0.9}$$

and for any $u_0\in\mathbb{E}^\pm$ we have[2]

$$\bigl\|\mathcal{U}_\varepsilon(t)u_0\bigr\|_s \le Ce^{\mp\mu t}\|u_0\|_s, \quad \pm t\ge 0. \tag{0.10}$$

(ii) *Let $f\in C_b(\mathbb{R},H^s)$. Then Eq.* (0.5) *has a unique solution $u\in C_b(\mathbb{R},H^s)$, which satisfies the inequality*

$$\sup_{t\in\mathbb{R}}\|u(t,\cdot)\|_s \le C\sup_{t\in\mathbb{R}}\|f(t,\cdot)\|_s. \tag{0.11}$$

If, in addition, the right-hand side $f(t,x)$ and the coefficients of P_ε are periodic (*almost periodic*) *in time, then so is the solution $u(t,x)$.*

Let us recall that the space $\mathbb{E}^+$ ($\mathbb{E}^-$) in decomposition (0.9) is called the *stable* (*unstable*) subspace for system (0.5) (with $f\equiv 0$).

We note that similar results hold in the case of nonlinear perturbations of an equation with constant coefficients. However, we will not present detailed proofs for nonlinear problems since they involve no new ideas compared with the case of scalar equations (see Section 4 and [25, 26, 27]).

5) It should be noted that the methods of this paper are quite similar to those applied in [23, 26] for studying strictly hyperbolic scalar equations: we first establish some energy estimates and then use them to investigate asymptotic properties of solutions. However, there are two distinctions that consist in the following.

The first distinction concerns the derivation of energy estimates. A general method allowing one to obtain a priori estimates for solutions of the Cauchy problem for high-order hyperbolic systems was suggested by Petrovskii [17]. It

[2]Here and henceforth, a formula involving the indices $\pm$ and $\mp$ is a brief notation for the two formulas corresponding to the upper and lower signs. When referring to the formula with the upper (lower) sign, we will use a number with subscript $+$ ($-$).

is based on reduction of the problem in question to a first-order system and transformation of the principal symbol of the latter to the diagonal form. In the case of scalar equations, Leray's separating operator technique substantially simplifies Petrovskii's arguments. A modification of Leray's approach was used in [23] to establish some energy estimates for equations satisfying the additional condition (0.8). However, Leray's separating operator has no analog in the case of systems, and here we use Petrovskii's idea to establish the required a priori estimates. Namely, we construct an indefinite metric in which the hyperbolic matrix under study is positive (cf. [12] where the stability is investigated with the help of a Lyapunov function). Moreover, the class of strongly hyperbolic systems is not stable under small perturbations, in contrast to strictly hyperbolic equations (cf. [23, Proposition 3.9]). This results in that the above-mentioned indefinite metric constructed for the unperturbed system (with constant coefficients) cannot be used to derive energy estimates for equations with variable coefficients (cf. [23, Section 5]).

The second distinction occurs in the proof of the property of ED. In the case of scalar equations, it was based on an "explicit" description of the stable and unstable subspaces in terms of the resolving operator of an initial value problem (IVP) with growth conditions at infinity. A similar IVP for hyperbolic systems turns out to be much more difficult, and to prove the existence of ED, we employ the fact that it is equivalent to the invertibility of the hyperbolic operator in the space of time-bounded functions. The latter follows from the solvability of the inhomogeneous system in the space of square integrable functions on the time axis with range in a Sobolev space H^s and from an a priori estimate for the H^s-norm of solutions.

Let us briefly describe the structure of this paper. In Section 1, we state the main results in more detail and show how they can be established with the help of some energy estimates. Section 2 is devoted to the derivation of the energy estimates. We substantially employ the existence of an indefinite metric in which the hyperbolic matrix satisfying condition (0.8) is positive. A construction of such a metric is given in Section 3. In Section 4, we discuss some generalizations of the main results to a broader class of nonlinear systems. In the Appendix (Section 5), we give an exposition of general properties of strongly hyperbolic matrices.

Notation

Throughout the paper, we denote by $P_\varepsilon(t,x,\xi) = P(z,\xi)$ a strongly hyperbolic $m \times m$ matrix symbol and by $P^0_\varepsilon(t,x,\xi) = P^0(z,\xi)$ its principal part. Here $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$ are the time and space variables, ξ is the variable dual to x, and $\varepsilon \in [-1,1]$ is a parameter. We denote by Σ_{n-1} the unit sphere in $\mathbb{R}^n$.

Let $J \subset \mathbb{R}$ be a closed interval and let X be a Banach space. We will use the following function spaces.

$H^s = H^s(\mathbb{R}^n)$ is the Sobolev space of order s with the scalar product

$$(u_1, u_2)_s = \int_{\mathbb{R}^n} u_1(\xi)\overline{\hat{u}_2(\xi)}\langle\xi\rangle^{2s} d\xi,$$

where $\hat{u}(\xi)$ is the Fourier transform of $u(x)$ and $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. The corresponding norm will be denoted by $\|\cdot\|_s$. The subscript s will be dropped if it is zero.
$C(J, X)$ is the space of continuous functions on J with range in X.
$C_b(J, X)$ is the space of bounded functions $f \in C(J, X)$.
$L^1_{\text{loc}}(J, X)$ is the space of Bochner-measurable functions $f(t) : J \to X$ whose norm is integrable on any finite interval $I \subset J$.

1 Main results and their reduction to a priori estimates

1.1 Formulation of the main results

Let us consider the problem

$$D_t u - P_\varepsilon(t, x, D_x)u = f(t, x), \quad (t, x) \in \mathbb{R}^{n+1}, \tag{1.1}$$

$$u(\theta, x) = u_0(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $\theta \in \mathbb{R}$ and $\varepsilon \in [-1, 1]$ is a parameter. We will assume that $P_\varepsilon(t, x, D_x)$ is an $m \times m$ matrix operator whose coefficients belong to $C_b^\infty([-1, 1] \times \mathbb{R}^{n+1}_{t,x})$. To simplify the notation, we will write $z = (\varepsilon, t, x)$, $\Omega = [-1, 1] \times \mathbb{R}^{n+1}_{t,x}$, and $P(z, D_x)$ instead of $P_\varepsilon(t, x, D_x)$.

Let $P^0_\varepsilon(t, x, \omega)$, $\omega \in \Sigma_{n-1}$, be the principal symbol of $P_\varepsilon(t, x, D_x)$ and let $\tau^0_j(\varepsilon, t, x, \omega)$, $j = 1, \dots, m$, be its characteristic roots.

Definition 1.1. An operator P_ε is said to be *strongly hyperbolic* if its principal symbol satisfies the following two conditions for any $(\varepsilon, t, x) \in \Omega$ and $\omega \in \Sigma_{n-1}$:

(a) the matrix $P^0_\varepsilon(t, x, \omega)$ has elementary divisors of degree no higher than 1, and the number of coinciding elementary divisors does not depend on the choice of the point $(\varepsilon, t, x, \omega)$;

(b) the characteristic roots $\tau^0_j(\varepsilon, t, x, \omega)$ are real.

We note that the class of operators described in Definition 1.1 was first introduced by Petrovskii [17, Section 3.1] while the term *strongly hyperbolic* is taken from [11].

Condition (a) of Definition 1.1 implies that the multiplicity of the roots for the principal symbol of a strongly hyperbolic operator does not depend on (z, ω). Let us denote by $\sigma^0_k(\varepsilon, t, x, \omega)$, $k = 1, \dots, l$, the pairwise distinct roots of $P^0_\varepsilon(t, x, \omega)$. In what follows, we will need a narrower class of *uniformly strongly hyperbolic* operators. By definition, this class consists of the operators for which there is $\varkappa > 0$ such that

$$|\sigma^0_j(z, \omega) - \sigma^0_k(z, \omega)| \geq \varkappa > 0 \quad \text{for} \quad (z, \omega) \in \Omega \times \Sigma_{n-1}, \quad j \neq k. \tag{1.3}$$

We will say that the coefficients of P_ε are *nearly constant* if they have the form (0.6), where p_0 is a constant and $p_1 \in C_b^\infty([-1,1] \times \mathbb{R}^{n+1})$.

The three theorems below are the main results of this paper.

For any $s \in \mathbb{R}$, we denote by $\mathbb{L}_s$ the space of functions $f(t,x) \in L^1_{\mathrm{loc}}(\mathbb{R}, H^s)$ such that

$$L_s(f) := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|f(t,\cdot)\|_s dt < \infty.$$

In the first theorem, we establish the invertibility of strongly hyperbolic operators in the space of time-bounded functions.

Theorem 1.2. *Let P_ε be a strongly hyperbolic operator with nearly constant coefficients that satisfies condition* (0.8) *with some $\delta > 0$. Then for any $s_0 > 0$ there are positive constants ε_0 and C such that, for $|s| \le s_0$ and $|\varepsilon| \le \varepsilon_0$, Eq.* (1.1) *with right-hand side $f \in \mathbb{L}_s$ has a unique solution $u \in C_b(\mathbb{R}, H^s)$, which satisfies the inequality*

$$F_s(u) := \sup_{t\in\mathbb{R}} \|u(t,\cdot)\|_s \le C L_s(f). \tag{1.4}$$

For any real numbers μ and s, we denote by $\mathbb{F}_{s,[\mu]}(\mathbb{R}_\pm)$ and $\mathbb{L}_{s,[\mu]}(\mathbb{R}_\pm)$ the spaces of functions $u(t,x) \in C(\mathbb{R}_\pm, H^s)$ and $f(t,x) \in L^1_{\mathrm{loc}}(\mathbb{R}_\pm, H^s)$, respectively, such that

$$F_{s,[\mu]}(u,\mathbb{R}_\pm) := \sup_{\pm t\ge 0} \Big(e^{\mu t}\|u(t,\cdot)\|_s\Big) < \infty, \quad L_{s,[\mu]}(f,\mathbb{R}_\pm) < \infty,$$

where for any interval J we set

$$L_{s,[\mu]}(f,J) := \sup_{t\in J}\Big(e^{\mu t}\int_{[t,t+1]\cap J} \|f(\theta,\cdot)\|_s d\theta < \infty\Big).$$

In the case $\mu = 0$, we will simply write $L_s(f,J)$.

The following theorem concerns the solvability of system (1.1) in the case when all characteristic roots of P_ε are stable. We note that assertion (i) of this theorem was proved in [12] for systems on a torus.

Theorem 1.3. *Let P_ε be a strongly hyperbolic operator with nearly constant coefficients such that*

$$\operatorname{Im}\tau_j(\varepsilon,t,x,\xi) \ge \delta \quad \textit{for} \quad (\varepsilon,t,x,\xi) \in [-1,1]\times\mathbb{R}^{n+1}\times\mathbb{R}^n_\xi, \quad j = 1\dots,m, \tag{1.5}$$

where $\delta > 0$. Then for any $s_0 > 0$ and $\mu < \delta$ there are positive constants ε_0 and C such that the following statements hold for $|\varepsilon| \le \varepsilon_0$ and $|s| \le s_0$.

(i) *For any right-hand side $f \in \mathbb{L}_{s,[\mu]}(\mathbb{R}_+)$ and initial function $u_0 \in H^s$ the problem* (1.1), (1.2) *with $\theta = 0$ has a unique solution $u \in \mathbb{F}_{s,[\mu]}(\mathbb{R}_+)$, which satisfies the inequality*

$$\|u(t,\cdot)\|_s \le C\, e^{-\mu t}\big(\|u_0\|_s + L_{s,[\mu]}(f,[0,t])\big), \quad t \ge 1. \tag{1.6}$$

In particular, the zero solution of the homogeneous equation is exponentially asymptotically stable as $t \to +\infty$.

(ii) *For any right-hand side $f \in \mathbb{L}_{s,[\mu]}(\mathbb{R}_-)$ Eq.* (1.1) *has a unique solution $u \in \mathbb{F}_{s,[\mu]}(\mathbb{R}_-)$, which satisfies the inequality*

$$\|u(t,\cdot)\|_s \leq C\, e^{-\mu t} L_{s,[\mu]}(f, \mathbb{R}_-), \quad t \leq 0. \tag{1.7}$$

In particular, the homogeneous equation has no nontrivial solution bounded for $t \leq 0$.

Finally, we consider the problem of exponential dichotomy for homogeneous systems.

Theorem 1.4. *Under the conditions of Theorem 1.2, for any $s_0 > 0$ and μ, $0 < \mu < \delta$, there are closed subspaces $\mathbb{E}_s^+(\theta), \mathbb{E}_s^-(\theta) \subset H^s$, $\theta \in \mathbb{R}$, and positive constants ε_0 and C depending on s_0 and μ such that the following assertions hold for $|s| \leq s_0$ and $|\varepsilon| \leq \varepsilon_0$.*

(i) *A vector function $u_0 \in H^s$ belongs to $\mathbb{E}_s^\pm(\theta)$ if and only if the solution of the problem* (1.1), (1.2) *with $f \equiv 0$ satisfies the inequality*

$$\|u(t,\cdot)\|_s \leq C\, e^{\mp\mu(t-\theta)} \|u_0\|_s, \quad \pm(t-\theta) \geq 0. \tag{1.8}$$

(ii) *The phase space H^s can be represented as the direct sum*

$$H^s = \mathbb{E}_s^+(\theta) \dotplus \mathbb{E}_s^-(\theta), \quad \theta \in \mathbb{R}. \tag{1.9}$$

Moreover, the projections $\mathsf{P}_s^+(\theta)$ and $\mathsf{P}_s^-(\theta)$ corresponding to the direct decomposition (1.9) *continuously depend on θ in the strong operator topology, and their norms are uniformly bounded with respect to θ, ε, and s.*

We will say that system (1.1) with $f \equiv 0$ possesses an ED in the space H^s if the above properties (i) and (ii) hold for some positives constants C and μ. Thus, the claim of Theorem 1.4 is that strongly hyperbolic first-order systems whose characteristic roots satisfy condition (0.8) for a positive δ and whose coefficients are close to being constant possesses an ED in H^s with any $\mu \in (0, \delta)$.

Proofs of Theorems 1.2 – 1.4 are discussed in Subsection 1.3. Here we present some corollaries of the above results.

We will need some functional spaces with exponential weights. For any real numbers μ_-, μ_+, and s, let $\mathbb{F}_{s,[\mu_-,\mu_+]}$ be the space of functions $u \in C(\mathbb{R}, H^s)$ such that

$$F_{s,[\mu_-,\mu_+]}(u) := \sup_{t\in\mathbb{R}} \Big(e_{\mu_-,\mu_+}(t) \|u(t,\cdot)\|_s \Big) < \infty,$$

where $e_{\mu_-,\mu_+}(t) = e^{\mu_- t}$ for $t \leq 0$ and $e_{\mu_-,\mu_+}(t) = e^{\mu_+ t}$ for $t \geq 0$. Similarly, let $\mathbb{L}_{s,[\mu_-,\mu_+]}$ be the space of functions $f(t,x) \in L^1_{\mathrm{loc}}(\mathbb{R}, H^s)$ such that

$$L_{s,[\mu_-,\mu_+]}(f) := \sup_{t\in\mathbb{R}} \Big(e_{\mu_-,\mu_+}(t) \int_t^{t+1} \|f(\theta,\cdot)\|_s d\theta \Big) < \infty.$$

Theorems 1.2 and 1.4 imply the following assertion on invertibility of hyperbolic operators in the above spaces.

Corollary 1.5. *Under the conditions of Theorem 1.2, for any* $\mu_-, \mu_+ \in (-\delta, \delta)$ *and* $s_0 > 0$ *there are positive constants* ε_0 *and* C *such that, for* $|s| \le s_0$ *and* $|\varepsilon| \le \varepsilon_0$*, Eq. (1.1) with right-hand side* $f \in \mathbb{L}_{s,[\mu_-,\mu_+]}$ *has a unique solution* $u \in \mathbb{F}_{s,[\mu_-,\mu_+]}$*, which satisfies the inequality*

$$F_{s,[\mu_-,\mu_+]}(u) \le C L_{s,[\mu_-,\mu_+]}(f). \tag{1.10}$$

Sketch of the proof. We first show that the homogeneous equation has no non-trivial solution in the space $\mathbb{F}_{s,[\mu_-,\mu_+]}$. To this end, we note that if a solution $u \in C(\mathbb{R}, H^s)$ satisfies the inequality

$$\|u(t,\cdot)\|_s \le \text{const } e^{-\mu_\pm t}, \quad \pm t \ge 0,$$

then

$$\|u(t,\cdot)\|_s \le \text{const } e^{-\mu|t|}, \quad t \in \mathbb{R}.$$

In particular, $u \in C_b(\mathbb{R}, H^s)$ and, hence, by Theorem 1.2, $u \equiv 0$.

To prove the existence, we first consider the case when $\mu_+ = \mu_- = \mu$. The solution of (1.1) is sought in the form $u = e^{-\mu t} v$. Substitution of this function into (1.1) results in the following equation for $v(t,x)$:

$$D_t v - (P_\varepsilon(t,x,D_x) - i\mu I)v = e^{\mu t} f. \tag{1.11}$$

The right-hand side of (1.11) belongs to $\mathbb{L}_s$, and the operator $P_\varepsilon - i\mu I$ satisfies the conditions of Theorem 1.2 with δ replaced by $\delta - |\mu|$. Therefore Eq. (1.11) has a unique solution $v \in C_b(\mathbb{R}, H^s)$ for sufficiently small ε. It is easy to see that $u = e^{-\mu t} v$ is the required solution.

