Ergodicity for the randomly forced 2D Navier–Stokes equations

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Abstract

We study space-periodic 2D Navier–Stokes equations perturbed by an unbounded random kick-force. It is assumed that Fourier coefficients of the kicks are independent random variables all of whose moments are bounded and that the distributions of the first N_0 coefficients (where N_0 is a sufficiently large integer) have positive densities against the Lebesgue measure. We treat the equation as a random dynamical system in the space of square integrable divergence-free vector fields. We prove that this dynamical system has a unique stationary measure and study its ergodic properties.

Contents

0 Introduction

We continue our study of the randomly forced 2D space-periodic Navier–Stokes system (NS), started in [KS1, KS2]. That is, we consider the equations

$$\dot{u} - \nu\Delta u + (u, \nabla)u + \nabla p = \eta^\omega(t, x), \qquad \operatorname{div} u = 0, \tag{0.1}$$

where $x \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, $0 < \nu \le 1$ is the viscosity, $u = u(t, x)$ is the velocity field, and $p = p(t, x)$ is the pressure. Equations (0.1) are supplemented by the conditions

$$\langle u \rangle \equiv \langle \eta \rangle \equiv 0, \qquad \operatorname{div} \eta = 0.$$

The brackets $\langle\cdot\rangle$ signify the space averaging. The right-hand side η^ω is a random process with range in the functional space

$$H = \{u \in L^2(\mathbb{T}^2, \mathbb{R}^2) : \operatorname{div} u = 0,\ \langle u \rangle = 0\},$$

and Equations (0.1) defines a random dynamical system in H. We provide H with the usual orthonormal basis $\{e_1, e_2, \dots\}$ formed by the trigonometric vector fields $C_s\binom{-s_2}{s_1}\sin(s \cdot x)$ and $C_s\binom{-s_2}{s_1}\cos(s \cdot x)$, $s \in \mathbb{Z}^2 \setminus \{0\}$. The e_j's are eigenvectors of the Laplacian, $-\Delta e_j = \alpha_j e_j$. We assume that the eigenvalues α_j are indexed in non-decreasing order.

In [KS1], we consider the NS equations forced by a bounded random kick-force

$$\eta^\omega = \sum_{k\in\mathbb{Z}} \delta(t - kT)\eta_k(x), \quad \eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j(x), \tag{0.2}$$

where $b_j \ge 0$ are some constants such that

$$b^2 := b_1^2 + b_2^2 + \cdots < \infty,$$

and $\{\xi_{jk}\}$ are independent random variables. It is assumed in [KS1] that the distribution $\mathcal{D}(\xi_{jk})$ of the random variable ξ_{jk} is k-independent and has the form

$$\mathcal{D}(\xi_{jk}) = p_j(r)dr \quad \text{for} \quad j \ge 1, \quad k \in \mathbb{Z}, \tag{0.3}$$

where p_j's are Lipschitz continuous functions such that $p_j(0) > 0$ and $\operatorname{supp} p_j \subset [-1, 1]$.

Let $\{S_t,\, t \geq 0\}$ be flow-maps of the free NS equation (0.1) with $\eta \equiv 0$. If $u(t, x)$ is a solution for (0.1) with a kick-force (0.2) normalised to be a continuous from the right curve in H, then for any integer k and for $t \in [Tk, T(k+1)]$ we have (see Figure 1 below)

$$u(t) = \begin{cases} S_{t-Tk}(u(Tk)), & t < T(k+1) \\ S_T(u(Tk)) + \eta_k, & t = T(k+1). \end{cases} \tag{0.4}$$

Accordingly, long-time behaviour of solutions for (0.1), (0.2) is described by long-time behaviour of solutions for the following random dynamical system with discrete time:

$$u_k = S(u_{k-1}) + \eta_k, \tag{0.5}$$

where $S = S_T$ and $u_k = u(Tk, \cdot) \in H$.

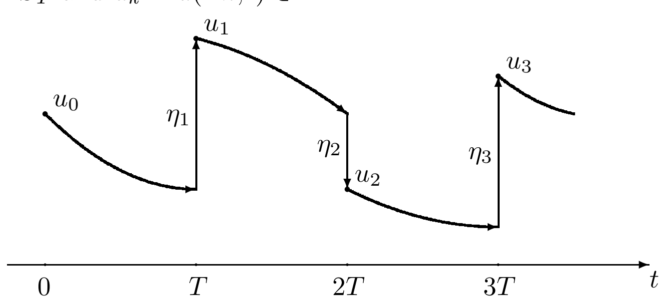

Figure 1: Evolution defined by (0.1), (0.2)

In [KS1], we show that if relations (0.3) hold with densities p_j as above and

$$b_j \neq 0 \quad \text{for} \quad 1 \leq j \leq N_0 \tag{0.6}$$

for some finite $N_0 = N_0(\nu, b) \geq 1$, then the random dynamical system (0.5) has in H a unique stationary measure λ. Moreover, if $(u_k,\, k \geq 0)$ satisfies (0.5) for $k > 0$ and $u_0 = u$, then

$$\mathcal{D}(u_k) \rightharpoonup \lambda \quad \text{as} \quad k \to \infty \tag{0.7}$$

for any choice of the initial vector $u \in H$.[1]

We note that if $b_j \neq 0$ for all $j \geq 1$ and $\sum b_j^2 < \infty$, then these results apply to Equation (0.1) with any $\nu > 0$, any $T > 0$ and with arbitrarily large kick-force η as above. See the Introduction to [KS1] for discussion of this result and see [G, KS1] for its relations with statistical hydrodynamics.

[1]In [KS1], we study in fact the system (0.3) restricted to the domain of attainability from zero $\mathcal{A}$, which is a compact subset of H, invariant for (0.3), and prove that the restricted system has a unique stationary measure and satisfies (0.7) for $u \in \mathcal{A}$. In the short paper [KS2] we show that this measure is a unique stationary measure for the system in the whole space H and prove that (0.7) holds for any $u \in H$.

Next, E, Mattingly, Sinai [EMS] and Bricmont, Kupiainen, Lefevere [BKL] considered the 2D NS equations perturbed by a white noise force

$$\eta = \sum_{j=1}^{\infty} b_j \dot{w}_j(t) e_j(x),$$

where $w_1, w_2, \dots$ are independent standard Brownian motions. Under the assumption that $b_j \neq 0$ for $1 \leq j \leq N_0(\nu)$ and $b_j = 0$ for $j > N$ with some $\infty > N \geq N_0(\nu)$, they obtained results similar to those reviewed above. We do not discuss these results here, but we mention that, as it is shown in [BKL], for almost all initial functions $u(0,x)$ the distribution of a solution converges to the stationary measure exponentially fast.

In this work we study the NS equation with unbounded random kick-forces. That is, we consider Equations (0.1), (0.2), where the independent random variables ξ_{jk} have k-independent distributions as in (0.3), the densities p_j are absolutely continuous and everywhere positive,

$$\int_{-\infty}^{\infty} \left| \frac{\partial p_j(r)}{\partial r} \right| dr < \infty \quad \text{for all} \quad j \geq 1; \quad p_j(r) > 0 \quad \text{for all} \quad j \geq 1, \quad r \in \mathbb{R}, \tag{0.8}$$

and decay at infinity faster than any negative degree of r. We consider in fact the following two extreme cases which are allowed by our techniques:

(A) (*finite moments*) the densities p_j satisfy (0.8) and

$$\int_{-\infty}^{\infty} |r|^m p_j(r)\, dr \leq C_m \quad \text{for all} \quad m \geq 1, \quad j \geq 1, \tag{0.9}$$

with some fixed constants C_m, $m \geq 1$.

(B) (*finite second exponential moments*) the densities p_j satisfy (0.8) and

$$\int_{-\infty}^{\infty} e^{\varkappa_0 r^2} p_j(r) dr \leq C_0 \quad \text{for any} \quad j \geq 1,$$

with some fixed positive constants $\varkappa_0$ and C_0;

We stress that in (A) and (B) it is not assumed that $\int p_j dr = 0$.

For any $s > 0$, we denote $H^s = H \cap H^s(\mathbb{T}^2; \mathbb{R}^2)$, where $H^s(\mathbb{T}^2; \mathbb{R}^2)$ is the Sobolev space of order s with the corresponding norm $\|\cdot\|_s$.

Main Theorem. *Let us assume that condition* (A) *is satisfied and*

$$\sum_{j=1}^{\infty} b_j^2 \alpha_j^s < \infty \quad \textit{for some} \quad s > 0.$$

Then there is an integer $N_0 < \infty$ with the following property: if (0.6) *holds, then the random dynamical system* (0.5) *has a unique stationary measure λ such that*

$$\int_H |u|^m \lambda(du) < \infty \quad \textit{for all} \quad m \geq 1.$$

Moreover, the following assertions hold:

a) $\lambda(H^s) = 1$;

b) *if* $(u_k, k \geq 0)$ *is a solution of Equation* (0.5) *with a deterministic initial function* $u_0 = u$*, then, for* λ*-almost all* u*, convergence* (0.7) *holds. Moreover,*

$$\mathbb{E}f(u_k) \to (\lambda, f) \quad \textit{as} \quad k \to \infty, \tag{0.10}$$

where f *is any continuous function on* H^s *such that* $|f(u)| \leq C_1 + C_2\|u\|_s^p$ *for some finite constants* C_1*,* C_2*, and* p*.*

c) *if* $b_j \neq 0$ *for all* j*, then* $\operatorname{supp}\lambda = H$*, and convergence* (0.10) *holds uniformly in* $u \in H^s$*,* $\|u\|_s \leq R$*, for any* $R > 0$*.*

Finally, if condition (B) *is also satisfied, then* $\int_H e^{\beta|u|^2}\lambda(du) < \infty$ *for some* $\beta > 0$*, and convergence* (0.10) *holds for* λ*-almost all* $u \in H$ *and any function* $f \in C(H^s)$ *such that* $|f(u)| \leq C\exp(\sigma\|u\|_s^\kappa)$*, where the positive constants* σ *and* κ *are sufficiently small.*

If $s > 1$, then the delta-function is a continuous functional on the space H^s. Accordingly, if $s > 1$ and $u(t,x)$ is a solution for (0.1) such that $u(0,x) = u_0 \in H$, then, for λ-almost all $u_0 \in H$, the correlation tensor of the solution $\mathbb{E}u^i(k,x)u^j(k,y)$ converges as $k \to \infty$ to the correlation tensor of the measure λ, equal to $\int u^i(x)u^j(y)\lambda(du)$. If u_0 is an arbitrary vector in H, then in this statement the convergence should be replaced by the Cesàro convergence.

The proof of the Main Theorem remains true if condition (A) is replaced by the following weaker assumption with $M \geq 20$:

($\mathbf{A}_M$) the densities p_j satisfy (0.8), and (0.9) holds for $m \leq M$ and all $j \geq 1$.

In this case, the stationary measure λ has $M' \leq M$ finite moments, where M' goes to infinity with M, and (0.10) holds for any continuous functional $f\colon H \to \mathbb{R}$ satisfying the inequality $|f(u)| \leq C_1 + C_2\|u\|_s^{M'}$.

The proof of the Main Theorem, which occupies Sections 1 – 5, follows the scheme developed in [KS1] to work with bounded kick-forces. It is based on a Foiaş–Prodi type reduction of (0.5) to a finite-dimensional abstract Gibbs system which has a unique stationary solution due to a version of the Ruelle–Perron–Frobenius theorem.

In fact, the Main Theorem can be strengthened as follows:

Amplification. *Under the assumption of the above theorem, convergence* (0.10) *holds for any* $u \in H$*, uniformly on bounded subsets of* H*.*

This result can be derived from the Main Theorem (and some intermediate assertions), using the methods of [KS2]. Since the corresponding arguments differ from those used in this work, we shall present them in another publication.

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Notation

We denote by $\mathbb{Z}$ be the set of all integers and by $\mathbb{Z}_0$ be the set of non-positive integers.

Let $\mathbf{X}$ be a topological space. We shall use the following notation.

$[B]_{\mathbf{X}}$ is the closure in the space $\mathbf{X}$ of its subset B.
$B_{\mathbf{X}}(\boldsymbol{x}, r)$ is a closed ball in $\mathbf{X}$ of radius r centred at $\boldsymbol{x} \in \mathbf{X}$.
$\mathcal{B}(\mathbf{X})$ is the σ-algebra of Borel subsets of $\mathbf{X}$.
$\mathcal{P}(\mathbf{X})$ is the set of probability measures on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$.
$\mathbf{C}(\mathbf{X})$ is the space of real-valued continuous functions on $\mathbf{X}$.
$\mathbf{C}_b(\mathbf{X})$ is the space of bounded functions $f \in \mathbf{C}(\mathbf{X})$. It is endowed with the supremum-norm $\|f\|_\infty$.
$L^1(\mathbf{X}, \mu)$ is the space of Borel functions on $\mathbf{X}$ with finite norm

$$\|f\|_\mu := \int_{\mathbf{X}} \big|f(\boldsymbol{x})\big| \, d\mu(\boldsymbol{x}).$$

The integral of a function $f(\boldsymbol{x})$ over the space $\mathbf{X}$ with respect to a measure μ will sometimes be denoted by (μ, f):

$$(\mu, f) = \int_{\mathbf{X}} f(\boldsymbol{x}) \, d\mu(\boldsymbol{x}) = \int_{\mathbf{X}} f \, d\mu.$$

$\mathcal{D}(\xi)$ is the distribution of a random variable ξ.
$a \vee b$ $(a \wedge b)$ is the maximum (minimum) of real numbers a and b.
We denote by C_i, $i = 1, 2, \ldots$, unessential positive constants.

1 Preliminaries: equations, estimates and the Markov chain

1.1 Description of the class of problems in question

Let us consider the Navier–Stokes (NS) system (0.1). Applying the L^2-orthogonal projection Π onto the space H of divergence-free vector fields with zero mean value (see the Introduction), we can write this system as

$$\dot{u} + \nu L u + B(u, u) = \eta(t, x), \quad x \in \mathbb{T}^2, \quad 0 < \nu \leq 1, \tag{1.1}$$

(for instance, see [CF]). Here $u(t)$ is a two-dimensional vector field with values in the functional space H. The operators L and B have the form

$$Lu = -\Delta u, \quad B(u, v) = \Pi(u, \nabla)v.$$

It is assumed that the right-hand side of (1.1) is a kick-force as in the Introduction. To simplify notations, we assume that $T = 1$. Then η takes the form

$$\eta(t, x) = \sum_{k=-\infty}^{+\infty} \delta(t - k)\eta_k(x), \tag{1.2}$$

where $\delta(\cdot)$ is the Dirac measure and $\eta_k, k \in \mathbb{Z}$, is a sequence of i.i.d. random variables with range in H. We note that if $g(t)\colon \mathbb{R} \to H$ is a continuous function with compact support, then

$$\int_{-\infty}^{+\infty} \langle \eta(t), g(t) \rangle dt = \sum_{k=-\infty}^{+\infty} \langle \eta_k, g(k) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in H.

We now turn to a description of the sequence $\{\eta_k\}$. Let $\alpha_1 \le \alpha_2 \le \cdots$ be eigenvalues of the positive self-adjoint operator L acting in H and let $e_j(x)$, $j \ge 1$, be the corresponding eigenfunctions as in the Introduction. We shall assume that the random vector η_k has the form

$$\eta_k(x) = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j(x), \tag{1.3}$$

where $\{\xi_{jk}\}$ is a family of independent scalar random variables satisfying condition (A) (see the Introduction), and $\{b_j\}$ is a sequence of real numbers such that

$$\sum_{j=1}^{\infty} b_j^2 \alpha_j^s < \infty, \quad s \ge 0. \tag{1.4}$$

In what follows, we always assume that inequality (1.4) and condition (A) are satisfied. In particular, it follows that

$$\mathbb{E}\,\|\eta_k\|_s^m < \infty \quad \text{for any} \quad m \ge 1, \tag{1.5}$$

where $\|\cdot\|_s$ stands for the sth Sobolev norm:

$$\|u\|_s = \left(\sum_{j=1}^{\infty} \alpha_j^s |u_j|^2 \right)^{1/2}.$$

Moreover, if $\{\xi_{jk}\}$ satisfies also condition (B), then

$$\mathbb{E} \exp\bigl(a\|\eta_k\|_s^2\bigr) < \infty$$

for any constant $a > 0$ such that

$$a b_j^2 \alpha_j^s \le \varkappa_0 \quad \text{for all} \quad j \ge 1.$$

To simplify notation, we shall write $|u|$ and $\|u\|$ instead of $\|u\|_0$ and $\|u\|_1$, respectively. In what follows, the constants b_j are assumed to be fixed, and we shall not specify dependence of different parameters on them.

We now define the notion of a solution for Equation (1.1). For any $s \ge 0$ we introduce the space $H^s = H \cap H^s(\mathbb{T}^2, \mathbb{R}^2)$ endowed with the norm $\|\cdot\|_s$. We note that the operator $\sqrt{L}$ defines an isomorphism $H^s \to H^{s-1}$, $s \ge 1$.

Let $I \subset \mathbb{R}$ be an open interval (which can be of infinite length).

Definition 1.1. A mapping $u(t)\colon I \to H$ is called a *regular curve* if it belongs to $L^1_{\mathrm{loc}}(I, H^1)$ and is continuous at non-integer points of I while at integer points it is continuous from the right and has a limit from the left.