To construct a solution in the general case, we choose $\varepsilon_0 > 0$ so small that, for $|s| \le s_0$ and $|\varepsilon| \le \varepsilon_0$, the homogeneous system (1.1) possesses an ED in the space H^s with $\mu = \max\{|\mu_-|, |\mu_+|\}$. Let us write the right-hand side of (1.1) in the form

$$f = f_+ + f_-, \quad f_+ = \zeta(t) f, \quad f_- = \big(1 - \zeta(t)\big) f,$$

where $\zeta \in C^\infty(\mathbb{R})$, $\zeta(t) = 0$ for $t \le 0$ and $\zeta(t) = 1$ for $t \ge 1$. It is easy to see that $f_- \in \mathbb{L}_{s,[\mu_-,\mu_-]}$ and $f_+ \in \mathbb{L}_{s,[\mu_+,\mu_+]}$. Let $u_\pm \in \mathbb{F}_{s,[\mu_\pm,\mu_\pm]}$ be the unique solution of system (1.1) with $f = f_\pm$. Repeating the argument used in the proof of uniqueness, we can show that

$$\|u_\pm(t,\cdot)\|_s \le \text{const } e^{-\mu|t|}, \quad \mp t \ge 0.$$

This implies that the function $u = u_+ + u_-$ belongs to $\mathbb{F}_{s,[\mu_-,\mu_+]}$ and satisfies (1.1). The details are left to the reader. ☐

We now turn to the problem of existence of periodic and almost periodic (AP) solutions. A trivial consequence of Theorem 1.2 is that the solution $u \in C_b(\mathbb{R}, H^s)$ of (1.1) is periodic in time if so are the right-hand side $f \in \mathbb{L}_s$ and the coefficients of P_ε. Indeed, if

$$f(t+T,x) = f(t,x), \quad P_\varepsilon(t+T,x,\xi) = P_\varepsilon(t,x,\xi),$$

then for any solution $u(t,x) \in C_b(\mathbb{R}, H^s)$ the function $u(t+T,x)$ also belongs to $C_b(\mathbb{R}, H^s)$ and satisfies (1.1). By uniqueness, it must coincides with $u(t,x)$.

The case of AP solutions is more complicated. However, the methods used in [23, Section 8] to construct AP solutions of scalar equations can be applied without changes to hyperbolic systems as well. Therefore, we confine ourselves to the statement (without proof) of a result concerning Bohr AP solutions for Eq. (1.1).

Let us introduce some notations. For a countable module $\mathfrak{M} \subset \mathbb{R}$ and a Fréchet space X, denote by $AP(X, \mathfrak{M})$ the class of Bohr AP functions $f(t): \mathbb{R} \to X$ whose module of Fourier exponents is contained in $\mathfrak{M}$ (see [13]). We note that, for any fixed ε, the coefficients of P_ε can be regarded as smooth functions of t with range in the polynormed space $C_b^\infty(\mathbb{R}_x^n)$, and therefore we can define the class $AP(C_b^\infty(\mathbb{R}_x^n), \mathfrak{M})$ of almost periodic coefficients.

Corollary 1.6. *Suppose that the conditions of Theorem 1.2 are satisfied. If $f \in AP(H^s, \mathfrak{M})$, and the coefficients of P_ε belong to $AP(C_b^\infty(\mathbb{R}_x^n), \mathfrak{M})$, where $\mathfrak{M} \subset \mathbb{R}$ is a countable module, then for sufficiently small ε the solution $u(t,x) \in C_b(\mathbb{R}, H^s)$ of* (1.1) *constructed in Theorem 1.2 belongs to $AP(H^s, \mathfrak{M})$.*

1.2 A priori estimates

In this subsection, we formulate some a priori estimates for solutions of Eq. (1.1). Their proofs will be postponed until Section 2.

Theorem 1.7. *Under the conditions of Theorem 1.3, for any $s_0 > 0$ and μ, $0 < \mu < \delta$, there are positive constants C and ε_0 such that, for $|s| \le s_0$, $|\varepsilon| \le \varepsilon_0$, and an arbitrary function $u(t,x)$ satisfying the inclusions*

$$u \in C_b(\mathbb{R}, H^s), \quad f = \mathcal{P}_\varepsilon(t,x,D)u \in \mathbb{L}_s, \tag{1.12}$$

we have the inequality

$$\big\|u(t,\cdot)\big\|_s^2 \le C\, L_s(\mathcal{P}_\varepsilon u, \mathbb{R}_-(t)) \int_{-\infty}^t e^{-2\mu(t-\theta)} \big\|\mathcal{P}_\varepsilon u(\theta,\cdot)\big\|_s d\theta, \quad t \in \mathbb{R}. \tag{1.13}$$

Let $L^2(\mathbb{R}, H^s)$ be the space of Bochner-measurable functions $u(t,\cdot)\colon \mathbb{R} \to H^s$ with finite norm

$$\big\|u\big\|_{L^2(\mathbb{R},H^s)} = \left(\int_{\mathbb{R}} \big\|u(t,\cdot)\big\|_s^2 dt\right)^{1/2}.$$

Theorem 1.8. *Under the conditions of Theorem 1.2, for any $s_0 > 0$ there are positive constants C and ε_0 such that the following statements hold for $|s| \le s_0$ and $|\varepsilon| \le \varepsilon_0$.*

(i) *Let $u(t,x)$ be an arbitrary function satisfying* (1.12). *Then* (1.4) *holds.*

(ii) *For any $T > 0$ and $\nu > 0$ there is $N = N(s_0, T, \nu) > 0$ such that*

$$\sup_{|t-r|\le T} \|u(t,\cdot)\|_s \le C\, L_s(f, [r-N, r+N]) + \nu L_s(f), \quad r \in \mathbb{R}. \tag{1.14}$$

(iii) *Let $u(t,x)$ be an arbitrary function satisfying the inclusions*

$$u \in L^2(\mathbb{R}, H^s), \quad f = \mathcal{P}_\varepsilon u \in L^2(\mathbb{R}, H^s). \tag{1.15}$$

Then

$$\|u\|_{L^2(\mathbb{R},H^s)} \le C\,\|f\|_{L^2(\mathbb{R},H^s)}. \tag{1.16}$$

1.3 Derivation of the main results from Theorems 1.7 and 1.8

First of all, we note that Theorem 1.3 can be derived from (1.13) by literal repetition of the arguments in Sections 6.1 and 6.2 of [23], and we will not dwell on it.

Proof of Theorem 1.2. Inequality (1.4), which implies the uniqueness of a solution, is included in Theorem 1.8 (see (i)). Let us prove the existence.

1) We first show that the operator $\mathcal{P}_\varepsilon(t,x,D)$ is invertible in the space $L^2(\mathbb{R}, H^s)$. To this end, let us note that its formal adjoint has the form

$$\big(\mathcal{P}_\varepsilon(t,x,D)\big)^* = D_t I - P_\varepsilon^*(t,x,D_x) - \varepsilon B_\varepsilon(t,x), \tag{1.17}$$

where $P_\varepsilon^*(t,x,\xi)$ is the adjoint matrix of $P_\varepsilon(t,x,\xi)$, $P_\varepsilon^*(t,x,D_x)$ is the operator with symbol $P_\varepsilon^*(t,x,\xi)$, and $B_\varepsilon(t,x)$ is an $m\times m$ matrix whose entries belong to $C_b^\infty([-1,1]\times\mathbb{R}^{n+1})$. It is easy to see that if $P_\varepsilon^0(t,x,\omega)$ satisfies conditions (a) and (b) of Definition 1.1, then so does the adjoint symbol $P_\varepsilon^{0*}(t,x,\omega)$. Moreover, the roots of $P_\varepsilon^*(t,x,\xi)$ are outside the strip $|\operatorname{Im}\tau| < \delta$ as soon as (0.8) holds for P_ε. Thus, P_ε^* satisfies the conditions of Theorem 1.2 and, hence, by assertion (iii) of Theorem 1.8, we have

$$\|u\|_{L^2(\mathbb{R},H^s)} \le C\,\|D_t u - P_\varepsilon^*(t,x,D_x)u\|_{L^2(\mathbb{R},H^s)}. \tag{1.18}$$

It follows that, for $|\varepsilon| \ll 1$, inequality (1.16) holds for $\mathcal{P}_\varepsilon^*$. Thus, we have a pair of dual estimates for the operator $\mathcal{P}_\varepsilon$ and its adjoint $\mathcal{P}_\varepsilon^*$. The invertibility of $\mathcal{P}_\varepsilon$ follows now from a well-known argument (e.g., see [8, Section 23.1]).

2) To prove the existence of a solution $u \in C_b(\mathbb{R}, H^s)$ for Eq. (1.1) with right-hand side $f \in \mathbb{L}_s$, we first assume that $f \in L^2(\mathbb{R}, H^{s+1})$. The invertibility of $\mathcal{P}_\varepsilon$ established above implies that Eq. (1.1) has a unique solution $u \in L^2(\mathbb{R}, H^{s+1})$. It follows from (1.1) that $D_t u \in L^2(\mathbb{R}, H^s)$ and therefore $u \in C_b(\mathbb{R}, H^s)$.

To construct a solution in the general case, we apply a standard regularization argument. Namely, we multiply the right-hand side $f \in \mathbb{L}_s$ by a truncating function in time and take the convolution with an averaging kernel in x to construct a sequence $f_k \in L^2(\mathbb{R}, H^{s+1})$ such that

$$L_s(f_k) \le L_s(f) \quad \text{for all} \quad k \ge 1,$$
$$L_s(f_k - f, [-N,N]) \to 0 \quad \text{as} \quad k \to \infty \quad \text{for any} \quad N > 0.$$

Denote by $u_k \in C_b(\mathbb{R}, H^s)$ the solution of Eq. (1.1) with $f = f_k$. Inequality (1.14) implies that the sequence $u_k(t,x)$ converges in $C_b([-T,T], H^s)$ for any $T > 0$. It is easy to see that the limiting function $u(t,x)$ is the required solution. □

Proof of Theorem 1.4. As is shown in [13, Chapter X], the property of ED (see the remark following Theorem 1.4) is equivalent to the invertibility of $\mathcal{P}_\varepsilon$ in the space $C_b(\mathbb{R}, H^s)$, which is established in Theorem 1.2. Hence, Eq. (1.1) with $f \equiv 0$ possesses an ED. To complete the proof, we must specify the rate of convergence, i. e., we must show that (1.8) holds for any μ, $0 < \mu < \delta$.

Consider the following equation resulting from (1.1) after the substitution $u = e^{-\mu t} v$:

$$D_t v - (P_\varepsilon(t,x,D_x) - i\mu I)v = 0. \tag{1.19}$$

It is clear that if $|\mu| < \delta$, then the operator $P_\varepsilon(t,x,D_x) - i\mu I$ satisfies the conditions of Theorem 1.2 with δ replaced by $\delta - |\mu|$. It follows that Eq. (1.19) possesses an ED for sufficiently small ε. In particular, we have the direct decomposition (cf. (1.9))

$$H^s = \mathbb{E}_s^+(\theta,\mu) \dotplus \mathbb{E}_s^-(\theta,\mu), \quad \theta \in \mathbb{R}, \tag{1.20}$$

where $\mathbb{E}_s^+(\theta,\mu)$ and $\mathbb{E}_s^-(\theta,\mu)$ are the stable and unstable subspaces for Eq. (1.19). We claim that for any positive $\delta' < \delta$ there is $\varepsilon_1 > 0$ such that

$$\mathbb{E}_s^\pm(\theta,\mu) = \mathbb{E}_s^\pm(\theta) \quad \text{for} \quad |\mu| \le \delta', \quad |\varepsilon| \le \varepsilon_1. \tag{1.21}$$

Indeed, let us show first that

$$\mathbb{E}_s^\pm(\theta) \subset \mathbb{E}_s^\pm(\theta,\mu) \quad \text{for} \quad |\mu| < \nu \ll 1. \tag{1.22}$$

We confine ourselves to the case of the index $+$. Let $u_0 \in \mathbb{E}_s^+(\theta)$ and let $u(t,x)$ be the solution of (1.1), (1.2). By construction, inequality (1.8) holds with a positive $\mu = \nu$. Now note that for any μ, $|\mu| < \nu$, the function $v = e^{\mu(t-\theta)}u$ satisfies (1.19) and decays exponentially as $t \to +\infty$. It follows that $v(\theta) = u_0 \in \mathbb{E}_s^+(\theta,\mu)$, which completes the proof of (1.22).

In view of the direct decompositions (1.9) and (1.20), the inclusion in (1.22) can be replaced by equality. Similar argument shows that for any $\mu_0 \in (-\delta,\delta)$ there are $\nu > 0$ and $\varepsilon_2 > 0$ such that

$$\mathbb{E}_s^\pm(\theta,\mu) = \mathbb{E}_s^\pm(\theta,\mu_0) \quad \text{for} \quad \mu \in (\mu_0 - \nu, \mu_0 + \nu), \quad |\varepsilon| \le \varepsilon_2. \tag{1.23}$$

Thus, the closed interval $[-\delta', \delta']$ can be covered by open neighborhoods of its points for which (1.23) holds. Choosing a finite subcovering, we obtain (1.21).

We now note that if $v(t,x)$ is a solution of (1.19), then $u = e^{-\mu(t-\theta)}v$ satisfies Eq. (1.1) with $f \equiv 0$. The required inequality (1.8) follows from (1.21) and the fact that the norm $\|v(t,\cdot)\|_s$ decays exponentially as $t \to \pm\infty$ if $v(\theta,\cdot) \in \mathbb{E}_s^\pm(\theta,\mu)$. □

2 Energy estimates

This section is devoted to the proof of Theorems 1.7 and 1.8. We first formulate an algebraic result on the existence of an indefinite metric in which a hyperbolic symbol without characteristic roots in the neighborhood of the real line is positive. Using this result, we establish two-sided a priori estimates for solutions of hyperbolic systems. Theorems 1.7 and 1.8 are simple consequences of these estimates.

2.1 Formulation of the result on indefinite metric

Let $\Omega \subset \mathbb{R}^d_z$ be an open convex region[3] and let $P(z,\xi)$ be a uniformly strongly hyperbolic symbol, i. e., an $m \times m$ matrix whose entries are first-order polynomials in $\xi \in \mathbb{R}^n$ with coefficients belonging to $C_b^\infty(\overline{\Omega})$ such that the matrix $P^0(z,\xi)$ consisting of the principal homogeneous parts of the entries of $P(z,\xi)$ satisfies conditions (a) and (b) in Definition 1.1 (with (ε,t,x) replaced by z) and inequality (1.3). We assume that the characteristic roots of $P(z,\xi)$ are outside the strip $|\operatorname{Im}\tau| < \delta$ for some $\delta > 0$ and denote by $\Pi^+(z,\xi)$ and $\Pi^-(z,\xi)$ the Riesz projections onto the eigenspaces of $P(z,\xi)$ corresponding to roots that lie in the half-planes $\operatorname{Im}\tau \geq \delta$ and $\operatorname{Im}\tau \leq -\delta$, respectively. The matrices of these projections can be represented as path integrals of the form

$$\Pi^\pm(z,\xi) = -(2\pi i)^{-1} \oint_{\gamma^\pm} \big(P(z,\xi) - \tau I\big)^{-1} d\tau, \tag{2.1}$$

where $\gamma^\pm = \gamma^\pm(z,\xi) \subset \mathbb{C}$ is a smooth path enclosing the roots of $P(z,\xi)$ in the half-plane $\operatorname{Im}\tau \geq \delta$ (accordingly, $\operatorname{Im}\tau \leq -\delta$).

For any $j \in \mathbb{R}$, denote by $S^j(\Omega)$ the space of functions $p(z,\xi) \in C^\infty(\overline{\Omega} \times \mathbb{R}^n_\xi)$ such that, for any multi-indices α and β,

$$[p,\Omega]_{j,\alpha,\beta} := \sup_{z\in\overline{\Omega},\, \xi\in\mathbb{R}^n} |\partial_\xi^\alpha \partial_z^\beta p(z,\xi)| \langle\xi\rangle^{|\alpha|-j} < \infty,$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, $|\xi|^2 = (\xi,\xi)$, and $(\cdot,\cdot)$ is the standard scalar product in $\mathbb{R}^n$. Let $S^j(\Omega,m)$ be the space of matrix symbols

$$P(z,\xi) = \big\| p_{ik}(z,\xi) \big\|_{i,k=1}^m, \quad z \in \Omega, \quad \xi \in \mathbb{R}^n, \tag{2.2}$$

whose elements belong to $S^j(\Omega)$.

We can now state the above-mentioned algebraic result generalizing a well-known Lyapunov theorem to the case of strongly hyperbolic matrices.