For a Banach space X, let $C^1_0(I, X)$ be the set of continuously differentiable functions $f(t)\colon I \to X$ with compact support.

Definition 1.2. A regular curve $u(t) : I \to H$ is called a *solution of Equation* (1.1) with a deterministic force of the form (1.2) if the left- and right-hand sides of (1.1) coincide as linear functionals on the space $C^1_0(I, H^1)$. That is,

$$\int_I \big(-\langle u, \dot v\rangle + \nu\langle \sqrt{L}u, \sqrt{L}v\rangle + \langle B(u,u), v\rangle\big)\, dt = \int_I \langle \eta, v\rangle dt = \sum_{k\in\mathbb{Z}\cap I} \langle \eta_k, v(k)\rangle \tag{1.6}$$

for any $v \in C^1_0(I, H^1)$.

A random process $u = u^\omega(t)$, $t \in I$, with range in H is called a *solution of Equation* (1.1) with a random force of the form (1.2), (1.3) if for almost all ω the mapping $u^\omega(t)\colon I \to H$ is a regular curve satisfying (1.1).

We note that if $u(t,x)$ is a solution of Equation (1.1), then, due to (1.6), we have

$$u(k,x) - u(k-0,x) = \eta_k(x) \quad \text{for any integer} \quad k \in I, \tag{1.7}$$

while on any interval not containing integer points the function $u(t,x)$ satisfies the free Navier–Stokes equations

$$\dot u + \nu L u + B(u,u) = 0, \quad u(t) \in H. \tag{1.8}$$

In particular, (0.4) holds with $T = 1$.

1.2 Cauchy problem and a priori estimates

We now consider the Cauchy problem for Equation (1.1):

$$u(0,x) = u^0(x), \tag{1.9}$$

where $u^0(x)$ is a random variable in H. We shall assume that it is independent of $\eta_1, \eta_2, \dots$ and that all of its moments are finite:

$$\mathbb{E}\,|u^0|^m < \infty \quad \text{for any} \quad m \ge 1. \tag{1.10}$$

We have the following theorem on the correctness of the Cauchy problem:

Theorem 1.3. *Assume that* (1.10) *is satisfied. Then the problem* (1.1), (1.9) *has a unique solution defined for* $t \ge 0$. *Moreover, for any* $m \ge 1$ *we have the estimate*

$$\mathbb{E}\,|u(k)|^m \le q^k \mathbb{E}\,|u^0|^m + C(m)\nu^{-(m-1)} d_\nu(k) \mathbb{E}\,|\eta_k|^m, \quad k \ge 1, \tag{1.11}$$

where $0 < \nu \le 1$, $q = e^{-\nu\alpha_1}$, $C(m) > 0$ *is a constant not depending on* u^0, k, *and* ν, *and*

$$d_\nu(k) = 1 + q + \cdots + q^{k-1} \le \alpha_1^{-1} e^{\alpha_1} \nu^{-1}.$$

Finally, if (1.4) *holds for some* $s > 0$ *and* $l = l(s) \ge 1$ *is the smallest integer no less than* s, *then there is a constant* $C(l,m) > 0$ *such that*

$$\mathbb{E}\,\|u(k)\|_s^m \le C(l,m) \begin{cases} \nu^{-m/2}\mathbb{E}\,|u_{k-1}|^m + \mathbb{E}\,\|\eta_k\|_s^m, & l = 1, \\ 1 + \nu^{-5lm/2}\mathbb{E}\,|u_{k-1}|^{m_l} + \mathbb{E}\,\|\eta_k\|_s^m, & l \ge 2, \end{cases} \tag{1.12}$$

where $k \ge 1$ *and* $m_l = m(2l+1)$.

In case the random variables u^0 and η_k have finite second exponential moments, stronger estimates for the solutions hold:

Theorem 1.4. *Suppose that the random variables* ξ_{jk} *satisfy condition* (B) *and there is* $\rho > 0$ *such that*

$$\mathbb{E}\exp\bigl(\rho\nu|u^0|^2\bigr) < \infty. \tag{1.13}$$

Then the solution of the problem (1.1), (1.9) *constructed in Theorem 1.3 satisfies the inequality*

$$\mathbb{E}\exp\bigl(\sigma_0\nu|u(k)|^2\bigr) \le d(k)\Bigl(\mathbb{E}\exp\bigl(\sigma_0\nu|u^0|^2\bigr)\Bigr)^{q^k}, \quad k \ge 1, \tag{1.14}$$

where $0 < \nu \le 1$, $q = e^{-\alpha_1\nu}$, $\sigma_0 = \rho \wedge (a\alpha_1 e^{-\alpha_1})$, *and*

$$d(k) = \Bigl(\mathbb{E}\exp(a|\eta_k|^2)\Bigr)^{1+q+\cdots+q^{k-1}} \le \Bigl(\mathbb{E}\exp(a|\eta_k|^2)\Bigr)^{\frac{1}{1-q}}.$$

Moreover, if (1.4) *holds for some* $s > 0$ *and* l *is the smallest integer no less than* s, *then there are positive constants* C_l *and* σ_l, *depending only on* σ_0 *and* l, *such that*

$$\mathbb{E}\exp\bigl(\sigma_l\nu^{p_l}\|u(k)\|_s^{2\kappa_l}\bigr) \le C_l\mathbb{E}\exp(a\|\eta_k\|_s^2)\,\mathbb{E}\exp\bigl(\sigma_0\nu|u(k-1)|^2\bigr), \quad k \ge 1. \tag{1.15}$$

Here $\kappa_1 = 1$, $p_1 = 2$, *and*

$$\kappa_l = \frac{1}{2l+1}, \quad p_l = \frac{7l+1}{2l+1} \quad \text{for} \quad l \ge 2.$$

The proof of Theorems 1.3 and 1.4 is carried out by standard methods and is given in the Appendix (see Section 6).

We shall also need some estimates for the rate of growth and for the mean value of solutions (and of the right-hand side of the equation).

For any sequence of non-negative numbers a_k and arbitrary integers $m \le n$, we set

$$\langle a_k\rangle_m^n = \frac{1}{n-m+1}\sum_{k=m}^{n} a_k.$$

In the case $m > n$, we set $\langle a_k\rangle_m^n = \langle a_k\rangle_n^m$.

Proposition 1.5. *Let $k_- \le k_0 \le k_+$ be some integers, where k_+ (k_-) can take the value $+\infty$ ($-\infty$), and let $u(t,x)$ be a solution of* (1.1) *that is defined for $k_- \le t \le k_+$ and satisfies the inequality*

$$\sup_{k_- \le k \le k_+} \mathbb{E}\,|u(k)|^m \le N_m \nu^{-m} \quad \textit{for} \quad 0<\nu\le 1, \quad m\ge 1, \tag{1.16}$$

where the constants $N_m > 0$ do not depend on ν. Then there is a constant $M > 1$, not depending on N_m and ν, and a non-negative random variable $T_\nu(\omega) \in \mathbb{Z}$ such that

$$\big\langle |u(k)|^2 + \|\eta_k\|_s^2 \big\rangle_{k_0}^T \le M\nu^{-2} \quad \textit{for} \quad k_- \le T \le k_+, \quad |T-k_0| \ge T_\nu(\omega). \tag{1.17}$$

Moreover, for any $m > 1$ there is a constant $C_m > 0$ such that

$$\mathbb{E}\,T_\nu^m \le C_m\big(N_{2m} + \mathbb{E}|\eta_1|^{4(m+2)}\big)\nu^{-m} \quad \textit{for} \quad 0<\nu\le 1. \tag{1.18}$$

In this proposition and everywhere below, we assume that $k < k_+$ if $k_+ = +\infty$ and $k > k_-$ if $k_- = -\infty$.

Remark 1.6. If in Proposition 1.5 we assume that condition (B) is also satisfied and replace inequality (1.16) by the stronger estimate

$$\sup_{k_- \le k \le k_+} \mathbb{E}\,e^{\sigma_0 \nu |u(k)|^2} \le N_0 \quad \text{for} \quad 0<\nu\le 1, \tag{1.19}$$

then (1.17) holds with a constant $M > 1$ (depending on σ_0 solely) and an integer-valued non-negative random variable $T_\nu(\omega) \in \mathbb{Z}$ such that

$$\mathbb{E}\,e^{\sigma T_\nu} \le C_0 \quad \text{for} \quad 0<\nu\le 1,$$

where the positive constants C_0 and σ depend only on N_0 and σ_0, respectively. Proof of this assertion follows the same scheme as that of Proposition 1.5, and we shall not dwell on it.

We also note that, due to Theorems 1.3 and 1.4, Proposition 1.5 and its modification above apply to any solution of the problem (1.1), (1.9), where the random variable u^0 satisfies condition (1.10) or (1.13).

Proof of Proposition 1.5. To simplify notation, we confine ourselves to the case when $k_0 = k_- = 0$ and $k_+ = +\infty$. Moreover, we shall only show that

$$\big\langle |u(k)|^2 \big\rangle_0^T \le M\nu^{-2} \quad \text{for} \quad T \ge T_\nu(\omega).$$

It will be clear from the proof that the same arguments apply in the general case.

1) We first note that

$$|u(T)|^2 + 2\nu \sum_{k=1}^{T} \int_{k-1}^{k} \|u(t)\|^2 dt = |u(0)|^2 + \sum_{k=1}^{T} \big(|\eta_k|^2 + 2\langle \eta_k, u(k-0)\rangle\big), \tag{1.20}$$

where $T \geq 1$ is an arbitrary integer. Indeed, since on any open interval $(k-1,k)$ the solution $u(t,x)$ satisfies the free NS equations (1.8), we have (see (6.3) with $l=0$)

$$|u(k-0)|^2 - |u(k-1)|^2 + 2\nu \int_{k-1}^{k} \|u(t)\|^2 dt = 0. \tag{1.21}$$

Besides, relation (1.7) implies that

$$|u(k)|^2 = |u(k-0)|^2 + |\eta_k|^2 + 2\langle \eta_k, u(k-0)\rangle. \tag{1.22}$$

Combining (1.21) and (1.22), we derive

$$2\nu \int_{k-1}^{k} \|u(t)\|^2 dt = |u(k-1)|^2 - |u(k)|^2 + |\eta_k|^2 + 2\langle \eta_k, u(k-0)\rangle.$$

Taking the sum over $k = 1, \dots, T$, we obtain (1.20).

2) We now recall that (see [CF])

$$|S_t(v)| \leq e^{-\nu\alpha_1 t}|v|, \quad t \geq 0, \tag{1.23}$$

where $\alpha_1 > 0$ is the first eigenvalue of L. It follows from (1.23) that

$$\begin{aligned} 2|\langle \eta_k, u(k-0)\rangle| &\leq (\nu\alpha_1)^{-1}|\eta_k|^2 + \nu\alpha_1|u(k-0)|^2 \\ &\leq (\nu\alpha_1)^{-1}|\eta_k|^2 + \nu \int_{k-1}^{k} \|u(t)\|^2 dt. \end{aligned}$$

Substitution of this inequality into (1.20) results in

$$|u(T)|^2 + \nu \sum_{k=1}^{T} \int_{k-1}^{k} \|u(t)\|^2 dt \leq |u(0)|^2 + (1 + \alpha_1^{-1}\nu^{-1}) \sum_{k=1}^{T} |\eta_k|^2. \tag{1.24}$$

Now note that, by (1.21) and (1.23), we have

$$\nu \int_{k-1}^{k} \|u(t)\|^2 dt \geq \frac{1 - e^{-2\alpha_1\nu}}{2} |u(k-1)|^2.$$

Combining this with (1.24), we derive

$$\sum_{k=0}^{T} |u(k)|^2 \leq c\nu^{-2}\bigl(\nu|u(0)|^2 + T\,\mathbb{E}\,|\eta_1|^2 + \Sigma(T)\bigr), \tag{1.25}$$

where $c = c(\alpha_1) > 0$ is a constant and

$$\Sigma(T) = \sum_{k=1}^{T} \bigl(|\eta_k|^2 - \mathbb{E}\,|\eta_k|^2\bigr).$$

Direct verification shows that, for any integer $p \geq 1$,

$$\mathbb{E}\,\bigl|\Sigma(T)\bigr|^{2p} \leq c_p\,(\mathbb{E}|\eta_1|^{4p})\,T^p, \tag{1.26}$$

where $c_p > 0$ is a constant depending only on p. We now set

$$t(\omega) = \min\Big\{t \in \mathbb{Z}_+ : \Sigma(t') \le t' \text{ for } t' \ge t\Big\}.$$

Using (1.26) and applying the Chebyshev inequality, we derive

$$\begin{aligned}\mathbb{E}\, t^m &= \sum_{j=1}^{\infty} \mathbb{P}\{t = j\} j^m \le \sum_{j=1}^{\infty} \mathbb{P}\Big\{\big|\Sigma(j-1)\big|^{2p} > j^{2p}\Big\} j^m \\ &\le c_p\, \mathbb{E}|\eta_1|^{4p} \sum_{j=1}^{\infty} j^{m-p} \le 2c_p\, \mathbb{E}|\eta_1|^{4p}.\end{aligned}$$

where $p = m+2$. Taking into account (1.25), we conclude that

$$\big\langle |u(k)|^2 \big\rangle_0^T \le c\big(\mathbb{E}\,|\eta_1|^2 + 1\big)\nu^{-2} \quad \text{for} \quad T \ge T_\nu(\omega),$$

where $T_\nu(\omega) = t(\omega) \vee \big(\nu |u(0)|^2\big)$. This completes the proof of inequality (1.17) in which $M = c\big(\mathbb{E}\,|\eta_1|^2 + 1\big)$. ☐

1.3 Markov chain

We recall that S_t denotes the semigroup generated by the free NS system (1.8). Consider a solution $u(t,x)$ of the problem (1.1), (1.9) and set $u_k = u(k,x)$, $k \ge 0$. Due to (0.4), we have

$$u_0 = u^0, \tag{1.27}$$

$$u_k = S(u_{k-1}) + \eta_k, \tag{1.28}$$

where $S = S_1$ and $k \ge 1$. Clearly, Equation (1.28) defines a random dynamical system (RDS) in H. Since the random variables η_k and u^0 are independent, the set of solutions corresponding to all $u^0 \in H$ is a family of Markov chains with the transition function

$$P(k, u^0, \Gamma) = \mathbb{P}\big\{u_k \in \Gamma\big\}, \quad u^0 \in H, \quad \Gamma \in \mathcal{B}(H).$$

Denote by

$$P_k : C_b(H) \to C_b(H), \quad P_k^* : \mathcal{P}(H) \to \mathcal{P}(H)$$

the Markov operators corresponding to $P(k, u^0, \Gamma)$.[2] It follows from Theorem 1.3 that if condition (1.4) is satisfied for some $s \ge 0$, then $P_1^* \mu(H^s) = 1$ for any $\mu \in \mathcal{P}(H)$. In particular, when μ is the delta-measure concentrated at u^0, we obtain

$$P(k, u^0, H^s) = 1 \quad \text{for any} \quad k \ge 1. \tag{1.29}$$

In what follows, we shall need some properties of the operators P_k and P_k^*. The following two lemmas show that P_k can be extended to a broader class of functionals.

[2]Since the map $S : H \to H$ is continuous, for any $f \in C_b(H)$ the function $P_k f(u) = \int_H P(k, u, dv) f(v)$ is continuous in u. Hence, P_k maps the space $C_b(H)$ into itself.

Lemma 1.7. *Suppose that condition* (1.4) *holds for some* $s > 0$. *Then* P_k *can be extended to a continuous operator from* $C_b(H^s)$ *to* $C_b(H)$ *whose norm is equal to* 1.

Proof. It suffices to consider the case $k = 1$. Let $f \in C_b(H^s)$. In view of Theorem 1.3, for any initial function $u^0 \in H$ the solution $u_1 = u(1, x)$ belongs to H^s with probability 1, so that the random variable $f(u_1)$ is well-defined. Moreover, since the operator $S: H \to H^s$ is continuous (see Lemma 6.1), we conclude from (1.28) that u_1 continuously depends (in H^s-norm) on $u^0 \in H$ for all ω. Therefore the function $f(u_1)$ is also continuous. The continuity of the function $P_1 f(u^0) = \mathbb{E} f(u_1)$ follows now from the Lebesgue theorem on dominated convergence. It remains to note that if $|f(u)| \le 1$ for all $u \in H$, then $|\mathbb{E} f(u_1)| \le 1$, that is, the norm of the operator $P_1 : C_b(H^s) \to C_b(H)$ does not exceed 1. □

We now show that the operators P_k can be continued to a class of functionals growing at infinity. For any increasing positive function $\beta(r)$, $r \ge 0$, we denote by $C(H^s; \beta)$ the space of continuous functions $f(u)\colon H^s \to \mathbb{R}$ such that

$$|f(u)| \le \text{const}\, \beta\big(\|u\|_s\big), \quad u \in H^s.$$

It is clear that $C(H^s; \beta)$ is a Banach space with respect to the norm

$$\|f\|_{s,\beta} := \sup_{u \in H^s} \Big\{ |f(u)| / \beta\big(\|u\|_s\big) \Big\}.$$

We recall that the integer $l = l(s) \ge 1$ and the constants m_l $(l \ge 2)$, κ_l, σ_l, and p_l are defined in Theorems 1.3 and 1.4, and set $m_l = m$ for $l = 0, 1$.

Lemma 1.8. *Under the conditions of Theorem 1.3, for any* $m > 1$ *and* m', $1 \le m' < m$, *the operator* P_k *can be extended to a continuous map from* $C(H^s, \beta_{m'})$ *to* $C(H; \beta_{m_l})$, *where* $\beta_d(r) = 1 + r^d$. *Moreover, for any* ν, $0 < \nu \le 1$, *the norms of the extended operators are bounded uniformly in* $k \ge 1$.

Remark 1.9. Under the assumptions of Theorem 1.4, the operator P_k extends to a bounded map from $C(H^s; \gamma)$ to $C(H; \gamma')$. Here $\gamma(r) = \exp(cr^{2\kappa_l})$ and $\gamma'(r) = \exp(c' r^2)$, where κ_l is defined in Theorem 1.4 and c and c' are some positive constants that can be easily recovered from Theorem 1.4.

Proof. The proofs of all assertions are similar, and to simplify notation, we confine ourselves to the case $s = 0$. Let $f \in C(H; \beta_{m'})$ and let $h_R(r)$ be a continuous function equal to 1 and 0 for $r \le R$ and $r \ge R + 1$, respectively. Obviously, the function $f_R(u) = h_R(|u|) f(u)$ belongs to $C_b(H)$. It follows from Lemma 1.7 and inequality (1.11) that for any $R_2 > R_1 \gg 1$ we have

$$\begin{aligned} \big|P_k f_{R_1}(u) - P_k f_{R_2}(u)\big| &\le \left| \int_H \big(h_{R_2}(|v|) - h_{R_1}(|v|)\big) f(v) \mu_k(dv) \right| \\ &\le \text{const} \int\limits_{R_1 \le |v| \le R_2 + 1} \big(1 + |v|\big)^{m'} \mu_k(dv) \qquad (1.30) \\ &\le \text{const}\, (1 + R_1)^{m' - m}, \qquad (1.31) \end{aligned}$$

where $\mu_k = P(k, u, \cdot)$. We note that inequality (1.31) holds uniformly in u from bounded subsets of H. Letting R_1 to go to infinity, we conclude that there is a limit

$$\lim_{R\to\infty} P_k f_R(u) =: P_k f(u),$$

and the limiting function $P_k f$ is continuous in $u \in H$. Moreover, it follows from (1.11) that

$$\begin{aligned}\left|P_k f(u)\right| &\le \left|\int_H f(v)\mu_k(dv)\right| \le \|f\|_{0,\beta_{m'}} \int_H \bigl(1+|v|\bigr)^{m'} \mu_k(dv)\\ &\le \text{const}\, \nu^{-m} \|f\|_{0,\beta_{m'}} \bigl(1+|u|\bigr)^m.\end{aligned}$$

This completes the proof in the case $s = 0$. ☐

We now turn to the problem of existence of a stationary measure.

Definition 1.10. A probability measure $\lambda \in \mathcal{P}(H)$ is said to be *stationary for Equation* (1.1) if $P_1^*\lambda = \lambda$.

We recall that the support $\operatorname{supp}\mu$ of a measure μ is defined as the minimal closed set of full measure and that $\mathcal{D}(\xi)$ denotes the distribution of a random variable ξ.

Proposition 1.11. *Suppose that condition* (1.4) *is satisfied for some* $s > 0$. *Then there is a stationary measure* $\lambda \in \mathcal{P}(H)$ *such that* $\lambda(H^s) = 1$ *and*

$$\int_H \|u\|_r^m \lambda(du) \le C(l, m) \begin{cases} \nu^{-m} & \text{for} \quad r = 0, \\ \nu^{-3m/2} & \text{for} \quad 0 < r \le 1, \\ \nu^{-(5l+2)m/2} & \text{for} \quad 1 < r \le s, \end{cases} \tag{1.32}$$

where $m \ge 1$, $0 < \nu \le 1$, $l = l(r)$ *is the smallest integer no less than* r, *and* $C(l, m)$ *is a constant not depending on* ν. *Moreover, there is a stationary Markov chain* $(u_k, k \in \mathbb{Z})$ *satisfying* (1.28) *for all* $k \in \mathbb{Z}$ *such that* $\mathcal{D}(u_k) = \lambda$. *Finally, if all the constants* b_j *in* (1.3) *are non-zero and* $\lambda_0 \in \mathcal{P}(H)$ *is an arbitrary stationary measure for* P_k^*, *then* $\operatorname{supp}\lambda_0 = H$.

Proof. The existence of a stationary measure and inequality (1.32) can easily be proved by the Bogolyubov–Krylov argument using Theorem 1.3 and the Prokhorov theorem on the weak compactness of a tight family of measures (cf. [DZ]). The fact that $\lambda(H^s) = 1$ follows immediately from (1.29) and the Chapman–Kolmogorov relation

$$\lambda(\Gamma) = \int_H P(1, u, \Gamma)\lambda(du), \quad u \in H, \quad \Gamma \in \mathcal{B}(H). \tag{1.33}$$

The existence of a stationary solution of (1.28) with distribution λ follows from the Prokhorov and Skorokhod theorems. (For the proof of this assertion in the case when the support of the distribution of η_k is a bounded subset in H, see [KS1, Section 1.2].)

To prove the last assertion of the theorem, we note that if γ is the distribution of the random variable η_k defined by the formula (1.3) in which all b_j are non-zero, then $\gamma(U) > 0$ for any open set $U \subset H$ (see Lemma 6.2 in the Appendix). It follows that $P(1, u, U) > 0$. Setting $\Gamma = U$ and $\lambda = \lambda_0$ in (1.33), we conclude that $\lambda(U) > 0$ for any open set U. □

Combining Propositions 1.5 and 1.11, we obtain the following assertion.

Proposition 1.12. *Suppose that* (1.4) *holds for some* $s > 0$. *Let* $\lambda_0 \in \mathcal{P}(H)$ *be a stationary measure for* P_k^* *that satisfies the condition*

$$\int_H |u|^m \lambda_0(du) \le N_m \nu^{-m} \quad \textit{for} \quad m \ge 1, \quad 0 < \nu \le 1, \tag{1.34}$$

where $N_m > 0$ *do not depend on* ν, *and let* $(u_k, k \in \mathbb{Z})$ *be a stationary solution of* (1.28) *such that* $\mathcal{D}(u_k) = \lambda_0$. *Then there is a constant* $M \ge 1$ *and for any* $k_0 \in \mathbb{Z}$ *there exists an integer-valued non-negative random variable* $T_\nu(\omega)$ *satisfying* (1.18) *such that* (1.17) *holds for* $|T - k_0| \ge T_\nu$.

Remark 1.13. Analogues of Propositions 1.11 and 1.12 are true in the case when condition (B) holds. In this situation, the stationary measure λ satisfies the inequalities

$$\int_{H^r} \exp\bigl(\sigma \nu^{p_l} \|u\|_s^{2\kappa_l}\bigr) \lambda(du) \le C_r, \quad 0 < \nu \le 1,$$

where $0 \le r \le s$, $l = l(r)$ is the smallest integer no less than r, $\kappa_0 = p_0 = 1$, the constants p_l and κ_l with $l \ge 1$ are defined in Theorem 1.4, and $C_r > 0$ is a constant not depending on ν. Moreover, the random variable $T_\nu(\omega)$ has a finite exponential moment.

2 Lyapunov–Schmidt reduction

In this section we prove a result of the Foiaş–Prodi type and show that if a Markov chain $\{u_k\}$ is a stationary solution of Equation (1.28), then sufficiently high Fourier modes of u_k are uniquely defined by low modes of the sequence $(u_l, l \le k)$. This will enable us to reduce the problem of uniqueness of a stationary solution for (1.28) to a similar question for a Gibbs system with a finite-dimensional phase space.

2.1 Formulation of the result

To simplify notations, from now on we assume that

$$\nu = 1.$$

Let us define $\mathcal{H}^s$ as the closure in H^s of the linear manifold spanned by those vectors e_j whose coefficients b_j in expansion (1.3) are non-zero. It is clear that if all the coefficients b_j are non-zero, then $\mathcal{H}^s = H^s$. For any integer $N \geq 2$, let H_N be the subspace in H spanned by the vectors e_j, $j = 1, \dots, N-1$, and let $H_N^\perp$ be its orthogonal complement. We set

$$\mathcal{H}_N^s = \mathcal{H}^s \cap H_N, \quad \mathcal{H}_N^{s\perp} = \mathcal{H}^s \cap H_N^\perp$$

and note that

$$|w| \leq \alpha_N^{-1/2} \|w\| \quad \text{for any} \quad w \in H_N^\perp, \tag{2.1}$$

where α_j, $j \geq 1$, are the eigenvalues of L indexed in increasing order. We denote by P_N and Q_N the orthogonal projections onto H_N and $H_N^\perp$, respectively. Finally, we set

$$\mathfrak{H}^s = H \times \mathcal{H}^s, \quad \mathfrak{H}_N^s = H_N \times \mathcal{H}_N^{s\perp},$$

and for any $s \geq 0$ and any integer $N \geq 1$ we define the projections

$$\Pi_N : \mathfrak{H}^s \to \mathfrak{H}_N^s, \quad \begin{pmatrix} u \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} \mathsf{P}_N u \\ \mathsf{Q}_N \eta \end{pmatrix}.$$

We shall also use the corresponding projections in the spaces of sequences:

$$\boldsymbol{\Pi}_N : \big(\mathfrak{H}^s\big)^{\mathbb{Z}_0} \to \big(\mathfrak{H}_N^s\big)^{\mathbb{Z}_0}, \quad \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\eta} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{P}_N \boldsymbol{u} \\ \mathbf{Q}_N \boldsymbol{\eta} \end{pmatrix},$$

where $\mathbf{P}_N \boldsymbol{u} = (\mathsf{P}_N u_l, l \leq 0)$ and $\mathbf{Q}_N \boldsymbol{\eta} = (\mathsf{Q}_N \eta_l, l \leq 0)$. In the case $N = \infty$, we set

$$\Pi_\infty : \mathfrak{H}^s \to H, \quad \begin{pmatrix} u \\ \eta \end{pmatrix} \mapsto u, \quad \boldsymbol{\Pi}_\infty : \big(\mathfrak{H}^s\big)^{\mathbb{Z}_0} \to H^{\mathbb{Z}_0}, \quad \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\eta} \end{pmatrix} \mapsto \boldsymbol{u}.$$

Applying Q_N to (1.28), we obtain

$$w_k = \mathsf{Q}_N S(v_{k-1} + w_{k-1}) + \psi_k, \tag{2.2}$$

where

$$v_k = \mathsf{P}_N u_k, \quad w_k = \mathsf{Q}_N u_k, \quad \psi_k = \mathsf{Q}_N \eta_k.$$

We wish to show that, for a sufficiently large class of sequences $(v_l, l \leq 0)$ and $(\psi_l, l \leq 0)$, Equation (2.2) with $k \leq 0$ has a unique solution $(w_l, l \leq 0)$, and the dependence of the zeroth component w_0 on v_l and ψ_l decays exponentially as a function of l. To formulate the corresponding results, we have to introduce some notations.

For a sequence $\boldsymbol{u} = (u_l, l \leq k)$ with $u_l \in H$ and integers $m \leq n \leq k$ we set

$$\big\langle |\boldsymbol{u}|^2 \big\rangle_m^n \equiv \big\langle |u_l|^2 \big\rangle_m^n = \frac{1}{n-m+1} \sum_{l=m}^n |u_l|^2.$$

In what follows, we shall need the following two-sided estimate for $\big\langle |\boldsymbol{u}|^2 \big\rangle_m^n$:

$$2\langle\!\langle \boldsymbol{u} \rangle\!\rangle_m^n \leq \langle |\boldsymbol{u}|^2 \rangle_m^n \leq c \langle\!\langle \boldsymbol{u} \rangle\!\rangle_m^n \tag{2.3}$$

where $c = 2(1-e^{-\alpha_1})^{-1}$ and

$$\langle\!\langle \boldsymbol{u} \rangle\!\rangle_m^n = \frac{1}{n-m+1}\sum_{l=m}^{n}\int_0^1 \big\|S_t(u_l)\big\|^2 dt.$$

To prove (2.3), we note that if $u(t)$ is a solution of the homogeneous NS system (1.8), then

$$|u(t)|^2 + 2\int_0^t \|u(\theta)\|^2 d\theta = |u(0)|^2, \quad t \ge 0. \tag{2.4}$$

This estimate immediately implies the left-hand inequality in (2.3). Combining (1.23) with $\nu = 1$ and (2.4), we derive

$$\int_0^t \|u(\theta)\|^2 d\theta \ge \frac{1}{2}(1-e^{-2\alpha_1 t})|u(0)|^2,$$

whence follows the right-hand estimate in (2.3).

For any $K > 0$ and any integer $R \ge 0$, we denote by $\mathbf{F}^s(K,R)$ the set of sequences[3]

$$\binom{\boldsymbol{u}}{\boldsymbol{\eta}} = \Big(\binom{u_k}{\eta_k}, k \le 0\Big), \quad u_k \in H, \quad \eta_k \in \mathcal{H}^s, \tag{2.5}$$

such that Equation (1.28) is satisfied for $k \le 0$, and the following inequality holds:

$$\big\langle |u_k|^2 + \|\eta_k\|_s^2 \big\rangle_T^0 \le K, \quad T \in \mathbb{Z}, \quad T \le -R. \tag{2.6}$$

It is clear that (2.6) is equivalent to the inequality

$$(|T|+1)\big\langle |u_k|^2 + \|\eta_k\|_s^2 \big\rangle_T^0 \le K(|T| \vee R + 1), \quad T \le 0, \tag{2.7}$$

which implies, in particular, that

$$|u_k|^2 + \|\eta_k\|^2 \le K(R \vee |k| + 1), \quad k \le 0. \tag{2.8}$$

We also introduce the space $\mathbf{F}^s(K)$ of sequences (2.5) that satisfy the inequality

$$\limsup_{T\to-\infty} \big\langle |u_k|^2 + \|\eta_k\|_s^2 \big\rangle_T^0 \le K.$$

It is clear that $\mathbf{F}^s(K,R) \subset \mathbf{F}^s(K)$ for any integer $R \ge 0$. The sets $\mathbf{F}^s(K)$ and $\mathbf{F}^s(K,R)$ are subsets of the linear space $\boldsymbol{\mathfrak{H}} = (\mathfrak{H}^0)^{\mathbb{Z}_0}$. We endow $\boldsymbol{\mathfrak{H}}$ with the Tikhonov topology. That is, a sequence $\binom{\boldsymbol{u}^k}{\boldsymbol{\eta}^k}$ converges to $\binom{\boldsymbol{u}}{\boldsymbol{\eta}}$ if $u_l^k \to v_l$ in H and $\eta_l^k \to \eta_l$ in $\mathcal{H}$ for each $l \le 0$. This topology metrisable; for instance, one can use the distance

$$\mathrm{dist}\Big(\binom{\boldsymbol{u}^1}{\boldsymbol{\eta}^1}, \binom{\boldsymbol{u}^2}{\boldsymbol{\eta}^2}\Big) = \sum_{l=-\infty}^{0}\big(|u_l^1 - u_l^2| + |\eta_l^1 - \eta_l^2|\big) \wedge 2^l.$$

[3]The choice of the space $\mathbf{F}^s(K,R)$ is implied by the fact that if $\{u_k\}$ is a stationary solution for (1.28) all of whose moments are finite, then with probability 1 the sequence $(u_l, \eta_l, l \le 0)$ belongs to $\mathbf{F}^s(M,R)$ for an integer $R \ge 0$, where $M > 0$ is the constant in Proposition 1.12.

The sets $\mathbf{F}^s(K)$ and $\mathbf{F}^s(K,R)$ are provided with the topology of $\mathfrak{H}$.

We stress that the topology in the spaces $\mathbf{F}^s(K)$ and $\mathbf{F}^s(K,R)$ is defined in terms of the L^2-norm $|\cdot|$, rather than the H^s-norm $\|\cdot\|_s$.

In the theorem below, we have compiled some properties of the spaces $\mathbf{F}^s(K)$ and $\mathbf{F}^s(K,R)$. We abbreviate $\mathbf{F}^0(K)=\mathbf{F}(K)$ and $\mathbf{F}^0(K,R)=\mathbf{F}(K,R)$.