[3]In what follows, the results of this subsection will be applied in the case when $d = n+2$, $z = (\varepsilon,t,x)$, and $\Omega = [-1,1] \times \mathbb{R}^{n+1}_{t,x}$. However, the actual structure of the parameter z does not play any role, and to shorten the notation, we formulate the results for symbols that depend on an abstract parameter z varying in a domain $\Omega \subset \mathbb{R}^d_z$.

Theorem 2.1. *Let $P(z,\xi)$ be a uniformly strongly hyperbolic matrix symbol whose characteristic roots satisfies the condition*

$$|\operatorname{Im}\tau_j(z,\xi)| \ge \delta \quad \text{for} \quad (z,\xi) \in \overline{\Omega} \times \mathbb{R}^n_\xi, \quad j = 1,\dots,m, \tag{2.3}$$

where $\delta > 0$. Then the matrix functions $\Pi^+(z,\xi)$ and $\Pi^-(z,\xi)$ belong to $S^0(\Omega,m)$, and for any positive $\delta' < \delta$ there are nonnegative Hermitian symbols $Q_\pm(z,\xi) \in S^0(\Omega,m)$, depending on δ', such that the following assertions hold for $(z,\xi) \in \overline{\Omega} \times \mathbb{R}^n$.

(i) *We have the inclusion*

$$P^*(z,\xi)Q_\pm^2(z,\xi) - Q_\pm^2(z,\xi)P(z,\xi) \in S^0(\Omega,m). \tag{2.4}$$

(ii) *We have the inequalities*

$$i\big(P^*(z,\xi)Q_+^2(z,\xi) - Q_+^2(z,\xi)P(z,\xi)\big) \ge 2\delta' Q_+^2(z,\xi), \tag{2.5}$$

$$-i\big(P^*(z,\xi)Q_-^2(z,\xi) - Q_-^2(z,\xi)P(z,\xi)\big) \ge 2\delta' Q_-^2(z,\xi). \tag{2.6}$$

(iii) *There is a constant $M = M(\delta') > 1$ such that*

$$M^{-1}I \le Q_+^2(z,\xi) + Q_-^2(z,\xi) \le MI. \tag{2.7}$$

(iv) *The following relations hold:*

$$Q_+^2(z,\xi)\Pi^+(z,\xi) = Q_+^2(z,\xi), \quad Q_-^2(z,\xi)\Pi^-(z,\xi) = Q_-^2(z,\xi). \tag{2.8}$$

Theorem 2.1 will be proved in Section 3. Here we confine ourselves to some remarks.

Remark 2.2. We note that the converse of Theorem 2.1 is also true. Namely, assume that a matrix function P satisfies the inequalities (the dependence of P on the parameters z and ξ is unessential here)

$$\pm i(P^*T_\pm - T_\pm P) \ge 2\delta T_\pm, \tag{2.9}$$

$$T_+ + T_- > 0, \tag{2.10}$$

where $\delta > 0$, and $T_\pm$ are Hermitian matrices (not necessarily nonnegative). Then the characteristic roots of P are outside the strip $|\operatorname{Im}\tau| < \delta$. Indeed, let $q \in \mathbb{C}^m$ be an eigenvector of P corresponding to an eigenvalue $\tau \in \mathbb{C}$. It follows from (2.10) that either $(T_+q,q) > 0$ or $(T_-q,q) > 0$. To be precise, assume that the first inequality holds. In this case (2.9_+) implies that

$$i\big((P^*T_+ - T_+P)q,q\big) = 2\operatorname{Im}\tau(T_+q,q) \ge 2\delta(T_+q,q),$$

whence follows that $\operatorname{Im}\tau \ge \delta$. Similarly, if $(T_-q,q) > 0$, then $\operatorname{Im}\tau \le -\delta$.

Let us define the Hermitian matrix function $W(z,\xi) := Q_+^2(z,\xi) - Q_-^2(z,\xi)$ and the corresponding scalar product $[q,q] = \big(W(z,\xi)q,q\big)$. It follows from (2.5) – (2.7) that

$$\mathrm{Re}\big[iP(z,\xi)q,q\big] = \frac{1}{2}\big(i(WP - P^*W)(z,\xi)q,q\big) \le -c(q,q),$$

where $c > 0$ is a constant. This means that the matrix function $iP(z,\xi)$ is dissipative in the scalar product $[q,q]$ uniformly with respect to (z,ξ).

Thus, Theorem 2.1 combined with the above observation is an analog (for matrix functions) of a well-known result according to which the characteristic roots of a matrix P lie outside a strip $|\mathrm{Im}\,\tau| < \delta$ if and only if there is an indefinite metric in which iP is dissipative (see [15, 24, 1]).

Remark 2.3. In the particular case when the characteristic roots belong to the half-plane $\mathrm{Im}\,\tau \ge \delta > 0$, Theorem 2.1 is proved in [12, Section 3]. In this situation assertions (i) – (iv) of the theorem take the form

$$P^*(z,\xi)Q^2(z,\xi) - Q^2(z,\xi)P(z,\xi) \in S^0(\Omega,m), \tag{2.11}$$

$$i\big(P^*(z,\xi)Q^2(z,\xi) - Q^2(z,\xi)P(z,\xi)\big) \ge 2\delta' Q^2(z,\xi), \tag{2.12}$$

$$M^{-1}I \le Q^2(z,\xi) \le MI, \tag{2.13}$$

where $0 < \delta' < \delta$ and $Q(z,\xi) \in S^0(\Omega,m)$ is a positive Hermitian symbol. Thus, Theorem 2.1 with $\delta_- = -\infty$ is an analog of the classical Lyapunov theorem (e. g., see [4, Chapter 15]).

Remark 2.4. The proof of Theorem 2.1 (see also Remarks 5.2 and 5.4) will imply that if the symbol $P(z,\xi)$ does not depend on z in a convex subset $\Omega' \subset \Omega$, then the symbols $Q_\pm(z,\xi)$ can be chosen so that they are also independent of $z \in \Omega'$. In particular, if the coefficients of $P_\varepsilon(t,x,\xi)$ are nearly constant, then so are the coefficients of the corresponding symbols $Q_\pm(\varepsilon,t,x,\xi)$.

2.2 Energy estimates: the case of stable roots

Let $C_b^\infty(\mathbb{R},H^s)$ be the space of vector functions $u(t,\cdot) : \mathbb{R} \to H^s$ that are uniformly bounded on $\mathbb{R}$ together with all their derivatives and let $C_b^\infty(\mathbb{R},H^\infty)$ be the projective limit of these spaces. For a symbol $Q(z,\xi)$, a vector function $u \in C_b^\infty(\mathbb{R},H^\infty)$, and real numbers $s \in \mathbb{R}$ and $\mu > 0$, we set

$$J^\pm_{s,\mu,Q}(u,t) = \\ = -\,\mathrm{Im} \int_{\pm\infty}^t e^{-2\mu|t-\theta|}\big(Q(\varepsilon,\theta,\cdot,D_x)(\mathcal{P}_\varepsilon u)(\theta,\cdot), Q(\varepsilon,\theta,\cdot,D_x)u(\theta,\cdot)\big)_s d\theta. \tag{2.14}$$

We also introduce the seminorm

$$\mathcal{E}^2_{s,\mu}(u,\mathbb{R}_\pm(t)) = \|u(t,\cdot)\|_s^2 + \left|\int_{\pm\infty}^t e^{-2\mu|t-\theta|}\|u(\theta,\cdot)\|_s^2 d\theta\right|,$$

where $R_\pm(t) = [t,\pm\infty)$.

Theorem 2.5. *Suppose that the conditions of Theorem 1.3 hold. Let* $0<\delta'<\delta$ *and let* $Q(z,\xi)\in S^0(\Omega,m)$ *be the nearly constant symbol in Remark 2.3. Then for any* $s_0>0$ *and* μ, $0<\mu<\delta'$, *there are positive constants* ε_0 *and* K *such that*

$$K^{-1}\mathcal{E}^2_{s,\mu}(u,\mathbb{R}_-(t))\le J^-_{s,\mu,Q}(u,t)\le K\mathcal{E}^2_{s,\mu}(u,\mathbb{R}_-(t)),\quad t\in\mathbb{R}, \tag{2.15}$$

where $|\varepsilon|\le\varepsilon_0$, $|s|\le s_0$, *and* $u\in C_b^\infty(\mathbb{R},H^\infty)$.

A similar assertion holds if the characteristic roots of $P(z,\xi)$ lie in the half-plane $\operatorname{Im}\tau\le-\delta$ for some $\delta>0$. In this case, inequality (2.15) takes the form

$$K^{-1}\mathcal{E}^2_{s,\mu}(u,\mathbb{R}_+(t))\le J^+_{s,\mu,Q}(u,t)\le K\mathcal{E}^2_{s,\mu}(u,\mathbb{R}_+(t)),\quad t\in\mathbb{R}.$$

Inequality (2.15) easily implies (1.13). To see this, it suffices to repeat word-for-word the argument in [23, Theorem 5.5].

Proof of Theorem 2.5. In what follows, we denote by $\Phi_k(\varepsilon,t;u)$, $k=1,2,\dots$, continuous quadratic forms on $H^s(\mathbb{R}^n_x)$ that continuously depend on the parameters (ε,t) and admit an estimate from above by const $\|u\|_s^2$ uniformly for $t\in\mathbb{R}$, $|\varepsilon|\le\varepsilon_1$, $|s|\le s_0$.

1) To outline the main ideas, we begin with the case of operators with constant coefficients. As was mentioned in Remark 2.4, in this case the symbol Q satisfying (2.11) – (2.13) depends on ξ solely.

We claim that

$$\begin{aligned} J^-_{s,\mu,Q}(u,t) &= -\operatorname{Im}\int_{-\infty}^t e^{-2\mu(t-\theta)}\big(Q(D_x)(D_\theta u-P(D_x))u,Q(D_x)u\big)_s d\theta\\ &=\frac12\big\|Q(D_x)u(t,\cdot)\big\|_s^2-\mu\int_{-\infty}^t e^{-2\mu(t-\theta)}\big\|Q(D_x)u(\theta,\cdot)\big\|_s^2 d\theta\\ &\quad+\frac12\int_{-\infty}^t e^{-2\mu(t-\theta)}\big(T(D_x)u(\theta,\cdot),u(\theta,\cdot)\big)_s^2 d\theta, \end{aligned}\tag{2.16}$$

where

$$T(\xi)=i\big(P^*(\xi)Q^2(\xi)-Q^2(\xi)P(\xi)\big). \tag{2.17}$$

Indeed, we have

$$\begin{aligned} -\operatorname{Im}\big(Q(D_x)D_\theta u,Q(D_x)u\big)_s &=\operatorname{Re}\big(Q(D_x)\partial_\theta u,Q(D_x)u\big)_s\\ &=\frac12\partial_\theta\big\|Q(D_x)u\big\|_s^2. \end{aligned}$$

Multiplying this relation by $e^{-2\mu(t-\theta)}$ and integrating over the half-line $(-\infty,t]$, we obtain the first two terms on the right-hand side of (2.16); the third term results from

$$\operatorname{Im}\int_{-\infty}^t e^{-2\mu(t-\theta)}\big(Q(D_x)P(D_x)u,Q(D_x)u\big)_s d\theta.$$

In view of (2.11), the right-hand side of (2.16) can be estimated from above by

$$\frac{1}{2}\big\|Q(D_x)u(t,\cdot)\big\|_s^2+(C_1-\mu)\int_{-\infty}^{t}e^{-2\mu(t-\theta)}\big\|Q(D_x)u(\theta,\cdot)\big\|_s^2d\theta,$$

where the constant $C_1>0$ does not depend on u. Similarly, inequality (2.12) implies that the right-hand side of (2.16) is no less than

$$\frac{1}{2}\big\|Q(D_x)u(t,\cdot)\big\|_s^2+(\delta'-\mu)\int_{-\infty}^{t}e^{-2\mu(t-\theta)}\big\|Q(D_x)u(\theta,\cdot)\big\|_s^2d\theta.$$

Taking into account (2.13), we obtain (2.15).

2) We now consider the case of variable coefficients with $s=0$. Let us write

$$P_\varepsilon(y,\xi)=P(\xi)+\varepsilon P'(\varepsilon,y,\xi),\quad Q(\varepsilon,y,\xi)=Q(\xi)+\varepsilon Q'(\varepsilon,y,\xi),\tag{2.18}$$

where $y=(t,x)$ and

$$P'(\varepsilon,y,\xi)=\int_0^1\partial_\lambda P_\lambda(y,\xi)\big|_{\lambda=r\varepsilon}dr,\quad Q'(\varepsilon,y,\xi)=\int_0^1\partial_\lambda Q(\lambda,y,\xi)\big|_{\lambda=r\varepsilon}dr.$$

We have

$$\begin{aligned}-\operatorname{Im}\big(Q(z,D_x)D_\theta u,Q(z,D_x)u\big)&=\operatorname{Re}\big(Q(z,D_x)\partial_\theta u,Q(z,D_x)u\big)\\&=\frac{1}{2}\partial_\theta\big\|Q(z,D_x)u\big\|^2-\operatorname{Re}\big((\partial_\theta Q)(z,D_x)u,Q(z,D_x)u\big)\\&=\frac{1}{2}\partial_\theta\big\|Q(z,D_x)u\big\|^2+\varepsilon\Phi_1(\varepsilon,t;u),\end{aligned}\tag{2.19}$$

where $z=(\varepsilon,\theta,x)$. Furthermore,

$$\operatorname{Im}\big(Q(z,D_x)P(z,D_x)u,Q(z,D_x)u\big)=\frac{1}{2}(\mathcal{T}_\varepsilon u,u),\tag{2.20}$$

where

$$\mathcal{T}_\varepsilon=i\big(P(z,D_x)^*Q(z,D_x)^*Q(z,D_x)-Q(z,D_x)^*Q(z,D_x)P(z,D_x)\big).$$

We claim that

$$(\mathcal{T}_\varepsilon u,u)=\big(T(\varepsilon,\theta,\cdot,D_x)u,u\big)+\varepsilon\Phi_2(\varepsilon,t;u),\tag{2.21}$$

where $T(\varepsilon,\theta,x,D_x)$ is the pseudodifferential operator with matrix symbol

$$T(\varepsilon,\theta,x,\xi)=i\big(P^*(z,\xi)Q^2(z,\xi)-Q^2(z,\xi)P(z,\xi)\big).\tag{2.22}$$

To prove (2.21), we will need the following remark on ΨDO's with nearly constant symbols.

Let us consider nearly constant symbols $A_k\in S^{j_k}(\Omega,m)$, $k=1,2$, and write them in the form

$$A_k(\varepsilon,y,\xi)=A_k(\xi)+A_k'(\varepsilon,y,\xi).$$

Explicit formulas for the composition of pseudodifferential operators and for the adjoint operator [8, Theorems 18.1.8 and 18.1.7] imply that

$$A_1(\varepsilon,y,D_x)A_2(\varepsilon,y,D_x) = (A_1A_2)(\varepsilon,y,D_x) + \varepsilon\mathcal{B}_1^{(j_1+j_2-1)}(\varepsilon,t), \tag{2.23}$$

$$A_k(\varepsilon,y,D_x)^* = A_k^*(\varepsilon,y,D_x) + \varepsilon\mathcal{B}_2^{(j_k-1)}(\varepsilon,t), \tag{2.24}$$

where $A_k^*(\varepsilon,y,\xi)$ is the matrix adjoint to $A_k(\varepsilon,y,\xi)$ and $\mathcal{B}_k^{(r)}(\varepsilon,t)$ stands for a family of continuous linear operators from $H^{(s)}$ into $H^{(s-r)}$ that continuously depend on the parameters (ε,t) and whose norms are uniformly bounded for $t\in\mathbb{R}$, $|\varepsilon|\le\varepsilon_1$, $|s|\le s_0$ (for arbitrary $s_0>0$). Relation (2.21) is easily implied by (2.23) and (2.24).

It follows from (2.19) – (2.21) that for $s=0$ the middle term in (2.15) differs from

$$\frac{1}{2}\big\|Q(z,D_x)u\big\|^2 - \mu\int_{-\infty}^{t} e^{-2\mu(t-\theta)}\big\|Q(z,D_x)u\big\|^2 d\theta + \\ + \int_{-\infty}^{t} e^{-2\mu(t-\theta)}\big(T(z,D_x)u,u\big)\,d\theta$$

by a function whose absolute value is no greater than $C|\varepsilon|\mathcal{E}_{0,\mu}(u,\mathbb{R}_-(t))$. Since the symbol $T(z,\xi)$ belongs to $S^0(\Omega,m)$ (see (2.11)), we obtain the right-hand inequality in (2.15). To prove the left-hand estimate, note that (cf. (2.18))

$$T(\varepsilon,y,\xi) = T(\xi) + \varepsilon T'(\varepsilon,y,\xi).$$

Hence,

$$\big(T(\varepsilon,\theta,\cdot,D_x)u,u\big) = \big(T(D_x)u,u\big) + \Phi_3(\varepsilon,\theta;u),$$
$$\big\|Q(\varepsilon,\theta,\cdot,D_x)u\big\|^2 = \big\|Q(D_x)u\big\|^2 + \Phi_4(\varepsilon,\theta;u).$$

Applying now inequality (2.15) for operators with constant symbols and choosing sufficiently small ε, we obtain (2.15) with $s=0$.

3) Finally, consider the case of an arbitrary $s\in\mathbb{R}$. Let us replace the function $u(t,x)$ in inequality (2.15) with $s=0$ by $\langle D_x\rangle^s u$. It is clear that

$$\mathcal{E}_{0,\mu}(\langle D_x\rangle^s u,\mathbb{R}_-(t)) = \mathcal{E}_{s,\mu}(u,\mathbb{R}_-(t)).$$

We now transform the middle term in (2.15). To this end, we recall relations (2.23) and (2.24) and also note that

$$\langle D_x\rangle^s A_k(\varepsilon,y,D_x) - A_k(\varepsilon,y,D_x)\langle D_x\rangle^s = \varepsilon\mathcal{B}_3^{(s+j_k-1)}(\varepsilon,t).$$

It follows that

$$\big\|Q(z,D_x)\langle D_x\rangle^s u\big\|^2 = \big\|\langle D_x\rangle^s Q(z,D_x)u + \mathcal{B}_1^{(s-1)}(\varepsilon,t)u\big\|^2 \\ = \big\|\langle D_x\rangle^s Q(z,D_x)u\big\|^2 + \varepsilon\Phi_5(\varepsilon,\theta;u).$$

Similarly, we have

$$\operatorname{Im}\bigl(Q(z,D_x)P(z,D_x)\langle D_x\rangle^s u, Q(z,D_x)\langle D_x\rangle^s u\bigr) = \\ = \operatorname{Im}\bigl(Q(z,D_x)P(z,D_x)u, Q(z,D_x)u\bigr)_s + \varepsilon\Phi_6(\varepsilon,\theta;u).$$

The required inequality is implied by the above relations. The proof of the theorem is complete. □

2.3 Energy estimates: the general case

We recall that the quadratic forms $J^{\pm}_{s,\mu,Q}(u,t)$ are defined by (2.14).

Theorem 2.6. *Suppose that the conditions of Theorem 1.2 hold. Let* $0<\delta'<\delta$ *and let* $Q_{\pm}(z,\xi)\in S^0(\Omega,m)$ *be the nearly constant symbols constructed in Theorem 2.1. Then for any* $s_0>0$ *and* μ, $0<\mu<\delta'$, *there are positive constants* ε_0 *and* K *such that*

$$K^{-1}\sup_{t\in\mathbb{R}}\|u(t,\cdot)\|_s^2 \le \sup_{t\in\mathbb{R}}\bigl(J^-_{s,\mu,Q_+}(u,t)+J^+_{s,\mu,Q_-}(u,t)\bigr) \le K\sup_{t\in\mathbb{R}}\|u(t,\cdot)\|_s^2, \tag{2.25}$$

where $|\varepsilon|\le\varepsilon_0$, $|s|\le s_0$, *and* $u\in C_b^\infty(\mathbb{R},H^{(\infty)})$.

Proof of Theorem 2.6. We confine ourselves to the case of constant symbols because inequality (2.25) in the general case can be derived from the former by repetition of the arguments in the proof of Theorem 2.5 (see Steps 2 and 3). Arguing as in the foregoing subsection, we conclude that the expression under the supremum sign in (2.25) can be written in the form

$$\begin{aligned} J^-_{s,\mu,Q_+}(u,t)+J^+_{s,\mu,Q_-}(u,t) &= \frac{1}{2}\bigl\|Q_+(D_x)u(t,\cdot)\bigr\|_s^2+\frac{1}{2}\bigl\|Q_-(D_x)u(t,\cdot)\bigr\|_s^2 \\ -\mu\Bigl(\int_{-\infty}^t e^{-2\mu(t-\theta)}\bigl\|Q_+(D_x)u\bigr\|_s^2 d\theta &+ \int_t^{+\infty} e^{-2\mu(t-\theta)}\bigl\|Q_-(D_x)u\bigr\|_s^2 d\theta\Bigr) \\ +\frac{1}{2}\Bigl(\int_{-\infty}^t e^{-2\mu(t-\theta)}\bigl(T_+(D_x)u,u\bigr)_s d\theta &+ \int_t^{+\infty} e^{-2\mu(t-\theta)}\bigl(T_-(D_x)u,u\bigr)_s d\theta\Bigr), \end{aligned} \tag{2.26}$$

where $T_+(\xi)$ is defined by formula (2.17) with $Q=Q_+$ and

$$T_-(\xi) = -i\bigl(P^*(\xi)Q_-^2(\xi)-Q_-^2(\xi)P(\xi)\bigr). \tag{2.27}$$

The right-hand inequality in (2.25) follows immediately from (2.4). To prove the left-hand inequality, note that, in view of (2.5), (2.6), and (2.26), we have

$$
\begin{aligned}
&J^{-}_{s,\mu,Q_+}(u,t) + J^{+}_{s,\mu,Q_-}(u,t) \geq \frac{1}{2}\Big(\big\|Q_+(D_x)u(t,\cdot)\big\|_s^2 + \big\|Q_-(D_x)u(t,\cdot)\big\|_s^2\Big) \\
&+(\delta'-\mu)\Big(\int_{-\infty}^{t} e^{-2\mu(t-\theta)}\big\|Q_+(D_x)u\big\|_s^2 d\theta + \int_{t}^{+\infty} e^{-2\mu(t-\theta)}\big\|Q_-(D_x)u\big\|_s^2 d\theta\Big) \\
&\qquad \geq \frac{1}{2}\Big(\big\|Q_+(D_x)u(t,\cdot)\big\|_s^2 + \big\|Q_-(D_x)u(t,\cdot)\big\|_s^2\Big) \\
&\qquad = \frac{1}{2}\Big(\big(Q_+^2(D_x) + Q_-^2(D_x)\big)u(t,\cdot), u(t,\cdot)\Big)_s,
\end{aligned}
\tag{2.28}
$$

where the inequality $\delta' \geq \mu$ was used. Applying (2.7), we can estimate the right-hand side of (2.28) from below by the expression $(2M)^{-1}\|u\|_s^2$. This completes the proof of (2.25) in the case of constant symbols. □

We now establish the a priori estimates in Theorem 1.8. We confine ourselves to (1.16) since (1.4) and (1.14) can be derived from (2.25) by literal repetition of the arguments in [23] (see Theorem 5.7 and Proposition 5.8).

Proof of (1.16). We confine ourselves to the case of constant coefficients. The general case can be treated using the arguments in Section 2.2 (see Steps 2 and 3).

It follows from (2.4) and (2.28) that

$$\|u(t,\cdot)\|_s^2 \leq 2M\big(J^{-}_{s,\mu,Q_+}(u,t) + J^{+}_{s,\mu,Q_-}(u,t)\big). \tag{2.29}$$

By the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
\int_{-\infty}^{+\infty}\big|J^{-}_{s,\mu,Q_+}(u,t)\big|\,dt &\leq \int_{-\infty}^{+\infty}\int_{-\infty}^{t} e^{-2\mu(t-\theta)}\|\mathcal{P}_\varepsilon u\|_s\|u\|_s d\theta\,dt \\
&\leq \int_{-\infty}^{+\infty}\int_{-\infty}^{t} e^{-2\mu(t-\theta)}\Big(\frac{M}{\mu}\|\mathcal{P}_\varepsilon u\|_s^2 + \frac{\mu}{4M}\|u\|_s^2\Big)d\theta\,dt \\
&\leq (8M)^{-1}\big\|u\big\|^2_{L^2(\mathbb{R},H^s)} + M\mu^{-2}\big\|\mathcal{P}_\varepsilon u\big\|^2_{L^2(\mathbb{R},H^s)}.
\end{aligned}
\tag{2.30}
$$

Similar argument shows that

$$\int_{-\infty}^{+\infty}\big|J^{+}_{s,\mu,Q_-}(u,t)\big|\,dt \leq (8M)^{-1}\big\|u\big\|^2_{L^2(\mathbb{R},H^s)} + M\mu^{-2}\big\|\mathcal{P}_\varepsilon u\big\|^2_{L^2(\mathbb{R},H^s)}. \tag{2.31}$$

Integrating (2.29) with respect to $t \in \mathbb{R}$ and using (2.30) and (2.31), we arrive at the required inequality. □

3 Construction of the indefinite metric

This section is devoted to the proof of Theorem 2.1. To clarify the main ideas, we first discuss the case of constant matrices. A detailed construction of the symbols Q_+ and Q_- is given in Subsection 3.2.

3.1 The case of constant dissipative matrices

Let P be an $m \times m$ matrix whose characteristic roots belong to the half-plane $\operatorname{Im} \tau \geq \delta$, where $\delta > 0$. We claim that for any $\delta' < \delta$ there exists a positive Hermitian matrix Q such that

$$i(P^* Q^2 - Q^2 P) \geq 2\delta' Q^2. \tag{3.1}$$

Indeed, let us fix an arbitrary positive $\delta' < \delta$. Since the characteristic roots of P lie in the half-plane $\operatorname{Im} \tau \geq \delta$, we have

$$\left|e^{isP}\right| + \left|e^{-isP^*}\right| \leq \operatorname{const}(1+s)^m e^{-\delta s}, \quad s \geq 0. \tag{3.2}$$

We now define the matrix [4]

$$Q^2 = \int_0^\infty e^{-isP^*} e^{isP} e^{2\delta' s} ds. \tag{3.3}$$

It follows from (3.2) that the matrix Q^2 is well-defined. Moreover, for any $q \in \mathbb{C}^m$ we have

$$\left(Q^2 q, q\right) = \int_0^\infty \left|e^{isP} q\right|^2 e^{2\delta' s} ds \geq \int_0^1 \left|e^{isP} q\right|^2 e^{2\delta' s} ds \geq c|q|^2, \tag{3.4}$$

which means that Q^2 is a positive Hermitian matrix and therefore has a square root. Furthermore, simple calculation shows that

$$\left(Q^2 e^{itP} q, e^{itP} q\right) = e^{-2\delta' t} \int_t^\infty \left|e^{isP} q\right|^2 e^{2\delta' s} ds.$$

Differentiating this relation with respect to t and setting $t = 0$, we derive

$$\left(i(Q^2 P - P^* Q^2) q, q\right) = -2\delta' \left(Q^2 q, q\right) - |q|^2,$$

which implies (3.1).

Let us note that the matrix Q^2 is not invariant under the change of basis. Indeed, the right-hand side of (3.3) with P replaced by $C^{-1}PC$ (where C is a nonsingular matrix) defines a different matrix.

There appear two essential difficulties when one tries to extend the above construction to matrix symbols without characteristic roots in a strip $|\operatorname{Im} \tau| < \delta$. The first of them consists in that the integral on the right-hand side of (3.3) with P replaced by a symbol $P(z, \xi) \in S^1_{\text{hom}}(\Omega, m)$ does not belong to the class $S^0(\Omega, m)$ (although it is a smooth function of the parameters (z, ξ)). The second difficulty is related to the fact that the analogs $Q^2_\pm(z, \xi)$ of the matrix Q^2 are not strictly positive and vanish on some subspaces of $\mathbb{C}^m$, and it is not quite clear why they possess smooth nonnegative square roots. (Indeed, a smooth square

[4] Here Q^2 is (temporarily) understood as a notation for the right-hand side of (3.3) and not as the square of the matrix Q.

root may not exist even in the one-dimensional case; for instance, if $Q^2(\xi) = \xi^2$ in the neighborhood of $\xi = 0$, then there is no smooth nonnegative square root.)

To overcome the first difficulty, we remark that formula (3.3) defines a symbol of the class $S^0(\Omega, m)$ if $P(z,\xi)$ has a block-diagonal form described in Theorem 5.3 (see (5.11) and (5.12)). Moreover, if Q^2 satisfies (3.1) and C is a nonsingular matrix, then $(C^{-1})^* Q^2 C^{-1}$ also satisfies inequality (3.1) with P replaced by CPC^{-1}. Using these two observations, the fact that $P(z,\xi)$ can be transformed (locally) to a block-diagonal form, and an appropriate partition of unity, we construct nonnegative Hermitian symbols Q_+^2 and Q_-^2 satisfying (2.4) – (2.8). The existence of smooth square roots is a separate question, and a sufficient condition for it is given in Proposition 3.3.

3.2 Proof of Theorem 2.1

To prove Theorem 2.1, we first establish (see Propositions 3.1 and 3.2 below) the existence of matrix functions $Q_\pm^2$ possessing the required properties in regions of the form

$$U_0 := \big\{(z,\xi) \in \overline{\Omega} \times \mathbb{R}^n_\xi : |\xi| < \rho + 1\big\}, \tag{3.5}$$

$$U := \big\{(z,\xi) \in \overline{\Omega} \times \mathbb{R}^n_\xi : |z - z^0| < \nu,\ |\xi/|\xi| - \omega^0| < \nu,\ |\xi| > \rho\big\} \tag{3.6}$$

and then, using a partition of unity, we construct the symbols $Q_\pm$ globally.

Proposition 3.1. *Under the conditions of Theorem 2.1, for any positive $\delta' < \delta$ there are nonnegative Hermitian matrix functions $Q^2_{0\pm}(z,\xi) \in C_b^\infty(\overline{U_0})$ satisfying (2.5) – (2.8) for $(z,\xi) \in \overline{U_0}$.*

Proof. 1) We set

$$Q^2_{0+}(z,\xi) = \int_0^{+\infty} \big(\Pi^+(z,\xi)\big)^* e^{-isP^*(z,\xi)} e^{isP(z,\xi)} \Pi^+(z,\xi) e^{2\delta' s}\, ds, \tag{3.7}$$

$$Q^2_{0-}(z,\xi) = \int_{-\infty}^{0} \big(\Pi^-(z,\xi)\big)^* e^{-isP^*(z,\xi)} e^{isP(z,\xi)} \Pi^-(z,\xi) e^{-2\delta' s}\, ds. \tag{3.8}$$

Let us show that $Q^2_{0\pm}(z,\xi) \in C_b^\infty(\overline{U_0})$. We confine ourselves to the case of the index $+$.

We choose a smooth path $\gamma \subset \mathbb{C}$ enclosing the characteristic roots of $P(z,\xi)$ lying in the half-plane $\operatorname{Im}\tau \geq \delta$ such that $\operatorname{Im}\tau \geq \mu := (\delta + \delta')/2$ for $\tau \in \gamma$ and the distance between γ and the roots of $P(z,\xi)$ is bounded from below by the constant $(\delta - \mu)/2$ for $(z,\xi) \in \overline{U_0}$. We have

$$e^{isP(z,\xi)} \Pi^+(z,\xi) = -(2\pi i)^{-1} \oint_\gamma e^{is\tau} (P(z,\xi) - \tau I)^{-1} d\tau.$$

Hence, by Lemma 5.9 with $f(\tau) = e^{is\tau}$, for $s \geq 0$ and $|\alpha| + |\beta| \leq N$, we have

$$\big|\partial_\xi^\alpha \partial_z^\beta \big(e^{isP(z,\xi)} \Pi^+(z,\xi)\big)\big| + \big|\partial_\xi^\alpha \partial_z^\beta \big((\Pi^+(z,\xi))^* e^{-isP^*(z,\xi)}\big)\big| \leq C_N e^{-\mu s}, \tag{3.9}$$

where the constant $C_N > 0$ depends only on δ' and the expression

$$\max_{|\alpha|+|\beta|\le N} \sup_{(z,\xi)\in\overline{U_0}} \big|\partial_\xi^\alpha \partial_z^\beta P(z,\xi)\big|. \tag{3.10}$$

It follows from (3.9) that the integrand in (3.7) and its derivatives with respect to (z,ξ) up to the order N can be estimated by a function of the form $C'_N e^{-2(\mu-\delta')s}$, and therefore the corresponding integrals converge uniformly with respect to (z,ξ). Hence, $Q_{0+}^2(z,\xi) \in C^\infty(\overline{U_0})$ and the derivatives of Q_{0+}^2 up to the order $N \ge 0$ are bounded by constants depending on δ' and expression (3.10).

2) We now prove (2.5) – (2.8). Relations (2.8) are obvious, and we confines ourselves to proving inequalities (2.5) and (2.7). Since

$$\begin{aligned}\big((Q_{0+}^2(z,\xi) + Q_{0-}^2(z,\xi))q, q\big) = \\ = \int_0^{+\infty} \big|e^{isP(z,\xi)}\Pi^+(z,\xi)q\big|^2 e^{2\delta' s} ds + \int_{-\infty}^0 \big|e^{isP(z,\xi)}\Pi^-(z,\xi)q\big|^2 e^{-2\delta' s} ds,\end{aligned} \tag{3.11}$$

the right-hand inequality (2.7) follows immediately from (3.9) and a similar estimate for $e^{isP}\Pi^-$. On the other hand, by Corollary 5.6, we have

$$\big|e^{isP(z,\xi)}\Pi^\pm(z,\xi)q\big| \ge c|\Pi^\pm(z,\xi)q| \quad \text{for} \quad |s| \le 1, \quad (z,\xi) \in \overline{\Omega} \times \mathbb{R}^n_\xi,$$

where the constant $c > 0$ does not depend on $q \in \mathbb{C}^m$, ξ, z, and s. Taking into account (3.11) and the estimate

$$\big|\Pi^+(z,\xi)q\big|^2 + \big|\Pi^-(z,\xi)q\big|^2 \ge |q|^2/2,$$

we derive the left-hand inequality in (2.7).