Theorem 2.1. (i) *Let $s>0$ and let $M>0$ be the constant defined in Proposition 1.5. Then for any $K\ge M$ the space $\mathbf{F}^s(K)$ is nonempty. Moreover, for any integer $R\ge 0$, $\mathbf{F}(K,R)$ is closed in $\mathfrak{H}$ and $\mathbf{F}^s(K,R)$ is compact in $\mathfrak{H}$.*

(ii) *There is a constant $C_*>0$ such that if $N\in[N_0,\infty]$, where $N_0=N_0(K)\ge 1$ is the smallest integer satisfying the condition*

$$\log\alpha_{N_0}>C_*K,$$

then the restriction of the projection $\mathbf{\Pi}_N$ to $\mathbf{F}(K)$ is injective. Moreover, for any integer $l\le 0$ the operator

$$\mathcal{W}_l\colon \mathbf{F}_N(K)\equiv\mathbf{\Pi}_N\mathbf{F}(K)\to H_N^\perp$$

taking each $\boldsymbol{\Upsilon}=\binom{\boldsymbol{v}}{\boldsymbol{\psi}}=\mathbf{\Pi}_N\binom{\boldsymbol{u}}{\boldsymbol{\eta}}$ to $w_l=\mathsf{Q}_N u_l$ satisfies the inequality

$$\begin{aligned}\big|\mathcal{W}_l(\boldsymbol{\Upsilon}^1)-\mathcal{W}_l(\boldsymbol{\Upsilon}^2)\big| &\le |\psi_l^1-\psi_l^2|+\\ &+\sum_{k=-\infty}^{l-1}\big(C\alpha_N^{-1/2}\big)^{l-k}\exp\Big\{C(l-k)\Big(\big\langle|\boldsymbol{u}^1|^2\big\rangle_k^{l-1}+\big\langle|\boldsymbol{u}^2|^2\big\rangle_k^{l-1}\Big)\Big\}|\Upsilon_k^1-\Upsilon_k^2|.\end{aligned} \tag{2.9}$$

Here $\boldsymbol{\Upsilon}^i=\binom{v_k^i}{\psi_k^i}\in\mathbf{F}_N(K)$, $i=1,2$, $C>0$ is a constant not depending on K, N, and $\boldsymbol{\Upsilon}^i$, and $|\Upsilon_k^1-\Upsilon_k^2|=|v_k^1-v_k^2|+|\psi_k^1-\psi_k^2|$.

Theorem 2.1 will be proved in Subsection 2.3. In the next subsection, we use this result to establish equivalence of two families of Markov chains related to a stationary measure for the original equation.

2.2 Theorem on isomorphism

In what follows, we assume that condition (1.4) is satisfied for some $s>0$. According to assertion (i) of Theorem 2.1, in this case $\mathbf{F}^s(K,R)$ is a compact subset of $\mathfrak{H}$ for any $R\ge 0$ and $K\ge M$. We denote by $\mathcal{B}^s(K)$ the Borel σ-algebra on the topological space $\mathbf{F}^s(K)$ and by $\mathcal{P}^s(K)$ the set of all probability measures on $(\mathbf{F}^s(K),\mathcal{B}^s(K))$. In the case $s=0$ we shall simply write $\mathcal{B}(K)$ and $\mathcal{P}(K)$.

We recall that to Equation (1.28) there corresponds an RDS and a family of Markov chains $\{\theta^k\}$ in H given by the formulas

$$\theta^0=u, \tag{2.10}$$

$$\theta^k=S(\theta^{k-1})+\eta_k, \tag{2.11}$$

where $k \geq 1$. Let us fix arbitrary stationary measure $\lambda_0 \in \mathcal{P}(H)$ for (2.10), (2.11) with finite moments (see (1.34)) and denote by $M > 0$ the constant in Proposition 1.12. Along with $\{\theta^k\}$, let us consider another family of Markov chains in $\mathbf{F}^s(K)$, $K \geq M$, defined by the rule

$$\mathbf{\Theta}^0 = \binom{\boldsymbol{u}}{\boldsymbol{\eta}}, \tag{2.12}$$

$$\mathbf{\Theta}^k = \Big(\mathbf{\Theta}^{k-1}, \boldsymbol{S}(\mathbf{\Theta}^{k-1}) + \binom{\eta_k}{\eta_k}\Big), \tag{2.13}$$

where $k \geq 1$ and

$$\boldsymbol{S} : \boldsymbol{\mathfrak{H}^s} \equiv \left(\mathfrak{H}^s\right)^{\mathbb{Z}_0} \to \mathfrak{H}^s, \quad \boldsymbol{S}(\boldsymbol{U}) = \begin{pmatrix} S(u_0) \\ 0 \end{pmatrix} \quad \text{for} \quad \boldsymbol{U} \in \boldsymbol{\mathfrak{H}^s}.$$

It is easy to see that (2.13) defines an RDS in $\mathbf{F}^s(K)$ in the sense that if $\mathbf{\Theta}^k \in \mathbf{F}^s(K)$, $K \geq M$, then $\mathbf{\Theta}^{k+1} \in \mathbf{F}^s(K)$ for all $\omega \in \Omega$. Accordingly, Equations (2.12) and (2.13) define a family of Markov chains in $\mathbf{F}^s(K)$. Moreover, if $(u_k, k \in \mathbb{Z})$ is a stationary solution for (2.11) such that $\mathcal{D}(u_k) = \lambda_0$ (see Propositions 1.5 and 1.11), then the random vector $\big(\binom{u_k}{\eta_k}, k \leq 0\big)$ belongs to $\mathbf{F}^s(K)$ with probability 1, and its distribution $\mathbf{\Lambda}_0$ is a stationary measure for (2.12), (2.13).

We now consider the image of $\{\mathbf{\Theta}^k\}$ under the projection $\mathbf{\Pi}_N$. Here and everywhere below, we assume that

$$N_0 \leq N \leq \infty, \quad \log \alpha_{N_0} > C_* K, \tag{2.14}$$

where $C_* > 0$ is the constant in Theorem 2.1. We shall see that all these projections are equivalent to the original chain $\{\mathbf{\Theta}^k\}$.

For any integers $K \geq M$ and $N \geq N_0$, we set

$$\mathbf{F}^s_N(K,R) = \mathbf{\Pi}_N \mathbf{F}^s(K,R), \quad \mathbf{F}^s_N(K) = \mathbf{\Pi}_N \mathbf{F}^s(K).$$

Thus, for $N < \infty$ the set $\mathbf{F}^s_N(K,R)$ consists of those sequences $\boldsymbol{\Upsilon} = \binom{\boldsymbol{v}}{\boldsymbol{\psi}}$ for which there is $\binom{\boldsymbol{u}}{\boldsymbol{\eta}} \in \mathbf{F}^s(K,R)$ such that $\boldsymbol{v} = \mathsf{P}_N \boldsymbol{u}$ and $\boldsymbol{\psi} = \mathsf{Q}_N \boldsymbol{\eta}$. By assertion (ii) of Theorem 2.1, the pair $\binom{\boldsymbol{u}}{\boldsymbol{\eta}}$ is uniquely determined. Similarly, $\mathbf{F}^s_\infty(K,R)$ consists of the sequences $\boldsymbol{u} = (u_k, k \leq 0)$ that are the first component of an element $\binom{\boldsymbol{u}}{\boldsymbol{\eta}} \in \mathbf{F}^s(K,R)$, which is also unique since $\eta_k = u_k - S(u_{k-1})$. The spaces $\mathbf{F}^s_N(K)$ and $\mathbf{F}^s_\infty(K)$ can be described in a similar way.

In what follows, we assume that $\mathbf{F}^s_N(K)$ is endowed with the Tikhonov topology of the space $\boldsymbol{\mathfrak{H}_N} = (\mathfrak{H}_N)^{\mathbb{Z}_0}$. We confine ourselves to the case $K = 2M$ (although the arguments below remain valid for all $K \geq M$). To simplify notations, we shall write $\mathbf{F}^s$ and $\mathbf{F}^s_N$ instead of $\mathbf{F}^s(2M)$ and $\mathbf{F}^s_N(2M)$, respectively. Since $\mathbf{\Pi}_N : \mathbf{F}^s \to \mathbf{F}^s_N$ is a one-to-one continuous mapping, we can define its inverse $\mathbf{\Pi}_N^{-1}$. We claim that for any integer $N \geq N_0 = N_0(2M)$ the mapping

$$\mathbf{\Pi}_N : (\mathbf{F}^s, \mathcal{B}(\mathbf{F}^s)) \to (\mathbf{F}^s_N, \mathcal{B}(\mathbf{F}^s_N))$$

is an isomorphism of measurable spaces. Indeed, the fact that $\mathbf{\Pi}_N$ is measurable (that is, $\mathbf{\Pi}_N^{-1}(\Gamma) \in \mathcal{B}(\mathbf{F}^s)$ for any $\Gamma \in \mathcal{B}(\mathbf{F}^s_N)$) follows from the continuity of $\mathbf{\Pi}_N$.

Therefore, it suffices to show that $\mathbf{\Pi}_N(\Gamma) \in \mathcal{B}(\mathbf{F}^s_N)$ for any $\Gamma \in \mathcal{B}(\mathbf{F}^s)$. To this end, we first note that

$$\mathbf{F}^s = \bigcap_{K>2M} \bigcup_{R=1}^{\infty} \mathbf{F}^s(K,R).$$

It follows that $\mathbf{F}^s$ is a Borel subset of $\mathfrak{H}$ and, hence, the Borel σ-algebra $\mathcal{B}(\mathbf{F}^s)$ coincides with the collection of subsets $\Gamma \subset \mathbf{F}^s$ for which there is a Borel set $\widetilde{\Gamma} \in \mathcal{B}(\mathfrak{H})$ such that $\Gamma = \widetilde{\Gamma} \cap \mathbf{F}^s$.

We now fix an arbitrary $\Gamma \in \mathcal{B}(\mathbf{F}^s)$. Since the restriction of $\mathbf{\Pi}_N$ to the compact set $\mathbf{F}^s(K,R)$ is continuous together with its inverse, the set $\mathbf{\Pi}_N(\Gamma) = \mathbf{\Pi}_N(\Gamma) \cap \mathbf{F}^s_N$ belongs to $\mathcal{B}(\mathbf{F}^s_N)$.

What has been proved implies, in particular, that the composition mapping

$$\Phi = \mathbf{\Pi}_\infty \circ \mathbf{\Pi}_N^{-1} : \mathbf{F}^s_N \to \mathbf{F}^s_\infty$$

defines an isomorphism of measurable spaces with the inverse

$$\Psi = \mathbf{\Pi}_N \circ \mathbf{\Pi}_\infty^{-1} : \mathbf{F}^s_\infty \to \mathbf{F}^s_N.$$

We also note that

$$\Phi(\boldsymbol{\Upsilon}) = \bigl(u_l, l \le 0\bigr), \qquad u_l = v_l + \mathcal{W}_0(\boldsymbol{\Upsilon}), \tag{2.15}$$

$$\Psi(\boldsymbol{u}) = \bigl(\bigl(\begin{smallmatrix} v_l \\ \psi_l \end{smallmatrix}\bigr), l \le 0\bigr), \qquad v_l = \mathsf{P}_N u_l, \quad \psi_l = \mathsf{Q}_N\bigl(u_l - S(u_{l-1})\bigr), \tag{2.16}$$

where the operator $\mathcal{W}_0$ is defined in Theorem 2.1.

We now describe the families of Markov chains resulting from application of $\mathbf{\Pi}_N$ to $\{\mathbf{\Theta}^k\}$. It is a matter of direct verification to show that for $N = \infty$ we obtain

$$\boldsymbol{\theta}^0 = \boldsymbol{u}, \tag{2.17}$$

$$\boldsymbol{\theta}^k = \bigl(\boldsymbol{\theta}^{k-1}, S(\theta_0^{k-1}) + \eta_k\bigr), \tag{2.18}$$

where $k \ge 1$ and $\boldsymbol{\theta}^k = (\theta_l^k, l \le 0)$, and for $N_0 \le N < \infty$ we have

$$\boldsymbol{\Upsilon}^0 = \bigl(\begin{smallmatrix} \boldsymbol{v} \\ \boldsymbol{\psi} \end{smallmatrix}\bigr), \tag{2.19}$$

$$\boldsymbol{\Upsilon}^k = \Bigl(\boldsymbol{\Upsilon}^{k-1}, T(\boldsymbol{\Upsilon}^{k-1}) + \bigl(\begin{smallmatrix} \varphi_k \\ \psi_k \end{smallmatrix}\bigr)\Bigr), \tag{2.20}$$

where $\varphi_k = \mathsf{P}_N \eta_k$, $\psi_k = \mathsf{Q}_N \eta_k$, and

$$T\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\psi} \end{pmatrix} = \begin{pmatrix} \mathsf{P}_N S(v_0 + \mathcal{W}_0(\boldsymbol{v}, \boldsymbol{\psi})) \\ 0 \end{pmatrix}. \tag{2.21}$$

We shall treat (2.18) and (2.20) as either random dynamical systems or Markov chains in the corresponding phase spaces. Note that the mapping Φ conjugates the two dynamical systems: if $\boldsymbol{\theta}^k = \Phi(\boldsymbol{\Upsilon}^k)$, then $\boldsymbol{\theta}^{k+1} = \Phi(\boldsymbol{\Upsilon}^{k+1})$.

Let us denote by $\mathbf{P}(k,\boldsymbol{u},\Gamma)$ and $\mathfrak{P}(k,\boldsymbol{\Upsilon},\Gamma)$ the transition probabilities for the families $\{\boldsymbol{\theta}^k\}$ and $\{\boldsymbol{\Upsilon}^k\}$, respectively, and by $\mathbf{P}_k$ and $\mathfrak{P}_k$ the Markov semigroups associated with them. The above construction implies that

$$\mathbf{P}(k,\Phi(\boldsymbol{u}),\Phi(\Gamma))=\mathfrak{P}(k,\boldsymbol{\Upsilon},\Gamma),\quad \boldsymbol{\Upsilon}\in\mathbf{F}_N^s,\quad \Gamma\in\mathcal{B}(\mathbf{F}_N^s),$$

and, hence,

$$(\mathbf{P}_kf)\circ\Phi=\mathfrak{P}_k(f\circ\Phi),\quad f\in\mathbf{C}_b(\mathbf{F}_N^s).$$

We now set

$$\boldsymbol{\Theta}^k=\binom{\boldsymbol{u}^k}{\boldsymbol{\eta}^k},\quad \boldsymbol{u}^k=(u_l,l\le k),\quad \boldsymbol{\eta}^k=(\eta_l,l\le k),$$

where $(u_k,k\in\mathbb{Z})$ is a stationary solution such that $\mathcal{D}(u_k)=\lambda_0$. It is clear that $\{\boldsymbol{\Theta}^k\}$ is a stationary Markov chain in $\mathbf{F}^s$ satisfying (2.13) for all $k\in\mathbb{Z}$. Let us consider its image under the projections $\boldsymbol{\Pi}_N$ and $\boldsymbol{\Pi}_\infty$:

$$\boldsymbol{\Upsilon}^k=\boldsymbol{\Pi}_N\boldsymbol{\Theta}^k=\Bigl(\binom{v_l}{\psi_l},l\le k\Bigr),\quad \boldsymbol{u}^k=\boldsymbol{\Pi}_\infty\boldsymbol{\Theta}^k=(u_l,l\le k),$$

where $v_l=\mathsf{P}_Nu_l$ and $\psi_l=\mathsf{Q}_N\eta_l$. What has been said implies that if N satisfies (2.14), then $\boldsymbol{\Upsilon}^k$ and $\boldsymbol{u}^k$ are stationary Markov chains in $\mathbf{F}_N^s$ and $\mathbf{F}_\infty^s$ that satisfy (2.20) and (2.18), respectively. Moreover, the distribution of each of the sequences $\boldsymbol{\Upsilon}^k$ and $\boldsymbol{u}^k$ uniquely determines the distribution of $\boldsymbol{\Theta}^k$. Thus, we obtain a one-to-one correspondence between some classes of stationary measures for (2.11), (2.18), and (2.20). More exactly, we have the following theorem.

Theorem 2.2. *Let $\lambda_0\in\mathcal{P}(H)$ be a stationary measure for* (2.11) *satisfying* (1.34) *and let $(u_k,k\in\mathbb{Z})$ be a stationary solution of* (2.11) *with distribution λ_0. Then the distribution μ of the corresponding stationary Markov chain $\boldsymbol{\Upsilon}^k$ in $\mathbf{F}_N^s$ is uniquely defined. Moreover, the measure μ uniquely determines λ_0. In particular, if Equation* (2.20) *has at most one stationary measure concentrated on the set*

$$\bigcup_{R=1}^{\infty}\mathbf{F}_N^s(2M,R)\subset\mathbf{F}_N^s(2M)\equiv\mathbf{F}_N^s,$$

then Equation (2.11) *has a unique stationary measure that satisfies* (1.34).

Remark 2.3. If $\{\boldsymbol{\Upsilon}^k=(\Upsilon_l^k,l\le0),k\in\mathbb{Z}\}$ is a stationary solution for (2.20), then $\{\Upsilon_0^k,k\in\mathbb{Z}\}$ is a stationary process. Its distribution in the space of sequences $\{\Upsilon_l,l\in\mathbb{Z}\}$ is an (abstract) Gibbs measure in the sense of Ruelle, Sinai and Bowen, see discussion in [KS1]. Therefore, uniqueness of a stationary solution for (2.20) which we prove in Section 4 below implies (is in fact equivalent to) uniqueness of the corresponding 1D Gibbs system.

2.3 Proof of Theorem 2.1

(i) Since $s>0$, Proposition 1.11 implies that there is a stationary solution $(u_k,k\in\mathbb{Z})$ of (1.28) whose distribution satisfies inequality (1.19) with $\nu=1$

and $k_\pm = \pm\infty$. By Proposition 1.5, almost every realisation of the random variable $\left(\binom{u_k}{\eta_k}, k \le 0\right)$ belongs to $\mathbf{F}^s(M)$, and therefore $\mathbf{F}^s(K) \ne \varnothing$ for $K \ge M$.