To prove (2.5), we fix an arbitrary $q \in \mathbb{C}^m$ and introduce the function

$$f_q(t) := \big(Q_{0+}^2(z,\xi)e^{itP(z,\xi)}q, e^{itP(z,\xi)}q\big).$$

Simple calculations based on (3.7) show that

$$f_q(t) = e^{-2\delta' t} \int_t^{+\infty} \big|e^{isP(z,\xi)}\Pi^+(z,\xi)q\big|^2 e^{2\delta' s} ds.$$

Differentiating $f_q(t)$ with respect to t and setting $t = 0$, we obtain

$$\begin{aligned} f'_q(t) &= \big(i(Q_{0+}^2(z,\xi)P(z,\xi) - P^*(z,\xi)Q_{0+}^2(z,\xi))q, q\big) \\ &= -\big|\Pi^+(z,\xi)q\big|^2 - 2\delta'(Q_{0+}^2(z,\xi)q, q) \le -2\delta'(Q_{0+}^2(z,\xi)q, q).\end{aligned}$$

Since $q \in \mathbb{C}^m$ is an arbitrary vector, we arrive at inequality (2.5). The proof is complete. □

Proposition 3.2. *Under the conditions of Theorem 2.1, for any region U of the form* (3.6) *and any positive $\delta' < \delta$ there are nonnegative Hermitian matrix functions $Q^2_{U\pm}(z,\xi) \in C^\infty_b(\overline{U})$ that satisfy* (2.5) – (2.8) *for $(z,\xi) \in \overline{U}$. Moreover, for any integer $N \geq 1$ there is a constant $C_N > 0$ not depending on (z^0, ω^0) such that*

$$\left|\partial_\xi^\alpha \partial_z^\beta Q^2_{U\pm}(z,\xi)\right| + \left|\partial_\xi^\alpha \partial_z^\beta (Q^2_{U\pm} P - P^* Q^2_{U\pm})(z,\xi)\right| \leq C_N |\xi|^{-|\alpha|}, \tag{3.12}$$

where $(z,\xi) \in \overline{U}$ and $|\alpha| + |\beta| \leq N$.

Proof. Before proceeding to the proof, we emphasize once again that formulas (3.7) and (3.8) do not lead to the desired result since the derivatives of the integrand can grow as $|\xi| \to +\infty$. However, if matrix functions $R^2_\pm(z,\xi)$ are defined by relations (3.7) and (3.8) in the basis in which $P(z,\xi)$ has a block-diagonal form, then the growing terms cancel out. The required symbols $Q^2_{U\pm}$ are the images of $R^2_\pm(z,\xi)$ in the original basis (see below).

1) We now turn to the accurate proof. Let us fix an arbitrary point $(z^0, \omega^0) \in \overline{\Omega} \times \Sigma_{n-1}$ and denote by $C(z,\xi)$ the matrix function transforming $P(z,\xi)$ to the block-diagonal form $\Lambda(z,\xi)$ in the region $\overline{U} = U_{\nu,\rho}(z^0,\omega^0)$ (see Theorem 5.3). We set

$$\widehat{\Pi}^\pm(z,\xi) := C^{-1}(z,\xi)\Pi^\pm(z,\xi)C(z,\xi) \tag{3.13}$$

and define the matrix functions

$$Q^2_{U\pm}(z,\xi) := \big(C^{-1}(z,\xi)\big)^* R^2_\pm(z,\xi) C^{-1}(z,\xi), \quad (z,\xi) \in \overline{U}, \tag{3.14}$$

where

$$R^2_\pm(z,\xi) = \pm \int_0^{\pm\infty} \big(\widehat{\Pi}^\pm(z,\xi)\big)^* e^{-is\Lambda^*(z,\xi)} e^{is\Lambda(z,\xi)} \widehat{\Pi}^\pm(z,\xi) e^{2\delta' s}\, ds. \tag{3.15}$$

We shall show that $R^2_\pm(z,\xi)$ possess the following properties.

(a) The matrix functions $R^2_\pm(z,\xi)$ are well-defined, nonnegative, and infinitely smooth on $\overline{U}$. Moreover,

$$\sup_{(z,\xi)\in\overline{U}} \left|\partial_\xi^\alpha \partial_z^\beta R^2_\pm(z,\xi)\right| \langle\xi\rangle^{|\alpha|} \leq C_N \quad \text{for} \quad |\alpha| + |\beta| \leq N,$$

where the constant C_N does not depend on (z^0, ω^0).

(b) There is a constant $K > 1$ not depending on (z^0, ω^0) such that

$$K^{-1} I \leq R^2_+(z,\xi) + R^2_-(z,\xi) \leq KI, \quad (z,\xi) \in \overline{U}. \tag{3.16}$$

(c) The following relations hold for $(z,\xi) \in \overline{U}$:

$$\pm i\big(\Lambda^* R^2_\pm - R^2_\pm \Lambda\big)(z,\xi) = \big(2\delta' R^2_\pm + (\widehat{\Pi}^\pm)^* \widehat{\Pi}^\pm\big)(z,\xi), \tag{3.17}$$

$$R^2_\pm(z,\xi)\widehat{\Pi}^\pm(z,\xi) = R^2_\pm(z,\xi). \tag{3.18}$$

2) Taking these assertions for granted, let us complete the proof of the proposition. The infinite smoothness of $Q^2_{U\pm}(z,\xi)$ (see (3.14)) and inequalities (3.12) follow from similar properties for C, C^{-1}, $(\widehat{\Pi}^{\pm})$, and $R^2_{\pm}$ and the relation

$$\begin{aligned}\pm i\bigl(P^* Q^2_{U\pm} - Q^2_{U\pm} P\bigr) &= (C^{-1})^*\bigl(\Lambda^* R^2_{\pm} - R^2_{\pm}\Lambda\bigr)C^{-1} \\ &= (C^{-1})^*\bigl(2\delta' R^2_{\pm} + (\widehat{\Pi}^{\pm})^*\widehat{\Pi}^{\pm}\bigr)C^{-1} \\ &= 2\delta' Q^2_{U\pm} + (C^{-1})^*(\widehat{\Pi}^{\pm})^*\widehat{\Pi}^{\pm}C^{-1},\end{aligned} \tag{3.19}$$

which is a consequence of (5.10), (3.14), and (3.17).

The estimates (2.5) and (2.6) follow easily from (3.19) and the inequality

$$\bigl((C^{-1})^*(\widehat{\Pi}^{\pm})^*\widehat{\Pi}^{\pm}C^{-1}q, q\bigr) = \bigl|\widehat{\Pi}^{\pm}C^{-1}q\bigr|^2 \geq 0.$$

To prove (2.7), we note that

$$\bigl(Q^2_{U\pm}(z,\xi)q, q\bigr) = \bigl(R^2_{\pm}(z,\xi)C^{-1}(z,\xi)q, C^{-1}(z,\xi)\bigr)$$

and therefore, in view of (3.16),

$$K^{-1}\bigl|C^{-1}(z,\xi)q\bigr|^2 \leq \bigl((Q^2_{U+}(z,\xi) + Q^2_{U-}(z,\xi))q, q\bigr) \leq K\bigl|C^{-1}(z,\xi)q\bigr|^2.$$

The required inequality follows now from the uniform boundedness of the matrix function $C^{-1}(z,\xi)$.

Relations (2.8) are implied by (3.18), (3.13), and (3.14).

3) Thus, it remains to establish properties (a) – (c). To this end, we note that the matrices $\Lambda(z,\xi)$ and $\widehat{\Pi}^{\pm}(z,\xi)$ entering the right-hand side of (3.15) have a block-diagonal form and consist of the $m_k \times m_k$ blocks

$$\Lambda_k(z,\xi) = |\xi|\sigma^0_k(z,\omega)I_{m_k} + \Lambda'_k(z,\xi), \quad \widehat{\Pi}^{\pm}_k(z,\xi), \quad k = 1,\dots,l.$$

It follows that the matrix $R^2_{\pm}(z,\xi)$ is also block-diagonal and is formed of the blocks

$$\begin{aligned}R^2_{k\pm}(z,\xi) &= \pm\int_0^{\pm\infty}\bigl(\widehat{\Pi}^{\pm}_k(z,\xi)\bigr)^* e^{-is\Lambda_k(z,\xi)^*}e^{is\Lambda_k(z,\xi)}\widehat{\Pi}^{\pm}_k(z,\xi)e^{\pm 2\delta' s}ds \\ &= \pm\int_0^{\pm\infty}\bigl(\widehat{\Pi}^{\pm}_k(z,\xi)\bigr)^* e^{-is\Lambda'_k(z,\xi)^*}e^{is\Lambda'_k(z,\xi)}\widehat{\Pi}^{\pm}_k(z,\xi)e^{\pm 2\delta' s}ds.\end{aligned}$$

Since a block-diagonal form is preserved under multiplication and addition of matrices, it suffices to establish (a) – (c) for each block.

To this end, we note that the matrix functions $\Lambda'_k(z,\xi)$ and $\widehat{\Pi}^{\pm}_k(z,\xi)$ considered in the region $\overline{U} = U_{\nu,\rho}(z^0,\omega^0)$ possesses the same properties as $P(z,\xi)$ and $\Pi^{\pm}(z,\xi)$ in $\overline{U_0}$. The only distinction is that the condition of uniform boundedness of derivatives should be replaced by finiteness of the seminorm

$$\sup_{(z,\xi)\in\overline{U}}\Bigl(\bigl|\partial^\alpha_\xi\partial^\beta_z\Lambda'_k(z,\xi)\bigr| + \bigl|\partial^\alpha_\xi\partial^\beta_z\widehat{\Pi}^{\pm}_k(z,\xi)\bigr|\Bigr)\langle\xi\rangle^{|\alpha|}$$

for any multi-indices α and β. Therefore, repeating the argument used in the proof of Proposition 3.1, we can establish the required assertions. The details are left to the reader. □

We have constructed symbols $Q^2_\pm(z,\xi)$ locally in the regions of the form (3.5) and (3.6). Construction of global symbols is based on a standard argument using a partition of unity, so that we only outline it.

1) Denote by ν and ρ the constants constructed in Proposition 3.2. Recall that the set U_0 is defined in (3.5). Denote by $\varphi_0(\xi) \in C^\infty(\mathbb{R}^n)$ an arbitrary function such that $0 \le \varphi \le 1$, $\varphi_0(\xi) = 1$ for $|\xi| < \rho$, and $\varphi_0(\xi) = 0$ for $|\xi| > \rho + 1$. Let $\{V_j\}$ be a finite covering of the unit sphere Σ_{n-1} by open sets of the form $\{\omega \in \Sigma_{n-1} : |\omega - \omega_j| < \nu\}$ and let $\big\{\psi_j(\omega) \in C_0^\infty(V_j)\big\}$ be a partition of unity subordinate to this covering. It is easy to construct a sequence $z_k \in \overline{\Omega}$ such that the family of open balls $W_k = \{z \in \mathbb{R}^d_z : |z - z_k| < \nu\}$ covers $\overline{\Omega}$, and each point is covered by $S = S(d) \ge 1$ sets at most. Let $\big\{\chi_k(z) \in C_0^\infty(W_k)\big\}$ be a partition of unity subordinate to $\{W_k\}$ such that the derivatives of $\chi_k(z)$ are bounded by constants not depending on k. (Here we use the fact that W_k are balls of the radius ν not depending on k.) Clearly, the open sets

$$U_0, \quad U_{jk} := \big\{(z,\xi) \in \mathbb{R}^d_z \times \mathbb{R}^n_\xi : z \in W_k, \xi/|\xi| \in V_j, |\xi| > \rho\big\}$$

form a covering of $\overline{\Omega} \times \mathbb{R}^n_\xi$ such that each point is covered by S sets at most. The family of functions

$$\varphi_0(\xi), \quad \varphi_{jk}(z,\xi) = (1 - \varphi_0(\xi))\psi_j(\xi/|\xi|)\chi_k(z)$$

forms a partition of unity subordinate to $\{U_0, U_{jk}\}$. It is easy to see that the derivatives of these functions are bounded by constants not depending on j and k.

2) Denote by $Q^2_{0\pm}(z,\xi)$ and $Q^2_{jk\pm}(z,\xi)$ the symbols constructed in Propositions 3.1 and 3.2 with $U = U_{jk} \cap (\overline{\Omega} \times \mathbb{R}^n_\xi)$. We extend the matrix functions $\varphi_0 Q^2_{0\pm}$ and $\varphi_{jk} Q^2_{jk\pm}$ by zero to the regions $(\overline{\Omega} \times \mathbb{R}^n_\xi) \setminus U_0$ and $(\overline{\Omega} \times \mathbb{R}^n_\xi) \setminus U_{jk}$, respectively, and set

$$T_\pm(z,\xi) = \varphi_0(\xi) Q^2_{0\pm}(z,\xi) + \sum_{j,k} \varphi_{jk}(z,\xi) Q^2_{jk\pm}(z,\xi), \quad (z,\xi) \in \overline{\Omega} \times \mathbb{R}^n_\xi. \tag{3.20}$$

For any point (z,ξ), this series has at most S nonzero terms. Hence, $T_\pm(z,\xi)$ are infinitely smooth matrix functions belonging to $S^0(\Omega, m)$. Furthermore, it is easily seen that (2.4) – (2.8) hold. Thus, it remains to prove that $T_\pm(z,\xi)$ has a nonnegative square root belonging to $S^0(\Omega, m)$. We will need the following auxiliary assertion.

Proposition 3.3. *Let $T(z,\xi) \in S^0(\Omega, m)$ be a nonnegative Hermitian symbol and let $\Pi(z,\xi)$ be a projection such that*

$$\Pi^*(z,\xi) T(z,\xi) \Pi(z,\xi) \ge c(\Pi^*\Pi)(z,\xi), \tag{3.21}$$

$$T(z,\xi)\Pi(z,\xi) = T(z,\xi), \tag{3.22}$$

where $c > 0$ does not depend on (z,ξ). Then there is a unique nonnegative Hermitian symbol $Q(z,\xi) \in S^0(\Omega, m)$ such that

$$Q^2(z,\xi) = T(z,\xi). \tag{3.23}$$

Moreover, for any integer $N \geq 1$ the seminorms $[Q, \Omega]_{0,\alpha,\beta}$ with $|\alpha| + |\beta| \leq N$ can be estimated by a constant depending on c and the expression

$$\max_{|\alpha|+|\beta|\leq N} [T, \Omega]_{0,\alpha,\beta}.$$

Thus, it suffices to show that $T_\pm$ satisfies the conditions of Proposition 3.3.

We confine ourselves to the case of the index $+$. Relation (3.22) with $\Pi = \Pi^+$ coincides with the first relation in (2.8). To prove (3.21), we multiply the left-hand inequality (2.7) by $(\Pi^+)^*$ from the left and by Π^+ from the right. Taking into account (2.8), we arrive at (3.21) with $c = M^{-1}$.

To complete the proof of Theorem 2.1, it only remains to establish Proposition 3.3.

Proof of Proposition 3.3. 1) The uniqueness of the nonnegative square root is established in [20, Theorem IV.9]. To prove the existence, let us set

$$C := \max_{q\in\mathbb{C}^m,\, |q|=1} \big(T(z,\xi)q, q\big)$$

and show that the spectrum of $T(z, \xi)$ consists of zero and a subset of the interval $[c, C]$, where c is the constant in (3.21). Indeed, the space $\mathbb{C}^m$ can be represented as the direct sum

$$\mathbb{C}^m = (I - \Pi)\mathbb{C}^m \oplus \Pi^*\mathbb{C}^m =: \mathbb{L} \oplus \mathbb{L}^\perp.$$

It follows from (3.22) that the restriction of $T(z, \xi)$ to $\mathbb{L}$ is a zero operator and hence its spectrum consists of the zero point. Thus, the required assertion will be proved if we show that

$$(Tp, p) \geq c|p|^2 \quad \text{for} \quad p \in \mathbb{L}^\perp. \tag{3.24}$$

To this end, we note that, in view of (3.21) and (3.22),

$$(Tp, p) = (\Pi^* T\Pi p, p) \geq c|\Pi p|^2. \tag{3.25}$$

Since $p \in \mathbb{L}^\perp = \Pi^*\mathbb{C}^m$, we have $p = \Pi^* p$ and therefore

$$|p|^2 = (p, p) = (p, \Pi^* p) = (\Pi p, p) \leq |\Pi p|\, |p|,$$

whence follows that $|\Pi p| \geq |p|$ for $p \in \mathbb{L}^\perp$. Combining this with (3.25), we arrive at (3.24)

2) We now choose a smooth path γ lying in the half-plane $\operatorname{Re} \tau > 0$ and containing the interval $[c, C]$ such that the distance between γ and the spectrum of $T(z, \xi)$ is bounded from below by the constant $c/2$. Denote by $\Pi_\gamma(z, \xi)$ the projection onto the eigenspace of $T(z, \xi)$ associated with the spectrum in $[c, C]$ and consider the analytic function $f(\tau) = \sqrt{\tau}$ defined for $\operatorname{Re} \tau > 0$ and normalized by the condition $f(1) = 1$. It is easy to see that the assumptions of Lemma 5.9 are fulfilled. Hence, the matrix function $Q(z, \xi) := f(\Pi_\gamma(z, \xi)T(z, \xi))$ belongs to the class $S^0(\Omega, m)$. It remains to note that $\Pi_\gamma(z, \xi)T(z, \xi) = T(z, \xi)$ and therefore relation (3.23) holds. ∎

4 Generalizations

In this section, some generalizations of the results in Section 1 are discussed. We first consider a quasilinear strongly hyperbolic system whose characteristic roots are outside a strip of the form $|\operatorname{Im}\tau| < \delta$ and describe a general scheme for constructing stable and unstable manifolds. We next consider semilinear systems and formulate a theorem on linearization of the phase portrait in the neighborhood of a stationary point. The proof of this result repeats the arguments used in [27] for the case of strictly hyperbolic scalar equations of high order. To simplify the notation, we will confine ourselves to autonomous systems.

4.1 Stable and unstable manifolds for quasilinear systems

Let us consider the quasilinear system [5]

$$\partial_t u = P_\varepsilon(x, u, \partial_x)u, \tag{4.1}$$

where P_ε is a matrix differential operator of the first order whose coefficients depend on $\varepsilon \in [-1, 1]$, $x \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$. We will assume that P_ε satisfies the following three hypotheses (cf. conditions of Theorem 1.2 with the time variable t replaced by $u \in \mathbb{R}^m$):

($\mathbf{H_1}$) The coefficients of P_ε are close to being constant, i. e., they are representable in the form

$$p(\varepsilon, x, u) = p_0 + \varepsilon p_1(\varepsilon, x, u),$$

where $p_0 \in \mathbb{R}$ is a constant and p_1 is a real-valued function that belongs to the space $C_b^\infty([-1, 1] \times \mathbb{R}^n \times B)$ for any ball $B \subset \mathbb{R}^m$.

($\mathbf{H_2}$) The operator P_ε is uniformly strongly hyperbolic, i. e., for any set of the parameters $(\varepsilon, x, u, \omega)$ the principal symbol $P_\varepsilon^0(x, u, i\omega)$ satisfies conditions (a) and (b) of Definition 1.1, and for any ball $B \subset \mathbb{R}^m$ the pairwise distinct characteristic roots of P_ε^0 are separated from one another uniformly with respect to $\varepsilon \in [-1, 1]$, $x \in \mathbb{R}^n$, $u \in B$, and $\omega \in \Sigma_{n-1}$.

($\mathbf{H_3}$) For any ball $B \subset \mathbb{R}^m$ there is $\delta > 0$ such that the characteristic roots of the full symbol $P_\varepsilon(x, u, i\xi)$ are outside the strip $|\operatorname{Im}\tau| < \delta$ for $(\varepsilon, x, u, \xi) \in [-1, 1] \times \mathbb{R}^n \times B \times \mathbb{R}^n$.

Consider the Cauchy problem for Eq. (4.1):

$$u(0, x) = u_0(x). \tag{4.2}$$

As is established by Petrovskii [17], the problem (4.1), (4.2) is locally well-posed. Namely, if s is a sufficiently large integer and $u_0 \in H^s$, then the problem (4.1), (4.2) has a unique solution $u(t, x)$ that is defined on a small time-interval $J \ni 0$ and satisfies the inclusions

$$\partial_t^j u \in C(J, H^{s-j}), \quad j = 0, \dots, s. \tag{4.3}$$

[5] In this section, we drop the factor $-i$ in front of the derivatives with respect to t and x since all the functions and operators are assumed to be real. In this case, the symbol of an operator is obtained on replacing the derivatives ∂_t and ∂_x by $i\tau$ and $i\xi$, respectively.

Theorem 4.1 below gives sufficient conditions for existence of global solutions.

Let us note that if the operator P_ε in (4.1) is replaced by $P_\varepsilon(x, 0, \partial_x)$, then we obtain a linear equation which satisfies the conditions of Theorem 1.4. In particular, for any integer $s \in \mathbb{R}$ and sufficiently small values of ε we can construct stable and unstable subspaces $\mathbb{E}_s^+$ and $\mathbb{E}_s^-$. We denote by $B_s(R)$ the closed ball in H^s of radius R centered at zero and set $\mathbb{E}_s^\pm(R) = \mathbb{E}_s^\pm \cap B_s(R)$.

The following assertion is a generalization of Theorem 1.4 to the case of quasilinear systems.

Theorem 4.1. *Suppose that conditions* (H_1) – (H_3) *are satisfied. Then for an arbitrary* $R > 0$, *sufficiently large integers* $s > 0$, *and any* μ, $0 < \mu < \delta$, *there are positive constants* ε_0 *and* C *such that the following statements hold for* $|\varepsilon| \le \varepsilon_0$.

(i) *There are weakly continuous injective mappings*

$$\mathcal{R}_s^\pm : \mathbb{E}_s^\pm(R) \to H^s \tag{4.4}$$

such that for any $u_0 \in \mathcal{M}_s^\pm(R) := \mathrm{Image}(\mathcal{R}_s^\pm)$ *the solution* $u(t, x)$ *of the problem* (4.1), (4.2) *can be continued to the half-line* $\mathbb{R}_\pm$ *and satisfies the inequality*

$$\sum_{j=0}^{s} \|\partial_t^j u(t, \cdot)\|_{s-j} \le C\, e^{\mp\mu t} \|u_0\|_s, \quad \pm t \ge 0. \tag{4.5}$$

(ii) *The sets* $\mathcal{M}_s^+(R)$ *and* $\mathcal{M}_s^-(R)$ *intersect only at zero. Moreover, the operators* $\mathcal{R}_s^+$ *and* $\mathcal{R}_s^-$ *are weakly differentiable at zero, and their derivatives are equal to the identity operators in* $\mathbb{E}_s^+$ *and* $\mathbb{E}_s^-$*, respectively. In particular, the manifold* $\mathcal{M}_s^\pm(R)$ *has a tangent space* $T\mathcal{M}_s^\pm$ *at zero, which coincides with* $\mathbb{E}_s^\pm$*, and hence* $H^s = T\mathcal{M}_s^+ \dotplus T\mathcal{M}_s^-$.

Outline of the proof. The main idea of the proof is the following. Theorem 1.4 remains valid for operators P_ε whose coefficients are of finite smoothness. In particular, it can be proved that if a function $u(t, x)$ satisfies the inclusions

$$\partial_t^j u \in C_b(\mathbb{R}, H^{s-j}), \quad j = 0 \dots, s, \tag{4.6}$$

where the integer s is sufficiently large, then the linear equation

$$\partial_t v = P_\varepsilon(x, u(t, x), \partial_x) v \tag{4.7}$$

possesses an ED. Therefore to any function $u(t, x)$ for which (4.6) holds there correspond projections $\mathsf{P}_s^+(u)$ and $\mathsf{P}_s^-(u)$ (in H^s) onto the stable and unstable subspaces associated with (4.7). Moreover, it is easy to see that $\mathsf{P}_s^\pm(u)$ depends only on the values of $u(t, x)$ for $\pm t \ge 0$.

Let us fix an arbitrary $u_0 \in \mathbb{E}_s^+$ and consider an operator $\mathcal{F}$ that takes each function $u(t, x)$ satisfying the inclusions

$$\partial_t^j u \in C_b(\mathbb{R}_+, H^{s-j}), \quad j = 0 \dots, s, \tag{4.8}$$

to the solution $v(t,x)$, $t \geq 0$, of Eq. (4.7) such that

$$v(0,x) = \mathsf{P}_s^+(u)u_0. \tag{4.9}$$

Since the right-hand side of (4.9) belongs to the stable subspace for (4.7), we conclude that $v(t,x)$ decays exponentially as $t \to +\infty$. Using this fact, it is not difficult to construct a subset K of the functional space defined by inclusions (4.8) such that $\mathcal{F}(K) \subset K$. Endowing K with the weak* topology and applying the Leray–Schauder theorem, we can show that $\mathcal{F}$ has a fixed point.

Thus, for any $u_0 \in \mathbb{E}_s^+$ there is a solution of the problem (4.1), (4.8) such that

$$u(0,x) = \mathsf{P}_s^+(u)u_0.$$

We denote by $\mathcal{G}_s^+$ the operator taking u_0 to $u(t,x)$ and set

$$\mathcal{R}_s^+(u_0) = \mathcal{G}_s^+(u_0)\big|_{t=0}.$$

The operator $\mathcal{R}_s^-$ is defined in a similar way. All the required properties can be verified with the help of the arguments used in the scalar case (cf. proof of Theorems 7.1 – 7.3 in [26]). □

4.2 Grobman–Hartman type theorem for semilinear systems

In this section, we study the semilinear system

$$\partial_t u = P_\varepsilon(x, \partial_x)u + \varepsilon R(\varepsilon, x, u), \tag{4.10}$$

where P_ε is a matrix differential operator satisfying the conditions of Theorem 1.2 and R is a real-valued function belonging to the space $C_b^\infty([-1,1] \times \mathbb{R}^n \times B)$ for any ball $B \subset \mathbb{R}^d$. Along with (4.10), let us consider the linear equation

$$\partial_t v = P_\varepsilon(x, \partial_x)v. \tag{4.11}$$

We denote by

$$\begin{aligned} \mathcal{U}_\varepsilon(t,\cdot) : H^s \to H^s, \quad u_0 \mapsto u(t,\cdot), \\ \mathcal{V}_\varepsilon(t) : H^s \to H^s, \quad u_0 \mapsto v(t,\cdot), \end{aligned}$$

the resolving operators of the Cauchy problem for Eqs. (4.10) and (4.11), respectively. Here $u(t,x)$ and $v(t,x)$ are the solutions of (4.10) and (4.11) starting from the initial function $u_0 \in H^s$. Thus, $\mathcal{U}_\varepsilon(t,\cdot)$ is defined in a small neighborhood of zero, while $\mathcal{V}_\varepsilon(t)$ exists for all $t \in \mathbb{R}$.

We denote by $\tau_j(\varepsilon, x, \xi)$ the characteristic roots of the full symbol $P_\varepsilon(x, i\xi)$ and set

$$\sigma_{\max} = \sup |\operatorname{Im} \tau_j(\varepsilon, x, \xi)|, \quad \sigma_{\min} = \inf |\operatorname{Im} \tau_j(\varepsilon, x, \xi)|,$$

where the supremum and infimum extend over the set $\varepsilon \in [-1,1]$, $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, $j = 1 \dots, m$. It is clear that $\sigma_{\max} \geq \sigma_{\min} > 0$. The proof of the following theorem repeat the scheme used in the case of scalar equations (see [27]).

Theorem 4.2. *Suppose that the conditions of Theorem 1.2 are satisfied. Then for any γ, $0 < \gamma < \sigma_{\min}/\sigma_{\max}$, sufficiently large integers $s > 0$, and an arbitrary $R > 0$ there are positive constants C and ε_0 such that the following statements hold for $|\varepsilon| \le \varepsilon_0$.*

(i) *There exists an open neighborhood of zero $O_R \subset H^s$ and a homeomorphism $\Phi_\varepsilon : B_s(R) \to O_R$ such that for any $u_0 \in B_s(R)$ we have*

$$\Phi_\varepsilon\big(\mathcal{U}_\varepsilon(t, u_0)\big) = \mathcal{V}_\varepsilon(t)\Phi_\varepsilon(u_0) \quad \textit{as long as} \quad \mathcal{U}_\varepsilon(t, u_0) \in B_s(R).$$

(ii) *The operator Φ_ε and its inverse Φ_ε^{-1} are uniformly Hölder continuous with exponent γ, i. e.,*

$$\big\|\Phi_\varepsilon(u_0) - \Phi_\varepsilon(v_0)\big\|_s \le C\|u_0 - v_0\|_s^\gamma, \qquad u_0, v_0 \in B_s(R),$$
$$\big\|\Phi_\varepsilon^{-1}(u_0) - \Phi_\varepsilon^{-1}(v_0)\big\|_s \le C\|u_0 - v_0\|_s^\gamma, \qquad u_0, v_0 \in O_R.$$

5 Appendix: strongly hyperbolic matrices

In this section, we have compiled some properties of strongly hyperbolic matrices. Although the results of this section are only applied to polynomial symbols, for the sake of completeness, we deal with standard Hörmander's classes of symbols. The presentation is organized as follows. We first recall some basic definitions and formulate (without proof) a refined version of Petrovskii's result on diagonalization of homogeneous strongly hyperbolic matrix symbols (see [17]). Using this assertion, we next show that inhomogeneous symbols can be transformation to block-diagonal form in which the blocks correspond to multiple characteristic roots of the principal symbol (cf. [11]). Finally, we prove that if the roots of a strongly hyperbolic matrix are divided into two groups separated by a strip, then the corresponding Riesz projections are symbols of order zero.

5.1 Homogeneous strongly hyperbolic symbols

Let $\Omega \subset \mathbb{R}^d_z$ be a convex open region and let Σ_{n-1} be the unit sphere in $\mathbb{R}^n_\xi$ centered at zero. We denote by $S_{\mathrm{h}}(\Omega)$ the space of functions $p^0(z,\omega) \in C^\infty(\overline{\Omega} \times \Sigma_{n-1})$ such that

$$[p^0, \Omega]_{\alpha,\beta} := \sup_{z\in\overline{\Omega},\, \omega\in\Sigma_{n-1}} |\partial_\omega^\alpha \partial_z^\beta p^0(z,\omega)| < \infty$$

for any multi-indices α and β. For any integer $m \ge 2$, denote by $S_{\mathrm{h}}(\Omega, m)$ the class of matrix symbols

$$P^0(z,\omega) = \big\|p^0_{ik}(z,\omega)\big\|_{i,k=1}^m, \quad z \in \Omega, \quad \omega \in \Sigma_{n-1}, \tag{5.1}$$

whose elements belong to $S_{\mathrm{h}}(\Omega)$. The spaces $S_{\mathrm{h}}(\Omega)$ and $S_{\mathrm{h}}(\Omega, m)$ are called classes of *homogeneous symbols.*

A homogeneous symbol $P^0(z,\omega) \in S_{\mathrm{h}}(\Omega, m)$ is said to be *uniformly strongly hyperbolic* if it satisfies conditions (a) and (b) of Definition 1.1 in which (ε, t, x)

is replaced by z, and inequality (1.3) holds for its pairwise distinct characteristic roots.

For an arbitrary symbol $P^0(z,\omega) \in S_{\mathrm{h}}(\Omega, m)$ of the form (5.1), we set

$$[P^0, \Omega]_{\alpha,\beta} := \max_{i,k=1,\dots,m} [p^0_{ik}, \Omega]_{\alpha,\beta}.$$

The following theorem is a refinement of a well-known Petrovskii's result on transformation of homogeneous strongly hyperbolic matrices to diagonal form (see [17, Section 3.1]). Its detailed proof can be found in [28, Section 4].

Theorem 5.1. *For any $B^0 > 0$ and $\varkappa > 0$ there is a constant $\nu > 0$ such that if a uniformly strongly hyperbolic symbol $P(z,\omega) \in S_{\mathrm{h}}(\Omega, m)$ satisfies the condition*

$$\max_{|\alpha|+|\beta|\le 1} [P^0, \Omega]_{\alpha,\beta} \le B^0, \tag{5.2}$$

and inequality (1.3) *holds for its characteristic roots, then the following assertions take place.*

(i) *For any $(z^0, \omega^0) \in \overline{\Omega} \times \Sigma_{n-1}$ there exists a nonsingular smooth matrix function $C^0(z,\omega)$ that is defined in the region*

$$U_\nu(z^0,\omega^0) = \big\{(z,\omega) \in \overline{\Omega} \times \Sigma_{n-1} : |z - z^0| \le \nu, |\omega - \omega^0| \le \nu\big\}$$

and satisfies relation

$$C^0(z,\omega)^{-1} P^0(z,\omega) C^0(z,\omega) = \Lambda^0(z,\omega) \quad \text{for} \quad (z,\omega) \in U_\nu(z^0,\omega^0), \tag{5.3}$$

where

$$\Lambda^0(z,\omega) = \begin{pmatrix} \sigma^0_1(z,\omega) I_{m_1} & & 0 \\ & \ddots & \\ 0 & & \sigma^0_l(z,\omega) I_{m_l} \end{pmatrix}, \tag{5.4}$$

and I_s is the $s \times s$ identity matrix. In particular, the multiplicities m_j of the characteristic roots of $P^0(z,\omega)$ do not depend on (z,ω).

(ii) *For any integer $N \ge 1$ there exists a constant $C_N > 0$ such that*

$$\sup_{(z,\omega)\in U_\nu(z^0,\omega^0)} \Big\{ \big|\partial^\alpha_\omega \partial^\beta_z C^0(z,\omega)\big| + \big|\partial^\alpha_\omega \partial^\beta_z C^0(z,\omega)^{-1}\big| \Big\} \le C_N \tag{5.5}$$

for $|\alpha| + |\beta| \le N$. Moreover, the constant C_N depends only on $\varkappa$ and the expression

$$\max_{|\alpha|+|\beta|\le N} [P^0, \Omega]_{\alpha,\beta}. \tag{5.6}$$

In particular, the characteristic roots of $P^0(z,\omega)$ belong to $C^\infty(\overline{\Omega} \times \Sigma_{n-1})$.