The proofs of the assertions on compactness and closedness are similar, and we confine ourselves to proving that $\mathbf{F}^s(K,R)$ is compact in the space $\mathfrak{H}$ with Tikhonov topology. Let $\binom{\boldsymbol{u}^i}{\boldsymbol{\eta}^i} \in \mathbf{F}^s(K,R)$ be an arbitrary sequence. The definition of $\mathbf{F}^s(K,R)$ implies that for any $l \le 0$ the sequence η_l^i is bounded in $\mathcal{H}^s$. Furthermore, it follows from Equation (1.28) and the continuity of the map S from H to H^s that the sequence u_l^i is contained in a bounded subset of H^s. Therefore, there are subsequences of $\{u_l^i\}$ and $\{\eta_l^i\}$ that converge in H. It is clear that the limiting pair of sequences $\binom{\boldsymbol{u}}{\boldsymbol{\eta}}$ satisfies (1.28) and belongs to $\mathbf{F}^s(K,R)$. This implies the required assertion.

(ii) The case $N = \infty$ is trivial, and therefore we shall assume that $N < \infty$. We shall need the following lemma whose proof is given in the Appendix (see Section 6.4).

Lemma 2.4. *There is a constant $C > 0$ such that the resolving semigroup of the free NS system* (1.8) *satisfies the inequalities*

$$\left|S_t(u_1^0) - S_t(u_2^0)\right| \le \left|u_1^0 - u_2^0\right| \exp\left\{C \int_0^t \left\|S_\theta(u_1^0)\right\|^2 d\theta\right\}, \tag{2.22}$$

$$\left\|S_t(u_1^0) - S_t(u_2^0)\right\| \le C\,(t^{-3/2} \vee 1)\left|u_1^0 - u_2^0\right| \times \\ \times \exp\left\{C \int_0^t \left(\left\|S_\theta(u_1^0)\right\|^2 + \left\|S_\theta(u_2^0)\right\|^2\right) d\theta\right\}, \tag{2.23}$$

where $t \ge 0$ and $u_1^0, u_2^0 \in H$.

Let

$$\boldsymbol{\Upsilon}^i = \begin{pmatrix} \boldsymbol{v}^i \\ \boldsymbol{\psi}^i \end{pmatrix} = \mathbf{\Pi}_N \begin{pmatrix} \boldsymbol{u}^i \\ \boldsymbol{\eta}^i \end{pmatrix} \in \mathbf{F}_N(K), \quad i = 1, 2.$$

We set $w_l^i = \mathsf{Q}_N u_l^i$ and $w_l^{i-} = \mathsf{Q}_N S(u_{l-1}^i)$. By (2.1), (2.3), and (2.23), for any $l \le 0$ we have

$$|w_l^1 - w_l^2| \le |w_l^{1-} - w_l^{2-}| + |\psi_l^1 - \psi_l^2| \le \alpha_N^{-1/2} \|w_l^{1-} - w_l^{2-}\| + |\psi_l^1 - \psi_l^2| \\ \le C\alpha_N^{-1/2} D(l-1,l)\left(|v_{l-1}^1 - v_{l-1}^2| + |w_{l-1}^1 - w_{l-1}^2|\right) + |\psi_l^1 - \psi_l^2|$$

where for any integers $p < q \le 0$ we set

$$D(p,q) = \exp\left\{C(q-p)\left(\left\langle|\boldsymbol{u}^1|^2\right\rangle_p^{q-1} + \left\langle|\boldsymbol{u}^2|^2\right\rangle_p^{q-1}\right)\right\}.$$

Arguing by induction, for any $m < l-1$ we derive

$$|w_l^1 - w_l^2| \le \sum_{k=m+1}^{l-1} \left(C\alpha_N^{-1/2}\right)^{l-k} D(k,l)\left(|v_k^1 - v_k^2| + |\psi_k^1 - \psi_k^2|\right) + \\ + |\psi_l^1 - \psi_l^2| + \left(C\alpha_N^{-1/2}\right)^{l-m} D(m,l)|u_m^1 - u_m^2|. \tag{2.24}$$

It follows from (2.7) and (2.8) that

$$D(k,l) \le \operatorname{const} e^{2KC|k|}, \quad |u_k^i| \le K^{1/2}|k|^{1/2} + \operatorname{const}, \quad i = 1,2.$$

Therefore, we can pass to the limit in (2.24) as $m \to -\infty$ on condition that

$$\log \alpha_N > 4KC + 2\log C.$$

This results in

$$|w_l^1 - w_l^2| \le |\psi_l^1 - \psi_l^2| + \sum_{k=-\infty}^{l-1} \bigl(C\alpha_N^{-1/2}\bigr)^{l-k} D(k,l)\bigl(|v_k^1 - v_k^2| + |\psi_k^1 - \psi_k^2|\bigr). \quad (2.25)$$

In particular, if $\boldsymbol{\Upsilon}^1 = \boldsymbol{\Upsilon}^2$, then $\boldsymbol{u}^1 = \boldsymbol{u}^2$ and, in view of (1.28), $\boldsymbol{\eta}^1 = \boldsymbol{\eta}^2$. It remains to note that (2.25) coincides with (2.9).

3 A version of the Ruelle–Perron–Frobenius theorem

In this section, we prove a version of the RPF-theorem which is a generalisation of the corresponding result from [KS1] to systems with unbounded phase space. Its application to the Markov semi-group corresponding to the family (2.19), (2.20) will give us the required uniqueness of a stationary measure.

3.1 Statement of the result

Let $\mathbf{X}_0 \subset \mathbf{X}_1 \subset \cdots$ be an increasing family of compact metric spaces which are subsets of a topological space $\mathbf{X}$. We assume that the embeddings $\mathbf{X}_R \subset \mathbf{X}_{R+1} \subset \mathbf{X}$ are isometries for any integer $R \ge 0$. Let $\mathcal{B}(\mathbf{X})$ be the Borel σ-algebra on $\mathbf{X}$ and let $\mathcal{P}(\mathbf{X})$ be the set of all probability measures on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$.

Let $\mathfrak{P}(k, \boldsymbol{v}, \Gamma)$ be a family of Feller transition probabilities on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ and let

$$\mathfrak{P}_k : \mathbf{C}_b(\mathbf{X}) \to \mathbf{C}_b(\mathbf{X}), \quad \mathfrak{P}_k^* : \mathcal{P}(\mathbf{X}) \to \mathcal{P}(\mathbf{X}), \quad k \ge 0,$$

be the corresponding Markov semi-groups. Recall that a subset $\mathcal{R} \subset \mathbf{C}_b(\mathbf{X})$ is called a *determining family for* $\mathcal{P}(\mathbf{X})$ if for arbitrary measures $\mu_1, \mu_2 \in \mathcal{P}(\mathbf{X})$ the condition

$$\int_{\mathbf{X}} f(\boldsymbol{v})\, d\mu_1(\boldsymbol{v}) = \int_{\mathbf{X}} f(\boldsymbol{v})\, d\mu_2(\boldsymbol{v}) \quad \text{for any} \quad f \in \mathcal{R}$$

implies that $\mu_1 = \mu_2$.

For any function $f(\boldsymbol{v})$, denote by f^+ and f^- its positive and negative parts, respectively:

$$f^+ = \frac{1}{2}\bigl(|f| + f\bigr), \quad f^- = \frac{1}{2}\bigl(|f| - f\bigr).$$

For a function $f \in \mathbf{C}_b(\mathbf{X})$, we shall write

$$f_k^+ = (\mathfrak{P}_k f)^+, \quad f_k^- = (\mathfrak{P}_k f)^-.$$

We shall assume that the condition below is satisfied (cf. hypothesis (H) in [KS1, Section 4.1]):

(H) *There is a determining family $\mathcal{R}$ for $\mathcal{P}(\mathbf{X})$ such that $f - c$ belongs to $\mathcal{R}$ for all $f \in \mathcal{R}$ and $c \in \mathbb{R}$, and for any $f \in \mathcal{R}$ and $\alpha > 0$ and arbitrary integers $R \geq 0$ and $\rho \geq 0$ there are $k_0 = k_0(\alpha, f, \rho, R) \in \mathbb{N}$ and $A = A_f(\alpha, \rho, R) > 1$ such that the following property holds: if*

$$\sup_{\boldsymbol{v} \in \mathbf{X}_\rho} f_k^+(\boldsymbol{v}) \geq \alpha \quad \textit{for all} \quad k \geq 0, \tag{3.1}$$

$$\sup_{\boldsymbol{v} \in \mathbf{X}_\rho} f_k^-(\boldsymbol{v}) \geq \alpha \quad \textit{for all} \quad k \geq 0, \tag{3.2}$$

then for any $k \geq k_0$ there is $l = l(k, \alpha, f, \rho, R) > 0$ such that

$$\sup_{\boldsymbol{v} \in \mathbf{X}_R} \big(\mathfrak{P}_l f_k^+\big)(\boldsymbol{v}) \leq A_f(\alpha, \rho, R) \inf_{\boldsymbol{v} \in \mathbf{X}_R} \big(\mathfrak{P}_l f_k^+\big)(\boldsymbol{v}), \tag{3.3}$$

$$\sup_{\boldsymbol{v} \in \mathbf{X}_R} \big(\mathfrak{P}_l f_k^-\big)(\boldsymbol{v}) \leq A_f(\alpha, \rho, R) \inf_{\boldsymbol{v} \in \mathbf{X}_R} \big(\mathfrak{P}_l f_k^-\big)(\boldsymbol{v}). \tag{3.4}$$

Sufficient conditions guaranteeing the validity of (H) are given in Section 3.3. The following result is a generalisation of Theorem 4.1 in [KS1].

Theorem 3.1. *Suppose that condition* (H) *is satisfied. Then the assertions below hold.*

(i) *Let $\mu \in \mathcal{P}(\mathbf{X})$ be a stationary measure of $\mathfrak{P}_k^*$ such that*

$$A_f(\alpha, \rho, R)\mu(\mathbf{X} \setminus \mathbf{X}_R) \to 0 \quad \textit{as} \quad R \to \infty \tag{3.5}$$

for all $f \in \mathcal{R}$, $\alpha > 0$, and $\rho \geq 0$. Then, for any $f \in \mathcal{R}$,

$$\mathfrak{P}_k f \to (\mu, f) \quad \textit{as} \quad k \to \infty \quad \textit{in} \quad L^1(\mathbf{X}, \mu). \tag{3.6}$$

(ii) *The operator $\mathfrak{P}_k^*$ has at most one stationary measure $\mu \in \mathcal{P}(\mathbf{X})$ satisfying* (3.5).

3.2 Proof of Theorem 3.1

1) As in the case of a single metric space (see [KS1]), (i) implies (ii). Indeed, if $\mu_1, \mu_2 \in \mathcal{P}(\mathbf{X})$ are two different stationary measures, then there is $f \in \mathcal{R}$ such that $(\mu_1, f) \neq (\mu_2, f)$. By (i),

$$\mathfrak{P}_k f \to (\mu_i, f) \quad \text{as} \quad k \to \infty \quad \text{in} \quad L^1(\mathbf{X}, \mu_i), \quad i = 1, 2.$$

Therefore, there is a sequence of integers k_s such that

$$\mathfrak{P}_{k_s} f \to (\mu_i, f) \quad \text{as} \quad s \to \infty \quad \mu_i\text{-almost everywhere.} \tag{3.7}$$

Let $C_i \subset \mathbf{X}$ be set of convergence in (3.7). We have $\mu_1(C_1) = \mu_2(C_2) = 1$ and $C_1 \cap C_2 = \varnothing$, and hence μ_1 and μ_2 are singular.

We now compare the measures μ_1 and $\mu = (\mu_1 + \mu_2)/2$. Applying the above argument to them, we see that μ_1 and μ are singular, which contradicts the definition of μ.

2) Thus, it suffices to establish (i). We can assume without loss of generality that $(\mu, f) = 0$. Since $\|\mathfrak{P}_k f\|_\mu$ is a non-increasing sequence, the required assertion will be established if we show that for any $\varepsilon > 0$ there is an integer $k_\varepsilon \geq 1$ such that

$$\|\mathfrak{P}_{k_\varepsilon} f\|_\mu \leq \varepsilon. \tag{3.8}$$

Let us assume that for any integer $\rho \geq 0$ there is a sequence $k_s(\rho)$ such that

$$\sup_{\boldsymbol{v} \in \mathbf{X}_\rho} f^+_{k_s(\rho)}(\boldsymbol{v}) \to 0 \quad \text{as} \quad s \to \infty.$$

In this case, we have

$$\begin{aligned} \int_{\mathbf{X}} (\mathfrak{P}_{k_s(\rho)} f)^+ d\mu(\boldsymbol{v}) &= \int_{\mathbf{X}} f^+_{k_s(\rho)} d\mu(\boldsymbol{v}) \\ &\leq \|f\|_\infty \mu(\mathbf{X} \setminus \mathbf{X}_\rho) + \sup_{\boldsymbol{v} \in \mathbf{X}_\rho} f^+_{k_s(\rho)}(\boldsymbol{v}). \end{aligned} \tag{3.9}$$

It is clear that the right-hand side of (3.9) can be made arbitrarily small by an appropriate choice of ρ and s. Moreover, it follows from the relation $(\mu, f) = 0$ that $(\mu, f_k^+) = (\mu, f_k^-)$, and therefore a subsequence of $(\mu, f_k^+) + (\mu, f_k^-) = \|f_k\|_\mu$ goes to zero. What has been said obviously implies (3.8).

Similar arguments apply in the case when, for any integer $\rho \geq 0$,

$$\sup_{\boldsymbol{v} \in \mathbf{X}_\rho} f^-_{k_s(\rho)}(\boldsymbol{v}) \to 0 \quad \text{as} \quad l \to \infty,$$

where $k_s(\rho)$ is a sequence going to $+\infty$ with s.

3) Thus, we can assume that inequalities (3.1) and (3.2) hold for some positive constants α and ρ. In this case, by condition (H), for any integers $R \geq 0$ and $k \geq k_0(\alpha, f, \rho, R)$ there is $l = l(k, \alpha, f, \rho, R) \geq 0$ such that (3.3) and (3.4) are satisfied.

We now fix arbitrary integer $R \geq 0$ and, repeating the scheme applied in [KS1], construct a sequence of integers $k_s = k_s(R)$ such that

$$\|\mathfrak{P}_{k_s} f\|_\mu \leq \varepsilon_f(R)\big(1 + a_f(R) + \cdots + a_f(R)^{s-1}\big)\|f\|_\infty + a_f(R)^s \|f\|_\mu, \tag{3.10}$$

where $s \geq 0$ and

$$\varepsilon_f(R) = \big(1 + 4A_f(R)^{-1}\big)\mu(\mathbf{X} \setminus \mathbf{X}_R), \quad a_f(R) = 1 - \mu(\mathbf{X}_R) A_f(R)^{-1} < 1. \tag{3.11}$$

Here and henceforth, the dependence on α and ρ is not indicated explicitly.

The proof of (3.10) is by induction on s. For $s=0$, in view of the relation $P_{k_0}^*\mu=\mu$, we have

$$\|\mathfrak{P}_{k_0}f\|_\mu=\|f\|_\mu,$$

which coincides with (3.10) for $s=0$.

Assuming that (3.10) is established for $s\le r$, we now prove it for $s=r+1$. We set $k_{r+1}=k_r+l_r$, where $l_r=l(k_r,\alpha,f,\rho,R)\ge 0$ is the integer entering condition (H). In view of (3.3) and (3.4), we have[4]

$$\begin{aligned}\int_{\mathbf{X}} f_{k_r}^{\pm}d\mu=\int_{\mathbf{X}}\mathfrak{P}_{l_r}f_{k_r}^{\pm}d\mu&=\int_{\mathbf{X}_R}+\int_{\mathbf{X}\setminus\mathbf{X}_R}\\&\le\Big\{\sup_{\boldsymbol{v}\in\mathbf{X}_R}\big(\mathfrak{P}_{l_r}f_{k_r}^{\pm}\big)(\boldsymbol{v})\Big\}\mu(\mathbf{X}_R)+\|f\|_\infty\mu(\mathbf{X}\setminus\mathbf{X}_R)\\&\le A_f(R)\Big\{\inf_{\boldsymbol{v}\in\mathbf{X}_R}\big(\mathfrak{P}_{l_r}f_{k_r}^{\pm}\big)(\boldsymbol{v})\Big\}\mu(\mathbf{X}_R)+\|f\|_\infty\mu(\mathbf{X}\setminus\mathbf{X}_R).\end{aligned}$$

It follows that

$$\mathfrak{P}_{l_r}f_{k_r}^{\pm}(\boldsymbol{v})-A_f(R)^{-1}\|f_{k_r}^{\pm}\|_\mu+A_f(R)^{-1}\|f\|_\infty\mu(\mathbf{X}\setminus\mathbf{X}_R)\ge 0\quad\text{for}\quad\boldsymbol{v}\in\mathbf{X}_R.$$

Let us estimate the expression $\|\mathfrak{P}_{k_{r+1}}f\|_\mu=\|\mathfrak{P}_{l_r}f_{k_r}\|_\mu$. We have

$$\int_{\mathbf{X}}\big|\mathfrak{P}_{l_r}f_{k_r}\big|d\mu=\int_{\mathbf{X}_R}+\int_{\mathbf{X}\setminus\mathbf{X}_R}\le D_r(f_{k_r}^+)+D_r(f_{k_r}^-)+\|f\|_\infty\mu(\mathbf{X}\setminus\mathbf{X}_R),\tag{3.12}$$

where

$$D_r(f_{k_r}^{\pm})=\int_{\mathbf{X}_R}\big|\mathfrak{P}_{l_r}f_{k_r}^{\pm}-A_f(R)^{-1}\|f_{k_r}^{\pm}\|_\mu\big|\,d\mu.$$

Now note that

$$\begin{aligned}D_r(f_{k_r}^{\pm})\le\int_{\mathbf{X}_R}\Big(\mathfrak{P}_{l_r}f_{k_r}^{\pm}(\boldsymbol{v})-A_f(R)^{-1}\|f_{k_r}^{\pm}\|_\mu+A_f(R)^{-1}\|f\|_\infty\mu(\mathbf{X}\setminus\mathbf{X}_R)\Big)\,d\mu\\+A_f(R)^{-1}\|f\|_\infty\mu(\mathbf{X}\setminus\mathbf{X}_R).\end{aligned}$$

This implies that

$$\begin{aligned}D_r(f_{k_r}^+)+D_r(f_{k_r}^-)&\le 4A_f(R)^{-1}\|f\|_\infty\mu(\mathbf{X}\setminus\mathbf{X}_R)+\\&+\int_{\mathbf{X}_R}\big\{\mathfrak{P}_{l_r}(f_{k_r}^++f_{k_r}^-)-A_f(R)^{-1}(\|f_{k_r}^+\|_\mu+\|f_{k_r}^-\|_\mu)\big\}\,d\mu\\&\le\big(1-\mu(\mathbf{X}_R)A_f(R)^{-1}\big)\|f_{k_r}\|_\mu+4A_f(R)^{-1}\|f\|_\infty\mu(\mathbf{X}\setminus\mathbf{X}_R).\end{aligned}$$

[4] Here and henceforth a formula involving the symbol $\pm$ is a brief notation for the two formulas corresponding to the upper and lower signs.

Substituting this expression into (3.12) and using the induction hypothesis, we obtain

$$\int_{\mathbf{X}} \big|\mathfrak{P}_{k_{r+1}} f\big| d\mu \leq \big(1-\mu(\mathbf{X}_R)A_f(R)^{-1}\big)^{r+1}\|f\|_\mu + \\ + \big(1+4A_f(R)^{-1}\big)\|f\|_\infty \mu(\mathbf{X}\setminus\mathbf{X}_R)\sum_{j=0}^{r}\big(1-\mu(\mathbf{X}_R)A_f(R)^{-1}\big)^j,$$

which completes the proof of (3.10).

It follows from (3.10) and (3.11) that

$$\|\mathfrak{P}_{k_s(R)} f\|_\mu \leq \frac{\varepsilon_f(R)}{1-a_f(R)}\|f\|_\infty + a_f(R)^s\|f\|_\mu \\ \leq \mu(\mathbf{X}\setminus\mathbf{X}_R)A_f(R)\big\{\mu(\mathbf{X}_R)(1+4A_f(R)^{-1})\big\}\|f\|_\infty + a_f(R)^s\|f\|_\mu. \tag{3.13}$$

The expression in the brackets on the right-hand side of (3.13) is no greater than 5. Hence, in view of (3.5), the right-hand side of (3.13) can be made arbitrarily small by a suitable choice of R and s. This completes the proof of (3.8).

3.3 Sufficient conditions for application of Theorem 3.1

Let $\mathfrak{P}(k,\boldsymbol{v},\Gamma)$, $\boldsymbol{v}\in\mathbf{X}$, $\Gamma\in\mathcal{B}(\mathbf{X})$, be a Feller transition function. Suppose that there is a determining family $\mathcal{R}$ for $\mathcal{P}(\mathbf{X})$ such that $\mathcal{R}$ is invariant with respect to addition of a constant, and the following two conditions hold:

(H₁) *For any $f\in\mathcal{R}$, $R\geq 0$, and $\beta>0$ and an arbitrary $\boldsymbol{v}\in\mathbf{X}_R$ there is an integer $k_0=k_0(f,R,\beta)\geq 1$, not depending on $\boldsymbol{v}$, and a Borel subset $O(f,\boldsymbol{v},R,\beta)\subset\mathbf{X}$ such that*

$$\big|\mathfrak{P}_k f(\boldsymbol{v}')-\mathfrak{P}_k f(\boldsymbol{v})\big|\leq\beta \quad \text{for} \quad k\geq k_0, \quad \boldsymbol{v}'\in O(f,\boldsymbol{v},R,\beta).$$

(H₂) *There is an integer $\rho_0\geq 0$ such that for any $\rho\geq\rho_0$, $R\geq 0$, $\beta>0$, and $f\in\mathcal{R}$ there is a constant $\varepsilon=\varepsilon(f,\rho,\beta)>0$, not depending on R, and an integer $l=l(f,\rho,\beta,R)\geq 1$ for which*

$$\mathfrak{P}\big(l,\boldsymbol{v}^0,O(f,\boldsymbol{v},\rho,\beta)\big)\geq\varepsilon \quad \text{for any} \quad \boldsymbol{v}^0\in\mathbf{X}_R, \quad \boldsymbol{v}\in\mathbf{X}_\rho, \tag{3.14}$$

where the set $O(f,\boldsymbol{v},\rho,\beta)$ is defined in condition (H_1).

Theorem 3.2. *Suppose that conditions* (H_1) *and* (H_2) *are satisfied. Then* (H) *holds for $\mathcal{R}$ with*

$$A_f(\alpha,\rho,R)=A_f(\alpha,\rho)=\frac{4\,\|f\|_\infty}{\alpha\,\varepsilon(f,\rho,\alpha/2)}, \tag{3.15}$$

where $\varepsilon(f,\rho,\alpha/2)$ is the constant in condition (H_2). *In particular, there is at most one stationary measure $\mu\in\mathcal{P}(\mathbf{X})$ concentrated on the union of $\mathbf{X}_R$, $R\geq 0$.*

Proof. Let $f \in \mathcal{R}$ be arbitrary function satisfying (3.1) and (3.2) for an integer $\rho \geq 1$. We must prove that (3.3) and (3.4) hold. To simplify notation, we confine ourselves to the case of the index $+$.

Without loss of generality, it can be assumed that $\rho \geq \rho_0$, where ρ_0 is the integer in condition (H$_2$). Let $\boldsymbol{v}_k \in \mathbf{X}_\rho$ be such that

$$f_k^+(\boldsymbol{v}_k) \geq \frac{\alpha}{2}, \quad k \geq 0.$$

By condition (H$_1$), there is an integer $k_0 = k_0(f, \rho, \alpha/2) \geq 1$ and a sequence of Borel sets $O_k = O(f, \boldsymbol{v}_k, \rho, \alpha/2)$ such that

$$f_k^+(\boldsymbol{v}') \geq \frac{\alpha}{4} \quad \text{for} \quad \boldsymbol{v}' \in O_k, \quad k \geq k_0. \tag{3.16}$$

Let $\varepsilon = \varepsilon(f, \rho, \alpha/2) > 0$ and $l = l(f, \rho, \alpha/2, R) \geq 1$ be the constants entering condition (H$_2$). In view of (3.14) and (3.16), we have

$$\sup_{\boldsymbol{v} \in \mathbf{X}_R} \bigl(\mathfrak{P}_l f_k^+\bigr)(\boldsymbol{v}) \leq \|f\|_\infty, \tag{3.17}$$

$$\begin{aligned}\inf_{\boldsymbol{v} \in \mathbf{X}_R} \bigl(\mathfrak{P}_l f_k^+\bigr)(\boldsymbol{v}) &= \inf_{\boldsymbol{v} \in \mathbf{X}_R} \int_{\mathbf{X}} \mathfrak{P}(l, \boldsymbol{v}, d\boldsymbol{v}') f_k^+(\boldsymbol{v}') \\ &\geq \inf_{\boldsymbol{v} \in \mathbf{X}_R} \int_{O_k} \mathfrak{P}(l, \boldsymbol{v}, d\boldsymbol{v}') f_k^+(\boldsymbol{v}') \\ &\geq \frac{\alpha}{4} \mathfrak{P}(l, \boldsymbol{v}, O_k) \geq \frac{\alpha\, \varepsilon(f, \rho, \alpha/2)}{4}.\end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18), we arrive at the required inequality.

We now prove the assertion on the uniqueness of a stationary measure. Since the constant $A_f(\alpha, \rho, R)$ is in fact independent of R (see (3.15)), there is at most one stationary measure such that

$$\mu(\mathbf{X} \setminus \mathbf{X}_R) \to 0 \quad \text{as} \quad R \to \infty. \tag{3.19}$$

It remains to note that (3.19) is equivalent to the condition that the measure μ is concentrated on the union of $\mathbf{X}_R$, $R \geq 0$. □

4 Uniqueness of a stationary measure for the reduced chain

4.1 Main result

We denote by $\mathfrak{P}(k, \boldsymbol{\Upsilon}, \Gamma)$ the transition probabilities for the family of Markov chains $\{\boldsymbol{\Upsilon}^k\}$ defined in the space measurable space $(\mathbf{F}_N^s, \mathcal{B}(\mathbf{F}_N^s))$ (see (2.19), (2.20)) and by $\mathfrak{P}_k$ and $\mathfrak{P}_k^*$ the corresponding Markov semi-groups. We shall also need the following metric generating the Tikhonov topology on $\boldsymbol{\mathfrak{H}}_{\boldsymbol{N}}$:

$$\operatorname{dist}\bigl(\boldsymbol{\Upsilon}^1, \boldsymbol{\Upsilon}^2\bigr) = \sum_{l=-\infty}^{0} |\Upsilon_l^1 - \Upsilon_l^2| \wedge 2^l.$$

Theorem 4.1. *Suppose that condition* (1.4) *is satisfied for some* $s > 0$. *There is a constant* $K_* \geq 2M$ *such that if a finite integer* N *satisfies* (2.14) *with* $K = K_*$ *and*

$$b_j \neq 0 \quad \textit{for} \quad j = 1, \ldots, N, \tag{4.1}$$

then $\mathfrak{P}_k^*$ *has a unique stationary measure* μ *that is concentrated on the union of the sets* $\mathbf{F}_N^s(2M, R)$, $R \geq 0$. *Moreover, for any* $f \in \mathbf{C}_b(\mathbf{F}_N^s)$ *and an arbitrary integer* $R \geq 0$, *we have*

$$\mathfrak{P}_k f(\boldsymbol{\Upsilon}) \to (\mu, f) \quad \textit{uniformly in} \quad \boldsymbol{\Upsilon} \in \mathbf{F}_N^s(2M, R) \quad \textit{as} \quad k \to \infty. \tag{4.2}$$

Proof. The existence of a stationary measure follows from Proposition 1.11 and Theorem 2.2. To prove the uniqueness and convergence (4.2), we apply the RPF type theorem established in Section 3.

1) We set

$$\mathbf{X}_R = \mathbf{F}_N^s(2M, R), \quad \mathbf{X} = \mathbf{F}_N^s.$$

Let $\mathcal{R} \subset \mathbf{C}_b(\mathbf{F}_N^s)$ be the set of continuous cylindrical functions on $\mathbf{F}_N^s$, i. e., the set of functions $f\colon \mathbf{F}_N^s \to \mathbb{R}$ for which there is an integer $m \geq 0$ and a bounded continuous function $F\colon (\mathfrak{H}_N)^{m+1} \to \mathbb{R}$ such that

$$f(\Upsilon) = F(v_{-m}, \psi_{-m}, \ldots, v_0, \psi_0), \quad \Upsilon = \binom{\boldsymbol{v}}{\boldsymbol{\psi}} \in \mathbf{F}_N^s. \tag{4.3}$$

Clearly, $\mathcal{R}$ is a determining family for $\mathcal{P}(\mathbf{F}_N^s)$.

It will be proved in Sections 4.2 and 4.3 that if an integer N satisfies (2.14) with sufficiently large $K \geq 2M$, then the transition function $\mathfrak{P}(k, \boldsymbol{\Upsilon}, \Gamma)$ obeys conditions (H_1) and (H_2) in which

$$O(f, \boldsymbol{\Upsilon}, R, \beta) = \bigl\{\boldsymbol{\Upsilon}' \in \mathbf{F}_N^s \cap \mathbf{F}_N(K, R) : \operatorname{dist}(\boldsymbol{\Upsilon}', \boldsymbol{\Upsilon}) \leq r\bigr\}, \tag{4.4}$$

where K is a fixed constant not depending on f, $\boldsymbol{\Upsilon}$, R, β, and N, while r depends only on f, R, and β. By Theorems 3.1 and 3.2, this will imply the uniqueness of a stationary measure concentrated on the union of $\mathbf{F}_N^s(2M, R)$, $R \geq 0$, and also convergence (4.2) in $L^1(\mathbf{X}, \mu)$-norm for any $f \in \mathcal{R}$. Moreover, as is shown in Proposition 4.4, the sequence formed of the restrictions of the functions $\mathfrak{P}_k f$ to $\mathbf{X}_R$ is uniformly equicontinuous for any integer $R \geq 0$. Therefore, by Arzelà–Ascoli theorem, a subsequence $\mathfrak{P}_{k_l} f$ converges uniformly on any $\mathbf{X}_R$. In view of the L_1-convergence, the limit is uniquely determined, and hence the whole sequence uniformly converges to (μ, f).

2) We now show that (4.2) holds for any function $f \in \mathbf{C}_b(\mathbf{F}_N^s)$. Since $\mathbf{X}_\rho$ is a compact subset of $\mathbf{X}$, the restriction of f to $\mathbf{X}_\rho$ is uniformly continuous for any integer $\rho \geq 0$. Let us denote by f_ρ an arbitrary uniformly continuous extension of $f\big|_{\mathbf{X}_\rho}$ to $\boldsymbol{\mathfrak{H}_N}$ such that

$$\|f_\rho\|_\infty \leq 3\|f\|_\infty.$$

For instance, we can take

$$f_\rho(\boldsymbol{\Upsilon}) = \inf_{\boldsymbol{\Upsilon}' \in \mathbf{X}_\rho} \bigl\{f(\boldsymbol{\Upsilon}') + \omega_\rho\bigl(d(\boldsymbol{\Upsilon}, \boldsymbol{\Upsilon}')\bigr)\bigr\},$$

where $\omega_\rho(r)$, $r \ge 0$, is the modulus of continuity of $f\big|_{\mathbf{X}_\rho}$:

$$\omega_\rho(r) = \sup\bigl\{|f(\boldsymbol{\Upsilon}^1) - f(\boldsymbol{\Upsilon}^2)| : \boldsymbol{\Upsilon}^1, \boldsymbol{\Upsilon}^2 \in \mathbf{X}_\rho, d(\boldsymbol{\Upsilon}^1, \boldsymbol{\Upsilon}^2) \le r\bigr\}.$$

Let us denote by $J_L : \mathfrak{H}_{\boldsymbol{N}} \to \mathfrak{H}_{\boldsymbol{N}}$ the operator taking each $\boldsymbol{\Upsilon} = (\Upsilon_l, l \le 0)$ to $(\dots, 0, \Upsilon_{-L}, \dots, \Upsilon_0)$. We define the function

$$f_{\rho L}(\boldsymbol{\Upsilon}) = f_\rho(J_L \boldsymbol{\Upsilon}), \quad \boldsymbol{\Upsilon} \in \mathfrak{H}_{\boldsymbol{N}}.$$

Clearly, we have $f_{\rho L} \in \mathcal{R}$. Thus, convergence (4.2) holds for $f = f_{\rho L}$.