Remark 5.2. It is easy to show that if a uniformly strongly hyperbolic symbol $P^0 \in S_{\mathrm{h}}(\Omega, m)$ does not depend on z in a convex subset $\Omega' \subset \Omega$, then the neighborhood $U_\nu(z^0,\omega^0)$ in Theorem 5.1 can be replaced by

$$U_\nu(\Omega',\omega^0) = \big\{(z,\omega) \in \overline{\Omega} \times \Sigma_{n-1} : \mathrm{d}(z,\Omega') \le \nu, |\omega - \omega^0| \le \nu\big\}$$

where

$$\mathrm{d}(z,\Omega') = \min_{z'\in\Omega'} |z - z'|.$$

In particular, the matrix function $C^0(z,\omega)$ and its inverse are defined for $(z,\omega) \in U_\nu(\Omega',\omega^0)$, and the supremum in (5.5) extends over the set $U_\nu(\Omega',\omega^0)$. Moreover, the matrix $C^0(z,\omega)$ can be chosen so that it does not depend on z in the region Ω'.

5.2 Inhomogeneous strongly hyperbolic symbols

As before, we denote by $\Omega \subset \mathbb{R}^d_z$ an open convex region. We recall that the classes $S^j(\Omega)$ and $S^j(\Omega,m)$ are defined in Section 2. Let $S^j_{\mathrm{p}}(\Omega)$ be the subspace in $S^j(\Omega)$ consisting of the symbols $p(z,\xi)$ for which there is $p^0(z,\omega) \in S_{\mathrm{h}}(\Omega)$ such that

$$p(z,\xi) - \chi(\xi)|\xi|^j p^0(z,\xi/|\xi|) \in S^{j-1}(\Omega),$$

where $\chi(\xi) \in C^\infty(\mathbb{R}^n)$, $\chi(\xi) = 0$ for $|\xi| \le 1$ and $\chi(\xi) = 1$ for $|\xi| \ge 2$. The function $p^0(z,\xi)$ is uniquely defined and called the *principal part* of $p(z,\xi)$. Similarly, we denote by $S^j_{\mathrm{p}}(\Omega,m)$ the space of matrix symbols

$$P(z,\xi) = \left\|p_{ik}(z,\xi)\right\|^m_{i,k=1}, \quad z \in \Omega, \quad \xi \in \mathbb{R}^n, \tag{5.7}$$

whose elements belong to $S^j_{\mathrm{p}}(\Omega)$. The matrix $P^0(z,\omega) = \left\|p^0_{ik}(z,\omega)\right\|$ consisting of the principal parts of p_{ik} is called the *principal part* of $P(z,\xi)$. A symbol $P(z,\xi) \in S^1_{\mathrm{p}}(\Omega,m)$ is said to be *uniformly strongly hyperbolic* if its principal part is uniformly strongly hyperbolic.

For an arbitrary symbol $P(z,\xi) \in S^j_{\mathrm{p}}(\Omega,m)$ of the form (5.7), we set

$$[P,\Omega]_{j,\alpha,\beta} := \max_{i,k=1,\dots,m} [p_{ik},\Omega]_{j,\alpha,\beta}.$$

The following theorem is a refinement of a result in [11].

Theorem 5.3. *For any $B > 0$ and $\varkappa > 0$ there is a constant $\nu > 0$ such that if a uniformly strongly hyperbolic symbol $P(z,\xi) \in S^1_{\mathrm{p}}(\Omega)$ satisfies the condition*

$$\max_{|\alpha|+|\beta|\le 1} [P,\Omega]_{1,\alpha,\beta} \le B, \tag{5.8}$$

and inequality (1.3) *holds for the roots of its principal part, then the following assertions take place.*

(i) *There exists $\rho > 0$ such that for any $(z^0,\omega^0) \in \overline{\Omega}\times\Sigma_{n-1}$ there is a smooth matrix function $C(z,\xi)$ that is defined on the set*

$$U_{\nu,\rho}(z^0,\omega^0) = \left\{(z,\xi) \in \overline{\Omega} \times \mathbb{R}^n_\xi : |z - z^0| \le \nu,\ |\xi/|\xi| - \omega^0| \le \nu,\ |\xi| \ge \rho\right\} \tag{5.9}$$

and transforms $P(z,\xi)$ to the block-diagonal form

$$C(z,\xi)^{-1}P(z,\xi)C(z,\xi) = \Lambda(z,\xi) \quad \textit{for} \quad (z,\xi) \in U_{\nu,\rho}(z^0,\omega^0). \tag{5.10}$$

Here $\Lambda(z,\xi)$ *is a smooth matrix function of the form*

$$\Lambda(z,\xi) = |\xi|\Lambda^0(z,\omega) + \Lambda'(z,\xi), \quad \omega = \xi/|\xi|, \tag{5.11}$$

where $\Lambda^0(z,\omega)$ *is defined in* (5.4),

$$\Lambda'(z,\xi) = \begin{pmatrix} \Lambda'_1(z,\xi) & & 0 \\ & \ddots & \\ 0 & & \Lambda'_l(z,\xi), \end{pmatrix} \tag{5.12}$$

and $\Lambda'_k(z,\xi)$ *is a smooth* $m_k \times m_k$ *matrix. Moreover, for any multi-indices* α *and* β *there is a constant* $C_{\alpha\beta} > 0$ *such that*

$$\left|\partial_\xi^\alpha \partial_z^\beta \Lambda'(z,\xi)\right| \le C_{\alpha\beta}|\xi|^{-|\alpha|}, \quad (z,\xi) \in U_{\nu,\rho}(z^0,\omega^0). \tag{5.13}$$

(ii) *The matrix* $C(z,\xi)$ *can be written as*

$$C(z,\xi) = C^0(z,\omega)C^1(z,\xi), \quad \omega = \xi/|\xi|, \tag{5.14}$$

where $C^0(z,\omega)$ *is defined in Theorem 5.1 and* $C^1(z,\xi)$ *is a smooth matrix satisfying the following inequality for any integer* $N \ge 1$ *and multi-indices* α *and* β, $|\alpha| + |\beta| \le N$:

$$\left|\partial_\xi^\alpha \partial_z^\beta (C^1(z,\xi) - I)\right| + \left|\partial_\xi^\alpha \partial_z^\beta (C^1(z,\xi)^{-1} - I)\right| \le C_N |\xi|^{-1-|\alpha|}, \tag{5.15}$$

where $(z,\xi) \in U_{\nu,\rho}(z^0,\omega^0)$. *Moreover, the constant* C_N *depends only on* $\varkappa$ *and the expression*

$$\max_{|\alpha|+|\beta|\le N} \Big(\big[P,\Omega\big]_{1,\alpha,\beta} + \big[P - \chi(\xi)P^0, \Omega\big]_{0,\alpha,\beta} \Big),$$

where $\chi(\xi) \in C^\infty(\mathbb{R}^n)$, $\chi(\xi) = 0$ *for* $|\xi| \le 1$ *and* $\chi(\xi) = 1$ *for* $|\xi| \ge 2$.

Theorem 5.3 will be proved in Subsection 5.3. Here we make a remark and derive two corollaries.

Remark 5.4. As in the case of Theorem 5.1, if a uniformly strongly hyperbolic symbol $P \in S^1_{\mathrm{p}}(\Omega, m)$ does not depend on z in a convex subset $\Omega' \subset \Omega$, then the neighborhood $U_{\nu,\rho}(z^0,\omega^0)$ in Theorem 5.3 can be replaced by

$$U_{\nu,\rho}(\Omega',\omega^0) = \big\{(z,\omega) \in \overline{\Omega} \times \Sigma_{n-1} : \mathrm{d}(z,\Omega') \le \nu, |\xi/|\xi| - \omega^0| \le \nu, |\xi| \ge \rho\big\}.$$

Moreover, the matrix function $C^1(z,\xi)$ can be chosen so that it does not depend on z in the region Ω'.

Theorem 5.3 implies the following estimate for the distance between the roots of a strongly hyperbolic symbols and those of its principal part.

Corollary 5.5. *Let $P(z,\xi) \in S^1_{\mathrm{p}}(\Omega)$ be a uniformly strongly hyperbolic symbol. Then its characteristic roots can be represented (under suitable indexing) in the form*

$$\tau_j(z,\xi) = |\xi|\sigma^0_k(z,\omega) + \mu_j(z,\xi), \quad j = s_{k-1}+1,\dots,s_k, \quad k = 1,\dots,l, \tag{5.16}$$

where $\omega = \xi/|\xi|$, $s_0 = 0$, $s_k = m_1 + \dots + m_k$, the functions μ_j satisfy the inequality

$$|\mu_j(z,\xi)| \le c \quad \textit{for} \quad z \in \overline{\Omega}, \quad |\xi| \ge 1, \tag{5.17}$$

and the constant $c > 0$ does not depend on z and ξ.

Proof. In view of (5.10) and (5.11), the characteristic roots of $P(z,\xi)$ coincides with those of the matrices

$$\Lambda_k(z,\xi) := |\xi|\Lambda^0_k(z,\omega) + \Lambda'_k(z,\xi), \quad k = 1,\dots,l,$$

where $\Lambda^0_k = \sigma^0_k I_{m_k}$ and Λ'_k is defined in Theorem 5.3. Therefore the functions $\mu_j(z,\xi)$ in (5.16) are the characteristic roots of $\Lambda'_k(z,\xi)$. Since $\Lambda'_k(z,\xi)$ is a bounded matrix function, we conclude that $\mu_j(z,\xi)$ are also uniformly bounded. □

The assertion below provides some uniform estimates for the exponent of a strongly hyperbolic symbol.

Corollary 5.6. *Let $P(z,\xi) \in S^1_{\mathrm{p}}(\Omega)$ be a uniformly strongly hyperbolic symbol. Then there are constants $A > 0$ and $C > 1$ such that*

$$C^{-1}e^{-A|t|} \le \left|e^{itP(z,\xi)}\right| \le C\,e^{A|t|}, \quad t \in \mathbb{R}, \quad (z,\xi) \in \overline{\Omega} \times \mathbb{R}^n_\xi. \tag{5.18}$$

Proof. We first prove the right-hand inequality. Let $\rho > 0$ be the constant defined in Theorem 5.3. The norm of $P(z,\xi)$ is uniformly bounded for $z \in \overline{\Omega}$, $|\xi| \le \rho$, and the required inequality follows from the expansion of the exponential into the Taylor series.

Now assume that $|\xi| \ge \rho$. In view of (5.10),

$$C(z,\xi)^{-1}e^{itP(z,\xi)}C(z,\xi) = e^{it\Lambda(z,\xi)}.$$

Since $\Lambda(z,\xi)$ has a block-diagonal form, it suffices to establish (5.18) for each block $\Lambda_k(z,\xi)$. The matrices $\Lambda^0_k = \sigma^0_k I_{m_k}$ and Λ'_k commute. Hence,

$$\left|e^{it\Lambda_k(z,\xi)}\right| = \left|e^{it\sigma^0_k(z,\xi)}e^{it\Lambda'_k(z,\xi)}\right| = \left|e^{it\Lambda'_k(z,\xi)}\right|.$$

Taking into account (5.13), we derive the right-hand inequality (5.18).

To establish the left-hand inequality, we replace t by $-t$ in the right-hand inequality (5.18) and derive

$$\left|e^{-itP(z,\xi)}q\right| \le C\,e^{A|t|}|q|, \quad q \in \mathbb{C}^m. \tag{5.19}$$

On setting $q = e^{itP(z,\xi)}p$ in (5.19), we obtain

$$|p| \le C\,e^{A|t|}\left|e^{itP(z,\xi)}p\right|,$$

which completes the proof. □

5.3 Proof of Theorem 5.3

To simplify the presentation, we will omit some technical details. A complete proof can be found in [28, Section 2.2].

1) We begin with a study of characteristic roots $\tau_j(z,\xi)$ of $P(z,\xi)$ for large $|\xi|$. We set

$$P'(z,\xi) = P(z,\xi) - |\xi| P^0(z,\omega).$$

It is clear that $\tau_j(z,\xi)/|\xi|$ are the roots of the matrix

$$|\xi|^{-1} P(z,\xi) = P^0(z,\omega) + |\xi|^{-1} P'(z,\xi), \quad \omega = \xi/|\xi|. \tag{5.20}$$

Matrix (5.20) is a small perturbation of $P^0(z,\omega)$, and its characteristic roots continuously depend on (z,ξ) and tend to the roots of $P^0(z,\omega)$ as $|\xi| \to \infty$. Let us derive a preliminary estimate for the difference between the corresponding roots (cf. (5.16)).

Let $\varkappa$ be the constant in (1.3) and let $\sigma_k^0(z,\omega)$, $k = 1,\dots,l$, be the pairwise distinct roots of $P^0(z,\omega)$ with multiplicity m_k. A standard argument based on the Rouché theorem shows that for $|\xi| \ge \rho \gg 1$ every disk Γ_k of radius $\varkappa/3$ centered at $\sigma_k^0(z,\omega)$ contains m_k roots of (5.20). Let $\tau_{jk}(z,\xi)$, $j = 1,\dots,m_k$, be the roots in Γ_k. We have

$$|\xi|^{-1}\tau_{jk}(z,\xi) \to \sigma_k^0(z,\omega) \quad \text{as} \quad |\xi| \to \infty.$$

Moreover, the function $|\xi|^{-1}\tau_{jk}(z,\xi)$ can be expanded into Puiseux series in powers of $|\xi|^{-\frac{1}{m_k}}$. Thus, for each z and ω, we obtain

$$\tau_{jk}(z,\xi) = \sigma_k^0(z,\omega) + \mu_{jk}(z,\xi), \tag{5.21}$$

where

$$|\mu_{jk}(z,\xi)| \le C\,|\xi|^{1-\frac{1}{m_k}}, \quad z \in \overline{\Omega}, \quad |\xi| \ge \rho, \tag{5.22}$$

and the constant C does not depend on z and ξ.

2) Let us proceed to a construction of the matrix $C^1(z,\xi)$. Denote by $\mathcal{E}(z,\omega) = \{e_1(z,\omega),\dots,e_m(z,\omega)\}$ the basis of $\mathbb{C}^m$ in which $P^0(z,\omega)$ has the diagonal form (5.4). We will define $C^1(z,\xi)$ as the transition matrix from $\mathcal{E}(z,\omega)$ to another basis $\mathcal{F}(z,\xi)$ in which $P(z,\xi)$ has a block-diagonal form. To this end, we introduce the Riesz projections

$$\Pi_k^0(z,\omega) = -(2\pi i)^{-1} \oint_{\gamma_k(z,\omega)} \big(P^0(z,\omega) - \tau I\big)^{-1} d\tau, \quad \omega \in \Sigma_{n-1}, \tag{5.23}$$

$$\Pi_k(z,\xi) = -(2\pi i)^{-1} \oint_{\gamma_k(z,\xi)} \big(P(z,\xi) - \tau I\big)^{-1} d\tau, \quad |\xi| \gg 1, \tag{5.24}$$

onto the eigenspaces of $P^0(z,\omega)$ and $P(z,\xi)$ corresponding to the eigenvalues $\sigma_k^0(z,\omega)$ and $\tau_{jk}(z,\xi)$, $j = 1,\dots,m_k$, respectively. Here $\gamma_k(z,\xi)$ is a circle of radius $\varkappa|\xi|/3$ centered at $|\xi|\sigma_k^0(z,\omega)$. In view of (1.3), (5.21) and (5.22), the circle $\gamma_k(z,\omega)$ encloses the root $\sigma_k^0(z,\omega)$ solely, and the circle $\gamma_k(z,\xi)$ encloses only the roots $\tau_{jk}(z,\xi)$, $j=1,\dots,m_k$, so that formulas (5.23) and (5.24) define the projections onto the eigenspaces described above.

Let $\chi_\rho(\xi)\in C^\infty(\mathbb{R}^n)$ be a function such that $\chi_\rho(\xi)=0$ for $|\xi|\le\rho-1$ and $\chi_\rho(\xi)=1$ for $|\xi|\ge\rho$. We claim that

$$\Pi_k^0(z,\omega)\in S_{\mathrm{h}}(\Omega), \tag{5.25}$$

$$\chi_\rho(\xi)\Pi_k(z,\xi)\in S^0_{\mathrm{p}}(\Omega,m), \tag{5.26}$$

where ρ is sufficiently large, and that Π_k^0 is the principal part of $\chi_\rho\Pi_k$.

To prove (5.25), we fix an arbitrary point (z^0,ω^0) and consider its small neighborhood $U=U(z^0,\omega^0)$ in $\overline{\Omega}\times\Sigma_{n-1}$. It suffices to show that (5.23) is infinitely smooth in U, and all its derivatives are bounded by constants not depending on (z^0,ω^0).