Let us fix arbitrary $R \ge 0$ and write

$$\bigl|\mathfrak{P}_k f(\boldsymbol{\Upsilon}) - (\mu, f)\bigr| \le \bigl|\mathfrak{P}_k f_{\rho L}(\boldsymbol{\Upsilon}) - (\mu, f_{\rho L})\bigr| + \bigl|(\mu, f - f_{\rho L})\bigr| + \\ + \bigl|\mathfrak{P}_k f(\boldsymbol{\Upsilon}) - \mathfrak{P}_k f_{\rho L}(\boldsymbol{\Upsilon})\bigr|. \quad (4.5)$$

As it was mentioned above,

$$\sup_{\boldsymbol{\Upsilon} \in \mathbf{X}_R} \bigl|\mathfrak{P}_k f_{\rho L}(\boldsymbol{\Upsilon}) - (\mu, f_{\rho L})\bigr| := \varepsilon_1(k, L, \rho), \quad (4.6)$$

where $\varepsilon_1(k, L, \rho) \to 0$ as $k \to \infty$ for any fixed $L \ge 1$ and $\rho \ge 0$. Furthermore, it is clear that

$$\sup_{\boldsymbol{\Upsilon} \in \mathbf{X}_\rho} d(\boldsymbol{\Upsilon}, J_L \boldsymbol{\Upsilon}) \to 0 \quad \text{as} \quad L \to \infty \quad \text{for any} \quad \rho \ge 0.$$

Therefore, in view of the uniform continuity of f_ρ, we have

$$\sup_{\boldsymbol{\Upsilon} \in \mathbf{X}_\rho} \bigl|f_{\rho L}(\boldsymbol{\Upsilon}) - f(\boldsymbol{\Upsilon})\bigr| = \sup_{\boldsymbol{\Upsilon} \in \mathbf{X}_\rho} \bigl|f_\rho(J_L \boldsymbol{\Upsilon}) - f_\rho(\boldsymbol{\Upsilon})\bigr| \le \varepsilon_2 = \varepsilon_2(L, \rho),$$

where $\varepsilon_2(L, \rho) \to 0$ as $L \to \infty$ for any $\rho \ge 1$. It follows that

$$\begin{aligned} \bigl|(\mu, f - f_{\rho L})\bigr| &\le \int_{\mathbf{X}} |f - f_{\rho L}|\, d\mu \le \int_{\mathbf{X}_\rho} + \int_{\mathbf{X} \setminus \mathbf{X}_\rho} \\ &\le \int_{\mathbf{X}_\rho} |f - f_{\rho L}|\, d\mu + 4\|f\|_\infty \mu(\mathbf{X} \setminus \mathbf{X}_\rho) \\ &\le \varepsilon_2(L, \rho) + 4\|f\|_\infty \mu(\mathbf{X} \setminus \mathbf{X}_\rho). \end{aligned} \quad (4.7)$$

Finally, to estimate the third term on the right-hand side of (4.5), we note that

$$\begin{aligned} \bigl|\mathfrak{P}_k f(\boldsymbol{\Upsilon}) - \mathfrak{P}_k f_{\rho L}(\boldsymbol{\Upsilon})\bigr| &\le \int_{\mathbf{X}} \mathfrak{P}(k, \boldsymbol{\Upsilon}, d\boldsymbol{\Upsilon}')\bigl|f(\boldsymbol{\Upsilon}') - f_{\rho L}(\boldsymbol{\Upsilon}')\bigr| \\ &\le \int_{\mathbf{X}_\rho} + \int_{\mathbf{X} \setminus \mathbf{X}_\rho} \le \varepsilon_2(L, \rho) + 4\|f\|_\infty \mathfrak{P}(k, \boldsymbol{\Upsilon}, \mathbf{X} \setminus \mathbf{X}_\rho). \end{aligned} \quad (4.8)$$

Combining (4.5), (4.6), (4.7), and (4.8), we derive

$$\bigl|\mathfrak{P}_k f(\boldsymbol{\Upsilon}) - (\mu, f)\bigr| \le \varepsilon_1(k, L, \rho) + \varepsilon_2(L, \rho) + 4\|f\|_\infty \bigl(\mathfrak{P}(k, \boldsymbol{\Upsilon}, \mathbf{X} \setminus \mathbf{X}_\rho) + \mu(\mathbf{X} \setminus \mathbf{X}_\rho)\bigr). \quad (4.9)$$

To conclude that the right-hand side of (4.9) goes to zero, we need the lemma below. We formulate two estimates the first of which is used here and the other will be needed in the next subsection.

Lemma 4.2. *For any integer $R \geq 0$ and any $m \geq 1$ there is a constant $C_{Rm} > 0$ such that*

$$\mathfrak{P}(k, \boldsymbol{\Upsilon}, \mathbf{F}_N^s \setminus \mathbf{F}_N^s(2M, \rho)) \leq C_{Rm}\rho^{-m} \quad \textit{for} \quad k \geq \rho \geq 1, \tag{4.10}$$

$$\mathfrak{P}(k, \boldsymbol{\Upsilon}, \mathbf{F}_N^s \setminus \mathbf{F}_N^s(3M, \rho)) \leq C_{Rm}\rho^{-m} \quad \textit{for} \quad k, \rho \geq 1, \tag{4.11}$$

where $\boldsymbol{\Upsilon} \in \mathbf{F}_N^s(2M, R)$.

Let us fix an arbitrary $\varepsilon > 0$. In view of (4.10) and the fact that μ is concentrated on $\cup_{\rho \geq 0}\mathbf{X}_\rho$, there is an integer $\rho \geq 0$ such that the third term on the right-hand side of (4.9) is less than ε for $k \geq \rho$. We then choose integers $L \geq 1$ and $k_0 \geq \rho$ so large that $\varepsilon_1(k, L, \rho) \leq \varepsilon$ for $k \geq k_0$ and $\varepsilon_2(L, \rho) \leq \varepsilon$. Combining all these estimates, we see that (4.9) does not exceed 4ε for $k \geq k_0$. Thus, to complete the proof of (4.2), it remains to establish Lemma 4.2. □

Proof of Lemma 4.2. Let us fix arbitrary $m \geq 1$. It is clear that it suffices to establish (4.10) and (4.11) for sufficiently large ρ. We fix an arbitrary $\boldsymbol{\Upsilon} \in \mathbf{F}_N^s(2M, R)$ and denote by $\boldsymbol{U} = \binom{\boldsymbol{u}}{\boldsymbol{\eta}}$ the element of $\mathbf{F}^s(2M, R)$ such that $\mathbf{\Pi}_N \boldsymbol{U} = \boldsymbol{\Upsilon}$. Let $(u_l, l \geq 0)$ be the solution of the problem (1.27), (1.28) with the initial function $u^0 = v_0 + \mathcal{W}_0(\boldsymbol{\Upsilon})$ (note that $|u^0| \leq (2MR)^{1/2}$) and let $a_l := |u_l|^2 + \|\eta_k\|_s^2$. Application of Proposition 1.5 with $k_- = 0$, $k_+ = k_0 = k$ and Remark 1.6 to the solution u_l, $0 \leq l \leq k$, shows that, with probability no less than $\varepsilon_{\rho m} := 1 - C_m\rho^{-m}$, we have

$$(k - T + 1)\langle a_l \rangle_T^k \leq 2M\big((k - T) \vee \rho + 1\big), \quad 0 \leq T \leq k. \tag{4.12}$$

Since $\boldsymbol{U} \in \mathbf{F}^s(2M, R)$, we conclude that if $\rho \geq R$, then

$$\langle a_l \rangle_T^0 \leq 2M \quad \text{for} \quad T \leq -R. \tag{4.13}$$

Combining (4.12) and (4.13), we see that for $T \leq 0$, with probability $\geq \varepsilon_{\rho m}$,

$$\begin{aligned}
\langle a_l \rangle_T^k &= (|T| + k + 1)^{-1}\big((|T| + 1)\langle a_l \rangle_T^0 + k\langle a_l \rangle_1^k\big) \\
&\leq M(|T| + k + 1)^{-1}\big(2(R \vee |T| + 1) + k \vee \rho\big) \\
&\leq \begin{cases} 2M & \text{if} \quad k \geq \rho \geq 2R, \\ 3M & \text{if} \quad |T| + k \geq \rho \geq R. \end{cases}
\end{aligned}$$

We have thus proved that

$$\begin{aligned}
\mathfrak{P}(k, \boldsymbol{\Upsilon}, \mathbf{F}_N^s(2M, \rho)) &\geq \varepsilon_{\rho m} = 1 - C_m\rho^{-m}, \quad k \geq \rho \geq 2R, \\
\mathfrak{P}(k, \boldsymbol{\Upsilon}, \mathbf{F}_N^s(3M, \rho)) &\geq \varepsilon_{\rho m} = 1 - C_m\rho^{-m}, \quad k \geq 1, \ \rho \geq R.
\end{aligned}$$

This implies the required inequalities (4.10) and (4.11). □

It Section 5, we shall need a corollary of Theorem 4.1. Let us recall that $\{\boldsymbol{\Upsilon}^k\}$ is isomorphic to the family of Markov chains $\{\boldsymbol{\theta}^k\}$ defined by (2.17), (2.18). We denote by $\mathbf{P}_k$ and $\mathbf{P}_k^*$ the Markov semigroups for $\{\boldsymbol{\theta}^k\}$.

Corollary 4.3. *Under the conditions of Theorem 4.1, the Markov semigroup $\mathbf{P}_k^*$ has a unique stationary measure $\boldsymbol{\lambda} \in \mathcal{P}(\mathbf{F}_\infty^s)$ that is concentrated on the union of $\mathbf{F}_\infty^s(2M,R)$, $R \geq 0$. Moreover, for any $f \in \mathbf{C}_b(\mathbf{F}_\infty^s)$ we have*

$$\mathbf{P}_k f(\boldsymbol{u}) \to (\boldsymbol{\lambda}, f) \quad \text{uniformly in} \quad \boldsymbol{u} \in \mathbf{F}_\infty^s(2M,R) \quad \text{as} \quad k \to \infty.$$

4.2 Checking condition (H_1)

For any integer $m \geq 0$, let $\mathcal{R}_m$ be the set of those $f \in \mathcal{R}$ for which the corresponding function F in (4.3) is defined on $(\mathfrak{H}_N)^{m+1}$. We recall that the set $O(f, \boldsymbol{\Upsilon}, R, \beta)$ is defined in (4.4).

Proposition 4.4. *Let the conditions of Theorem 4.1 be fulfilled and let $K \geq 2M$ be arbitrary constant. Then for any integer $R \geq 0$ and any $\beta > 0$ there is $r = r(R, \beta, K) > 0$ satisfying the following property: if $f \in \mathcal{R}_m$ for an integer $m \geq 1$, then*

$$\left|\mathfrak{P}_k f(\boldsymbol{\Upsilon}^1) - \mathfrak{P}_k f(\boldsymbol{\Upsilon}^2)\right| \leq \beta \|f\|_\infty \quad \text{for} \quad k \geq m+1,$$

where $\boldsymbol{\Upsilon}^1 \in \mathbf{F}_N^s(2M,R)$, $\boldsymbol{\Upsilon}^2 \in \mathbf{F}_N^s \cap \mathbf{F}_N(K,R)$, and $\operatorname{dist}(\boldsymbol{\Upsilon}^1, \boldsymbol{\Upsilon}^2) \leq r$. In particular, the sequence $\mathfrak{P}_k f\big|_{\mathbf{X}_R}$, $k \geq m+1$, is uniformly equicontinuous for any $R \geq 0$, and condition (H_1) holds with any domain $O(f, \boldsymbol{\Upsilon}, R, \beta)$ of the form (4.4).

Proof. 1) Let dv be the Lebesgue measure on the finite-dimensional space H_N and let $d\alpha(\psi)$ be the distribution of the random variables ψ_k on $\mathcal{H}_N^{s\perp}$. We denote by $D(v)$, $v \in H_N$, the density of the random variables φ_k with respect to dv. (It follows from (1.3) and the conditions imposed on ξ_{jk} that $D(v) = \prod_{j=1}^N p_j(b_j x_j)$, where $v = (x_1, \dots, x_N) \in H_N$.) Direct verification shows that for $f \in \mathcal{R}_m$ and $k \geq m+1$ we have (cf. [KS1, Section 1.3])

$$\mathfrak{P}_k f(\boldsymbol{\Upsilon}) = \int_{(\mathfrak{H}_N^s)^k} F(\Upsilon_{k-m}, \dots, \Upsilon_k) D_k(\boldsymbol{\Upsilon}; \overline{\Upsilon}_k)\, \ell_k(d\overline{\Upsilon}_k), \tag{4.14}$$

where $\overline{\Upsilon}_k = (\Upsilon_1, \dots, \Upsilon_k)$ and $\ell_k(d\overline{\Upsilon}_k) = dv_1 \dots dv_k d\alpha(\psi_1) \dots d\alpha(\psi_k)$,

$$D_k(\boldsymbol{\Upsilon}; \Upsilon_1, \dots, \Upsilon_k) = \prod_{l=1}^k D\big(v_l - T_0(\boldsymbol{\Upsilon}, \Upsilon_1, \dots, \Upsilon_{l-1})\big), \tag{4.15}$$

and T_0 is the first component of the operator T defined in (2.21), that is, $T_0(\boldsymbol{\Upsilon}) = \mathsf{P}_N S(v_0 + \mathcal{W}_0(\boldsymbol{v}, \boldsymbol{\psi}))$.

2) Now let $\boldsymbol{\Upsilon}^1 \in \mathbf{F}_N^s(2M,R)$ and $\boldsymbol{\Upsilon}^2 \in \mathbf{F}_N^s \cap \mathbf{F}_N(K,R)$. For any $k \geq 1$, we denote by $V_k = V_k(\boldsymbol{\Upsilon}^1, \boldsymbol{\Upsilon}^2)$ the doubled variational distance between the two

measure on $(\mathfrak{H}_N^s)^k$ defined by the densities $D_k(\boldsymbol{\varUpsilon}^i, \overline{\varUpsilon}_k)$, $i = 1, 2$. In other words,

$$V_k = \int_{(\mathfrak{H}_N^s)^k} \bigl| D_k(\boldsymbol{\varUpsilon}^1, \overline{\varUpsilon}_k) - D_k(\boldsymbol{\varUpsilon}^2, \overline{\varUpsilon}_k) \bigr| \, \ell_k(d\overline{\varUpsilon}_k).$$

Since $\|F\|_\infty = \|f\|_\infty$, it follows from (4.14) and (4.15) that

$$\bigl|\mathfrak{P}_k f(\boldsymbol{\varUpsilon}^1) - \mathfrak{P}_k f(\boldsymbol{\varUpsilon}^2)\bigr| \le \|f\|_\infty V_k.$$

Thus, it is sufficient to estimate V_k. To this end, we note that

$$V_k \le V_{k-1} + \int_{(\mathfrak{H}_N^s)^k} D_{k-1}(\boldsymbol{\varUpsilon}^1, \overline{\varUpsilon}_{k-1}) \Delta_k(\boldsymbol{\varUpsilon}^1, \boldsymbol{\varUpsilon}^2; \overline{\varUpsilon}_k) \, \ell_k(d\overline{\varUpsilon}_k) =: V_{k-1} + I_k, \tag{4.16}$$

where

$$\Delta_k(\boldsymbol{\varUpsilon}^1, \boldsymbol{\varUpsilon}^2; \overline{\varUpsilon}_k) = \bigl| D\bigl(v_k - T_0(\boldsymbol{\varUpsilon}^2, \overline{\varUpsilon}_{k-1}\bigr) - D\bigl(v_k - T_0(\boldsymbol{\varUpsilon}^1, \overline{\varUpsilon}_{k-1})\bigr)\bigr|.$$

We now derive an estimate for $I_k = I_k(\boldsymbol{\varUpsilon}^1, \boldsymbol{\varUpsilon}^2)$.

3) Let us fix arbitrary $K \ge 2M$ and $B \ge 1$. To estimate I_k, we represent the domain of integration $(\mathfrak{H}_N^s)^k$ as the union of a sequence of non-intersecting subsets on each of which the expression $\Delta_k(\boldsymbol{\varUpsilon}^1, \boldsymbol{\varUpsilon}^2; \overline{\varUpsilon}_k)$ admits a uniform estimate. Namely, for any integer $\rho \ge R$ we set

$$A^k(\rho) = \tilde{A}^k(\rho) \setminus \tilde{A}^k(\rho - 1),$$

where $\tilde{A}^k(R-1) = \varnothing$ and $\tilde{A}^k(\rho)$ is the set of those $(\varUpsilon_1, \dots, \varUpsilon_{k-1}) \in (\mathfrak{H}_N^s)^{k-1}$ for which $(\boldsymbol{\varUpsilon}^1, \varUpsilon_1, \dots, \varUpsilon_{k-1}) \in \mathbf{F}_N^s(3M, \rho)$. It is easy to see that the union of $A^k(\rho)$, $\rho \ge R$, coincides with $(\mathfrak{H}_N^s)^{k-1}$ for any $k \ge 1$. Let us write the integral I_k as

$$I_k = \sum_{\rho=R}^{\infty} I_{k\rho}, \tag{4.17}$$

where

$$I_{k\rho} = I_{k\rho}(\boldsymbol{\varUpsilon}^1, \boldsymbol{\varUpsilon}^2) = \int\limits_{A^k(\rho) \times \mathfrak{H}_N^s} D_{k-1}(\boldsymbol{\varUpsilon}^1, \overline{\varUpsilon}_{k-1}) \Delta_k(\boldsymbol{\varUpsilon}^1, \boldsymbol{\varUpsilon}^2; \overline{\varUpsilon}_k) \, \ell_k(d\overline{\varUpsilon}_k). \tag{4.18}$$

By the mean value theorem, we have

$$\Delta_k(\boldsymbol{\varUpsilon}^1, \boldsymbol{\varUpsilon}^2; \overline{\varUpsilon}_k) \le Q_k(v_k) \, \bigl| T_0(\boldsymbol{\varUpsilon}^1, \overline{\varUpsilon}_{k-1}) - T_0(\boldsymbol{\varUpsilon}^2, \overline{\varUpsilon}_{k-1}) \bigr|, \tag{4.19}$$

where

$$Q_k(v_k) = \int_0^1 \bigl| \nabla D\bigl(v_k - \theta T_0(\boldsymbol{\varUpsilon}^1, \overline{\varUpsilon}_{k-1}) - (1-\theta) T_0(\boldsymbol{\varUpsilon}^2, \overline{\varUpsilon}_{k-1})\bigr) \bigr| \, d\theta.$$

It is clear that

$$\int_{\mathfrak{H}_N^s} Q_k(v_k) \, \ell_1(d\varUpsilon_k) \le Q,$$

where $Q > 0$ is a constant not depending on $\boldsymbol{\Upsilon}^1$, $\boldsymbol{\Upsilon}^2$ and $\overline{\Upsilon}_{k-1}$. Therefore, by (4.17) – (4.19), we obtain

$$\begin{aligned}
I_k &\le Q \sum_{\rho=R}^{\infty} \int_{A^k(\rho)} D_{k-1}(\boldsymbol{\Upsilon}^1, \overline{\Upsilon}_{k-1}) \times \\
&\qquad\qquad \times \left| T_0(\boldsymbol{\Upsilon}^1, \overline{\Upsilon}_{k-1}) - T_0(\boldsymbol{\Upsilon}^2, \overline{\Upsilon}_{k-1}) \right| \ell_{k-1}(d\overline{\Upsilon}_{k-1}) \\
&\le Q \sum_{\rho=R}^{\infty} h_{k\rho} \int_{A^k(\rho)} D_{k-1}(\boldsymbol{\Upsilon}^1, \overline{\Upsilon}_{k-1}) \ell_{k-1}(d\overline{\Upsilon}_{k-1}) \\
&\le Q \sum_{\rho=R}^{\infty} h_{k\rho}\, \mathfrak{P}\big(k-1, \boldsymbol{\Upsilon}^1, \mathbf{A}^k(\rho)\big),
\end{aligned} \tag{4.20}$$

where $\mathbf{A}^k(\rho)$ is the set of elements in $\mathbf{F}_N^s$ of the form $(\boldsymbol{\Upsilon}^1, \overline{\Upsilon}_{k-1})$ with $\overline{\Upsilon}_{k-1} \in A^k(\rho)$, and

$$h_{k\rho} = h_{k\rho}(\boldsymbol{\Upsilon}^1, \boldsymbol{\Upsilon}^2) = \sup_{\overline{\Upsilon}_{k-1} \in A^k(\rho)} \left| T_0(\boldsymbol{\Upsilon}^1, \overline{\Upsilon}_{k-1}) - T_0(\boldsymbol{\Upsilon}^2, \overline{\Upsilon}_{k-1}) \right|.$$

4) We now estimate $h_{k\rho}$. To this end, we need the following lemma.