The integrand in (5.23) is a meromorphic function that has the unique pole $\tau=\sigma_k^0(z,\omega)$ inside the circle $\gamma_k(z,\omega)$. Since $\sigma_k^0(z,\omega)$ continuously depends on (z,ω), there is $\nu>0$ such that the circle $\gamma_k(z^0,\omega^0)$ encloses only the root $\sigma_k^0(z,\omega)$ if

$$(z,\omega)\in U_\nu(z^0,\omega^0)=\bigl\{(z,\omega)\in\overline{\Omega}\times\Sigma_{n-1}:|z-z^0|\le\nu,|\omega-\omega^0|\le\nu\bigr\}.$$

Hence, by the Cauchy theorem, we can replace the path of integration in (5.23) by $\gamma_k(z^0,\omega^0)$ and write

$$\Pi_k^0(z,\omega)=-(2\pi i)^{-1}\oint_{\gamma_k(z^0,\omega^0)}\bigl(P^0(z,\omega)-\tau I\bigr)^{-1}d\tau,\quad (z,\omega)\in U_\nu(z^0,\omega^0). \tag{5.27}$$

Now note that

$$\bigl(P^0(z,\omega)-\tau I)\bigr)^{-1}=C^0(z,\omega)^{-1}\bigl(\Lambda^0(z,\omega)-\tau I\bigr)^{-1}C^0(z,\omega). \tag{5.28}$$

The expression on the right-hand side of (5.28) is infinitely smooth with respect to $(z,\omega)\in U_\nu(z^0,\omega^0)$ and $\tau\in\gamma_k(z,\omega)$, and all the derivatives are bounded by constants not depending on (z^0,ω^0). This implies the required assertion.

To prove (5.26), we change the variable $\tau\mapsto\tau\,|\xi|$ in (5.24) and separate the principal part of $P(z,\xi)$. This results in

$$\Pi_k(z,\xi)=-(2\pi i)^{-1}\oint_{\gamma_k(z,\omega)}\bigl(P^0(z,\omega)+|\xi|^{-1}P'(z,\xi)-\tau I\bigr)^{-1}d\tau,\quad |\xi|\gg 1. \tag{5.29}$$

It is clear that $P'(z,\xi)$ is a smooth matrix function satisfying the inequalities

$$\bigl|\partial_\xi^\alpha\partial_z^\beta P'(z,\xi)\bigr|\le C_{\alpha\beta}|\xi|^{-|\alpha|},\quad |\xi|\gg 1.$$

For $\tau\in\gamma_k(z,\xi)$, we can represent the integrand in (5.29) as a power series:

$$\begin{aligned}&\bigl(P^0(z,\omega)+|\xi|^{-1}P'(z,\xi)-\tau I\bigr)^{-1}\\ &\quad=\bigl(P^0(z,\omega)-\tau I\bigr)^{-1}\Bigl(I+\sum_{j=1}^\infty(-1)^j|\xi|^{-j}\bigl(P'(z,\xi)(P^0(z,\omega)-\tau I)^{-1}\bigr)^j\Bigr).\end{aligned}$$

Substitution of this expression into (5.29) results in

$$\Pi_k(z,\xi) = \Pi_k^0(z,\omega) - (2\pi i)^{-1}\sum_{j=1}^{\infty}(-1)^j|\xi|^{-j}R_{jk}(z,\xi),$$

where

$$R_{jk}(z,\xi) = \oint_{\gamma_k(z,\omega)}(P^0(z,\omega)-\tau I)^{-1}\big(P'(z,\xi)(P^0(z,\omega)-\tau I)^{-1}\big)^j\,d\tau. \qquad (5.30)$$

Thus, the desired assertion will be proved if we show that the functions $R_{jk}(z,\xi)$ are infinitely differentiable for $|\xi|\gg 1$ and satisfy the inequalities

$$\big|\partial_\xi^\alpha\partial_z^\beta R_{jk}(z,\xi)\big| \le b_{\alpha\beta}^j|\xi|^{-|\alpha|} \quad \text{for} \quad z\in\overline{\Omega}, \quad |\xi|\gg 1, \qquad (5.31)$$

where the constants $b_{\alpha\beta}$ do not depend on z, ξ, and j. To this end, we repeat the argument in the proof of (5.25) and replace the path of integration in (5.30) by $\gamma_k(z^0,\omega^0)$. The fact that $R_{jk}(z,\xi)$ is infinitely smooth follows from the infinite smoothness of the integrand for $\tau\in\gamma_k(z^0,\omega^0)$. To prove inequalities (5.31), it suffices to note that the integrand in (5.30) satisfies similar estimates for $\tau\in\gamma_k(z^0,\omega^0)$. The details are left to the reader.

3) We now fix an arbitrary point $(z^0,\omega^0)\in\overline{\Omega}\times\Sigma_{n-1}$ and define a set of vectors $\mathcal{F}(z,\xi)=\{f_1(z,\xi),\dots,f_m(z,\xi)\}$ for $(z,\xi/|\xi|)\in U_\nu(z^0,\omega^0)$, where $\nu>0$ is the constant in Theorem 5.1:

$$f_j(z,\xi) = \Pi_k(z,\xi)e_j(z,\omega), \quad 1\le j-(m_1+\dots+m_{k-1})\le m_k, \qquad (5.32)$$

where $k=1,\dots,l$. Since Π_k^0 is the principal part of $\chi_\rho\Pi_k$, we have $\Pi_k=\Pi_k^0+\Pi_k'$, where Π_k' satisfies the inequalities

$$\big|\partial_\xi^\alpha\partial_z^\beta\Pi_k'(z,\xi)\big| \le C_{\alpha\beta}|\xi|^{-1-|\alpha|} \quad \text{for} \quad z\in\overline{\Omega}, \quad |\xi|\gg 1. \qquad (5.33)$$

It follows that the vectors $f_j(z,\omega)$ are representable in the form

$$f_j(z,\xi) = \big(\Pi_k^0(z,\omega)+\Pi_k'(z,\xi)\big)e_j(z,\omega) = e_j(z,\omega)+\Pi_k'(z,\xi)\,e_j(z,\omega).$$

Inequality (5.33) with $\alpha=\beta=0$ implies that for large $|\xi|$ the vectors $f_j(z,\omega)$ are linearly independent, and therefore $\mathcal{F}(z,\omega)$ is a basis in $\mathbb{C}^m$. By construction of the projections $\Pi_k(z,\xi)$, the matrix $P(z,\xi)$ has a block-diagonal form in this basis.

To prove (5.15), we expand the vectors $g_j(z,\xi)=\Pi_k'(z,\xi)e_j(z,\omega)$ in the basis $\mathcal{E}(z,\omega)$:

$$g_j(z,\xi) = \sum_{i=1}^m g_{ij}(z,\xi)\,e_i(z,\omega).$$

The matrix function

$$G(z,\xi) = \big\|g_{ij}(z,\xi)\big\|_{i,j=1}^m$$

satisfies inequalities (5.33) for large $|\xi|$. It follows that there is $\rho>0$ such that the matrix

$$C^1(z,\xi) = I+G(z,\xi)$$

is invertible for $|\xi|\ge\rho$, and (5.15) holds. The proof is complete.

5.4 Projections associated with separated groups of characteristic roots

Proposition 5.7. *Let $P(z,\xi) \in S^1_p(\Omega,m)$ be a uniformly strongly hyperbolic symbol whose characteristic roots are outside the strip $|\operatorname{Im}\tau| < \delta$, where $\delta > 0$ is a constant, and let $\Pi^+(z,\xi)$ and $\Pi^-(z,\xi)$ be the Riesz projections corresponding to the roots in the half-planes $\operatorname{Im}\tau \geq \delta$ and $\operatorname{Im}\tau \leq -\delta$, respectively* (*see* (2.1)). *Then $\Pi^\pm(z,\xi) \in S^0(\Omega,m)$.*

The proof of this assertion is based on the two lemmas below.

Lemma 5.8. *Let $R(z,\xi)$ be an infinitely smooth $m\times m$ matrix function that is defined on the closure of an open region $U \subset \mathbb{R}^d_z \times \mathbb{R}^n_\xi$ and satisfies the condition*

$$r_{\alpha\beta} := \sup_{(z,\xi)\in\overline{U}} \big|\partial^\alpha_\xi \partial^\beta_z R(z,\xi)\big| \langle\xi\rangle^{|\alpha|} < \infty \tag{5.34}$$

for any multi-indices α and β. Let $\gamma \subset \mathbb{C}$ be a smooth closed path without self-intersections such that the distance between γ and the characteristic roots of $R(z,\xi)$ is bounded from below by a constant $b > 0$ for $(z,\xi) \in \overline{U}$. Then

$$\big|\partial^\alpha_\xi \partial^\beta_z (R(z,\xi) - \tau I)^{-1}\big| \leq c_N \langle\xi\rangle^{-|\alpha|}, \quad (z,\xi) \in \overline{U}, \quad \tau \in \gamma, \quad |\alpha| + |\beta| \leq N, \tag{5.35}$$

where the constant $c_N > 0$ depends on $r_{\alpha\beta}$, $|\alpha| + |\beta| \leq N$, and b.

Proof. Transforming $R(z,\xi)$ to an upper-triangular form, we can easily prove that

$$\big|\det(R(z,\xi) - \tau I)\big| \geq b^m.$$

Combining this inequality with the explicit formula for the inverse matrix, we derive inequality (5.35) for $N = 0$.

To prove (5.35) for $N \geq 1$, we note that

$$\partial_{\xi_k}(R(z,\xi) - \tau I)^{-1} = (R(z,\xi) - \tau I)^{-1} (\partial_{\xi_k} R)(z,\xi) (R(z,\xi) - \tau I)^{-1},$$

whence follows that

$$\big|\partial_{\xi_k}(R(z,\xi) - \tau I)^{-1}\big| \leq c_0^2 \big|(\partial_{\xi_k} R)(z,\xi)\big| \leq \text{const}\langle\xi\rangle^{-1}.$$

The expression $\partial_{z_k}(R(z,\xi) - \tau I)^{-1}$ and higher derivatives can be estimated in a similar way. □

In the conditions of Lemma 5.8, we denote by $\Pi_\gamma(z,\xi)$ the projection onto the eigenspace of $R(z,\xi)$ that corresponds to the part of the spectrum enclosed by γ. Let $f(\tau)$ be an analytic function in a one-connected neighborhood of γ. Then, according to the Dunford–Taylor formula (see [9, Chapter I, Section 6]), we can define the operator function

$$f(\Pi_\gamma(z,\xi) R(z,\xi)) = -(2\pi i)^{-1} \oint_\gamma f(\tau) \big(R(z,\xi) - \tau I\big)^{-1} d\tau. \tag{5.36}$$

Lemma 5.9. *For any integer $N \geq 0$ there is a constant $C_N > 0$ such that*

$$\left|\partial_\xi^\alpha \partial_z^\beta f(\Pi_\gamma(z,\xi)R(z,\xi))\right| \leq C_N \langle\xi\rangle^{-|\alpha|} \sup_{\tau\in\gamma} |f(\tau)|, \tag{5.37}$$

where $(z,\xi) \in \overline{U}$ and $|\alpha| + |\beta| \leq N$.

Proof. In view of Lemma 5.8 and formula (5.36), we have

$$\left|f(\Pi_\gamma(z,\xi)R(z,\xi))\right| \leq C_0 \sup_{\tau\in\gamma} |f(\tau)|.$$

Differentiating (5.36) with respect to (z,ξ) and applying again Lemma 5.8, we can establish (5.37) for any $N \geq 1$. □

Proof of Proposition 5.7. The smoothness of $\Pi^\pm(z,\xi)$ is obvious. Therefore it suffices to show that

$$\left|\partial_\xi^\alpha \partial_z^\beta \Pi^\pm(z,\xi)\right| \langle\xi\rangle^{|\alpha|} \leq C_{\alpha\beta} \quad \text{for} \quad z \in \overline{\Omega}, \quad |\xi| \gg 1,$$

where α and β are arbitrary multi-indices. We confine ourselves to the case of the index $+$.

1) Let us prove that there are $\nu > 0$ and $\rho > 0$ such that

$$\left|\partial_\xi^\alpha \partial_z^\beta \Pi^+(z,\xi))\right| \leq C_N \langle\xi\rangle^{-|\alpha|} \quad \text{for} \quad (z,\xi) \in U_{\nu,\rho}(z^0,\omega^0), \quad |\alpha| + |\beta| \leq N, \tag{5.38}$$

where $(z^0,\omega^0) \in \overline{\Omega} \times \Sigma_{n-1}$ is an arbitrary point, and the constant $C_N > 0$ does not depend on (z^0,ω^0). To this end, we choose the path γ^+ entering (2.1_+) in the following way. According to Corollary 5.5, the roots of $P(z,\xi)$ can be represented in the form (5.16), where the functions μ_j satisfy inequality (5.17) with a constant $c > 0$. Let us denote by $\gamma_k^+ = \gamma_k^+(z,\xi)$ the boundary of the intersection of the half-plane $\operatorname{Im}\tau \geq \delta/2$ and the disk of radius $r = 2c$ centered at $|\xi|\sigma_k^0(z,\omega)$, where $\omega = \xi/|\xi|$. We note that the paths $\gamma_k^+(z,\xi)$ have no intersections for $|\xi| \gg 1$. Set

$$\gamma^+(z,\xi) = \bigcup_{k=1}^{l} \gamma_k^+(z,\xi).$$

By construction, $\gamma^+(z,\xi)$ encloses the roots of $P(z,\xi)$ lying in the half-plane $\operatorname{Im}\tau \geq \delta$. Hence, the matrix of the projection Π^+ can be written as

$$\Pi^+(z,\xi) = \sum_{k=1}^{l} \Pi_k^+(z,\xi), \tag{5.39}$$

where

$$\Pi_k^+(z,\xi) = -(2\pi i)^{-1} \oint_{\gamma_k^+} \big(P(z,\xi) - \tau I\big)^{-1} d\tau. \tag{5.40}$$

Clearly, it suffices to prove (5.38) for $\Pi_k^+(z,\xi)$.

2) We denote by $\nu > 0$ and $\rho > 0$ the constants constructed in Theorem 5.3 and fix an arbitrary point $(z^0, \omega^0) \in \overline{\Omega} \times \Sigma_{n-1}$. According to Theorem 5.3, we have

$$P(z,\xi) = C(z,\xi)\Lambda(z,\xi)C^{-1}(z,\xi), \quad (z,\xi) \in U_{\nu,\rho}(z^0,\omega^0). \tag{5.41}$$

Substituting this expression into (5.40), we derive

$$\Pi_k^+(z,\xi) = -(2\pi i)^{-1}C(z,\xi)\oint_{\gamma_k^+}\big(\Lambda(z,\xi)-\tau I\big)^{-1}d\tau\, C^{-1}(z,\xi). \tag{5.42}$$

The matrix $\Lambda(z,\xi)$ has a block-diagonal form and consists of the blocks

$$\Lambda_j(z,\xi) = |\xi|\sigma_j^0(z,\omega)I_{m_j} + \Lambda'_j(z,\xi), \quad j=1,\dots,l.$$

Consequently, the integrand in (5.42) has a similar structure and is formed of the blocks

$$\big(\Lambda'_j(z,\xi) + (|\xi|\sigma_j^0(z,\omega)-\tau)I_{m_j}\big)^{-1}, \quad j=1,\dots,l.$$

Let us substitute this expression into the path integral in (5.42). Since $\gamma_k^+(z,\xi)$ encloses only the roots lying in the neighborhood of $|\xi|\sigma_k^0(z,\omega)$, we conclude that this path integral has a block-diagonal form and that the only nonzero block is the kth block

$$\pi_k^+(z,\xi) = \oint_{\gamma_k^+}\big(\Lambda'_k(z,\xi) + (|\xi|\sigma_k^0(z,\omega)-\tau)I_{m_k}\big)^{-1}d\tau. \tag{5.43}$$

If we show that $\pi_k^+(z,\xi)$ satisfies the inequalities

$$\big|\partial_\xi^\alpha\partial_z^\beta\pi_k^+(z,\xi))\big| \le c_N\langle\xi\rangle^{-|\alpha|} \quad \text{for} \quad (z,\xi)\in U_{\nu,\rho}(z^0,\omega^0), \quad |\alpha|+|\beta|\le N, \tag{5.44}$$

then the required estimates for $\Pi_k^+(z,\xi)$ will follow from (5.42).

3) To prove (5.44), we perform the change $\tau \mapsto \tau + |\xi|\sigma_k^0(z,\omega)$ in (5.43). This results in

$$\pi_k^+(z,\xi) = \oint_\gamma\big(\Lambda'_k(z,\xi) - \tau I_{m_k}\big)^{-1}d\tau,$$

where γ is the boundary of the intersection of the half-plane $\operatorname{Im}\tau \ge \delta/2$ and the disk of radius $r = 2c$ centered at zero. Note that $\Lambda'_k(z,\xi)$ satisfies inequalities (5.13), and the distance between the characteristic roots of $\Lambda'(z,\xi)$ and γ is bounded from below by the expression $\min\{\delta/2, c\}$. The required estimates follow now from Lemma 5.9 with $f(\tau)\equiv 1$. □

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