Lemma 4.5. *There is a constant $C > 0$ such that for any $K \ge 2M$ and any integer $\rho \ge 0$ we have*

$$\left| T_0(\boldsymbol{\Upsilon}) \right| \le \big(K(\rho+1)\big)^{1/2}, \tag{4.21}$$

$$\left| T_0(\boldsymbol{\Upsilon}^1) - T_0(\boldsymbol{\Upsilon}^2) \right| \le C \sum_{q=-\infty}^{0} (C\alpha_N^{-1/2})^{-q} e^{CK(|q| \vee \rho+1)} |\Upsilon_q^1 - \Upsilon_q^2|, \tag{4.22}$$

where $\boldsymbol{\Upsilon}, \boldsymbol{\Upsilon}^1, \boldsymbol{\Upsilon}^2 \in \mathbf{F}_N(K,\rho) \cap \mathbf{F}_N^s(K)$.

Taking this assertion for granted, let us complete the proof of the proposition.

By definition, we have $(\boldsymbol{\Upsilon}^1, \overline{\Upsilon}_{k-1}) \in \mathbf{F}_N^s(3M,\rho) \cap \mathbf{F}_N^s$ for $\overline{\Upsilon}_{k-1} \in A^k(\rho)$. It follows that $(\boldsymbol{\Upsilon}^2, \overline{\Upsilon}_{k-1}) \in \mathbf{F}_N(3K,\rho) \cap \mathbf{F}_N^s$. Therefore, in view of inequality (4.21) with K replaced by $K_1 := 3K$, we have

$$h_{k\rho} \le \sup_{\overline{\Upsilon}_{k-1} \in A^k(\rho)} \big\{ |T_0(\boldsymbol{\Upsilon}^1, \overline{\Upsilon}_{k-1})| + |T_0(\boldsymbol{\Upsilon}^2, \overline{\Upsilon}_{k-1})| \big\} \le 2\big(K_1(\rho+1)\big)^{1/2}. \tag{4.23}$$

On the other hand, inequality (4.22) implies that

$$\begin{aligned}
h_{k\rho} &\le C \sum_{q=-\infty}^{1-k} (C\alpha_N^{-1/2})^{-q} e^{CK_1(|q| \vee \rho+1)} |\Upsilon_{q+k-1}^1 - \Upsilon_{q+k-1}^2| \\
&\le C \sum_{q=-\infty}^{0} (C\alpha_N^{-1/2})^{-q+k-1} e^{CK_1(|q|+k) + CK_1\rho}\, |\Upsilon_q^1 - \Upsilon_q^2| \\
&\le C_1(R)\, 2^{-k}\, e^{CK_1\rho}\, d,
\end{aligned} \tag{4.24}$$

where $d = d(\boldsymbol{\Upsilon}^1, \boldsymbol{\Upsilon}^2)$, $C_1(R) > 0$ is a constant depending only on R, and the constant K in (2.14) is chosen to be so large that $2(CK_1+\log C+\log 2) \le \log \alpha_N$. Note that the third inequality in (4.24) uses the estimate

$$d(\boldsymbol{\Upsilon}^1, \boldsymbol{\Upsilon}^2) \le C'(R) \sum_{q=-\infty}^{0} 2^q \,|\Upsilon_q^1 - \Upsilon_q^2|, \quad \boldsymbol{\Upsilon}^1, \boldsymbol{\Upsilon}^2 \in \mathbf{F}_N(K, R).$$

Combining (4.23) and (4.24), we derive

$$h_{k\rho} \le \big(C_1(R)\, 2^{-k}\, e^{CK_1\rho}\, d\big) \wedge \big(2K_1^{1/2}(\rho+1)^{1/2}\big). \tag{4.25}$$

5) We can now easily complete the proof of the proposition. We wish to show that $V_k \le \beta$ if $d(\boldsymbol{\Upsilon}^1, \boldsymbol{\Upsilon}^2) \le r$, where $r = r(\beta) > 0$ is sufficiently small. In view of inequality (4.11) with $m = 3$ and the inclusion $\mathbf{A}^k(\rho) \subset \mathbf{F}_N^s \setminus \mathbf{F}_N^s(3M, \rho-1)$ for $\rho \ge R+1$, we have

$$\mathfrak{P}\big(k-1, \boldsymbol{\Upsilon}^1, \mathbf{A}^k(\rho)\big) \le C_{R3}\rho^{-3} \tag{4.26}$$

for all $k \ge 1$, $\rho \ge R$ and $\boldsymbol{\Upsilon}^1 \in \mathbf{F}_N^s(2M, R)$. Substituting (4.25), (4.26) and (4.20) into (4.16) and iterating the resulting inequality, we arrive at

$$\begin{aligned} V_k &\le C_{R3} Q \sum_{j=1}^{k} \sum_{\rho=R}^{\infty} \rho^{-3} \Big\{ \big(C_1(R)\, 2^{-k}\, e^{CK_1\rho}\, d\big) \wedge \big(2K_1^{1/2}(\rho+1)^{1/2}\big) \Big\} \\ &\le \Sigma(d) := C_2 \sum_{j=1}^{\infty} \sum_{\rho=R}^{\infty} \rho^{-3} \big\{ (2^{-j}\, D^\rho\, d) \wedge \rho^{1/2} \big\}, \end{aligned}$$

where C_2 and D are positive constants. Thus, the expression V_k can be estimated by the double series $\Sigma(d)$ vanishing for $d = 0$. By the Lebesgue theorem on dominated convergence, the required assertion will be established if we show that the series converges uniformly in $d \in [0, 1]$. Since all the terms in the sum $\Sigma(d)$ are non-decreasing functions of d, it suffices to prove the convergence for $d = 1$.

To this end, we divide the domain of summation (i. e., $j \ge 1$, $\rho \ge R$) into two non-intersecting sets:

$$S_1 = \big\{(j, \rho) : 2^{-j} D^\rho \le \rho^{1/2} 2^{-j/2}\big\}, \quad S_2 = \big\{(j, \rho) : 2^{-j} D^\rho > \rho^{1/2} 2^{-j/2}\big\}$$

Let Σ_1 and Σ_2 be the sums corresponding to S_1 and S_2, respectively. Clearly,

$$\Sigma_1 \le C_2 \sum_{(j,\rho)\in S_1} \rho^{-5/2} 2^{-j/2} < \infty.$$

On the other hand, if $(j, \rho) \in S_2$, then $j \le c\rho$, where $c > 0$ depends only on D. Therefore,

$$\Sigma_2 \le C_2 \sum_{\rho=R}^{\infty} \sum_{j \le c\rho} \rho^{-5/2} \le C_2 c \sum_{\rho=R}^{\infty} \rho^{-3/2} < \infty.$$

Thus, it remains to establish Lemma 4.5. □

Proof of Lemma 4.5. Inequality (4.21) is a simple consequence of the definition of T_0 and $\mathbf{F}_N(K,\rho)$:

$$\big|T_0(\boldsymbol{\varUpsilon})\big| = \big|S(v_0+\mathcal{W}_0(\boldsymbol{\varUpsilon}))\big| \le |u| \le \big(K(\rho+1)\big)^{1/2}, \quad u = v_0+\mathcal{W}_0(\boldsymbol{\varUpsilon}).$$

Let us prove (4.22). Inequality (2.22) with $t=1$ implies that

$$\begin{aligned}\big|T_0(\boldsymbol{\varUpsilon}^1)-T_0(\boldsymbol{\varUpsilon}^2)\big| &= \big|S(v_0^1+\mathcal{W}_0(\boldsymbol{\varUpsilon}^1)) - S(v_0^2+\mathcal{W}_0(\boldsymbol{\varUpsilon}^2))\big| \\ &\le \big(|v_0^1-v_0^2| + |\mathcal{W}_0(\boldsymbol{\varUpsilon}^1)-\mathcal{W}_0(\boldsymbol{\varUpsilon}^2)|\big)\exp\Big\{C_1\int_0^1 \big\|S_t(u^1)\big\|^2 dt\Big\}, \end{aligned} \tag{4.27}$$

where $u^i = v_0^i+\mathcal{W}_0(\boldsymbol{\varUpsilon}^i)$, $i=1,2$. In view of (2.7), (2.9), (2.3) and the definition of the space $\mathbf{F}_N(K,\rho)$, we have

$$\begin{aligned} &|\mathcal{W}_0(\boldsymbol{\varUpsilon}^1)-\mathcal{W}_0(\boldsymbol{\varUpsilon}^2)| \\ &\le |\psi_0^1-\psi_0^2| + \sum_{q=-\infty}^{-1}\big(C_2\alpha_N^{-1/2}\big)^{-q}\exp\Big\{C_2|q|\big(\big\langle|\boldsymbol{u}^1|^2\big\rangle_q^{-1} + \big\langle|\boldsymbol{u}^2|^2\big\rangle_q^{-1}\big)\Big\}|\varUpsilon_q^1-\varUpsilon_q^2| \\ &\le |\psi_0^1-\psi_0^2| + \sum_{q=-\infty}^{-1}\big(C_2\alpha_N^{-1/2}\big)^{-q} e^{2C_2K(|q|\vee\rho+1)}|\varUpsilon_q^1-\varUpsilon_q^2|, \end{aligned} \tag{4.28}$$

where $\boldsymbol{\varUpsilon}^1,\boldsymbol{\varUpsilon}^2 \in \mathbf{F}_N(K,\rho)\cap\mathbf{F}_N^s(K)$ and $\boldsymbol{u}^i = \varPhi(\boldsymbol{\varUpsilon}^i)$, $i=1,2$. Moreover, by inequality (2.4),

$$\int_0^1 \big\|S_t(u^1)\big\|^2 dt \le \frac{1}{2}|u^1|^2 \le \frac{1}{2}K(\rho+1) \le \frac{1}{2}K(|q|\vee\rho+1), \quad q\le 0. \tag{4.29}$$

Combining (4.27)–(4.29), we derive (4.22). □

4.3 Checking condition $(\mathbf{H_2})$

We recall that $B_{\mathbf{X}}(\boldsymbol{\varUpsilon},r)$ denotes the ball of radius r in $\mathbf{X}$ centred at $\boldsymbol{\varUpsilon}$.

Proposition 4.6. *Under the conditions of Theorem 4.1, there is an integer $\rho_0\ge 1$ and positive constants K and C such that if $\rho\ge\rho_0$, then the following assertions hold:*

(i) *For any $R\ge 0$ there is an integer $l_1^* = l_1^*(R)\ge 1$ such that*

$$\mathfrak{P}(l_1,\boldsymbol{\varUpsilon}^0,\mathbf{F}_N^s(2M,\rho_0)) \ge 1/2 \quad \textit{for any} \quad l_1\ge l_1^*, \quad \boldsymbol{\varUpsilon}^0\in\mathbf{F}_N^s(2M,R). \tag{4.30}$$

(ii) *For any $r>0$, any integer $\rho\ge\rho_0$, and an arbitrary $\boldsymbol{\varUpsilon}\in\mathbf{F}_N^s(2M,\rho)$ there is $\varepsilon=\varepsilon(\rho,r)>0$ and an integer $l_2 = l_2(\boldsymbol{\varUpsilon},\rho,r)\ge 1$ such that*

$$\mathfrak{P}(l_2,\boldsymbol{\varUpsilon}^0,B_{\mathbf{X}}(\boldsymbol{\varUpsilon},r)\cap\mathbf{F}_N(K,\rho) \ge \varepsilon \quad \textit{for any} \quad \boldsymbol{\varUpsilon}^0\in\mathbf{F}_N^s(2M,\rho_0). \tag{4.31}$$

Moreover, there is an integer $l_2^* = l_2^*(\rho,r)\ge 1$ such that $l_2(\boldsymbol{\varUpsilon},\rho,r)\le l_2^*$ for all $\boldsymbol{\varUpsilon}\in\mathbf{F}_N^s(2M,\rho)$.

(iii) *The transition function* $\mathfrak{P}(k,\boldsymbol{\Upsilon},\Gamma)$ *satisfies condition* (H_2) *in which the set* $O(f,\boldsymbol{\Upsilon},R,\beta)$ *has the form* (4.4).

Proof. We first show that (i) and (ii) imply (iii). Indeed, let us fix any $r>0$ and any integers $R\ge 0$ and $\rho\ge\rho_0$. Choosing $l=l_1^*(R)+l_2^*(\rho,r)$ and $l_1=l-l_2(\boldsymbol{\Upsilon},\rho,r)$, from (4.30), (4.31), and the Chapman–Kolmogorov relation, we derive

$$\mathfrak{P}\big(l,\boldsymbol{\Upsilon}^0,O(\boldsymbol{\Upsilon},\rho,r)\big)\ge\int_{\mathbf{X}_{\rho_0}}\mathfrak{P}(l_1,\boldsymbol{\Upsilon}^0,d\boldsymbol{\Upsilon}')\mathfrak{P}(l_2,\boldsymbol{\Upsilon}',O(\boldsymbol{\Upsilon},\rho,r))$$
$$\ge\varepsilon(\rho,r)/2,$$

where $O(\boldsymbol{\Upsilon},\rho,r)=B_{\mathbf{X}}(\boldsymbol{\Upsilon},r)\cap\mathbf{F}_N(K,\rho)$. This proves the required assertion.

We now turn to the proof of (i) and (ii).

Proof of (i). For $\boldsymbol{\Upsilon}^0\in\mathbf{F}_N^s(2M,R)$, we set $\boldsymbol{U}=\binom{\boldsymbol{u}}{\boldsymbol{\eta}}=\boldsymbol{\Pi}_N^{-1}\boldsymbol{\Upsilon}^0\in\mathbf{F}^s(2M,R)$ (see Section 2.2). We denote by $(u_l,l\ge 0)$ the trajectory of the RDS (2.11) which starts from u_0 (the zeroth component of $\boldsymbol{u}$) and set $\boldsymbol{u}^k=(u_l,l\le k)$ and $a_l=|u_l|^2+\|\eta_k\|_s^2$. Since $|u^0|^2\le 2M(R+1)$, inequality (1.11) implies that there is an integer $L_1=L_1(R)\ge 1$ such that

$$\mathbb{E}\,|u_k|\le\begin{cases}C_1(R+1), & 1\le k\le L_1-1,\\ C_1, & k\ge L_1,\end{cases}\tag{4.32}$$

where the constant $C_1>0$ does not depend on $R\ge 0$. Let us fix arbitrary integer $R_1\ge C_1(R+1)$ and estimate the probability of the event

$$|u_k|\le R_1,\quad \|\eta_k\|_s\le R_1\quad k=1,\dots,L_1-1.\tag{4.33}$$

In view of (4.32), (1.5) and the Chebyshev inequality, we have

$$\mathbb{P}\{(4.33)\text{ holds}\}\ge 1-\sum_{k=1}^{L_1-1}\big(\mathbb{P}\{|u_k|\ge R_1\}+\mathbb{P}\{\|\eta_k\|_s\ge R_1\}\big)\ge 1-p_1,$$

where $p_1=p_1(R,R_1)\to 0$ as $R_1\to\infty$ for any fixed R. Furthermore, let us fix sufficiently large integers $\rho_0\ge 1$ and $L_0\ge\rho_0$, set $l_1^*:=L_1+L_0$, and take an arbitrary $l_1\ge l_1^*$. Applying Proposition 1.5 to the solution u_k, $k_-\le k\le k_+$, where $k_-=L_1$ and $k_+=k_0=l_1$, we conclude that there is a constant $C_0>0$, not depending on R, such that with probability no less than $p_0=1-C_0\rho_0^{-1}$,

$$\langle a_l\rangle_T^{l_1}\le M,\quad L_1\le T\le l_1-\rho_0.\tag{4.34}$$

Hence, we have shown that

$$\mathbb{P}\{(4.33)\text{ and }(4.34)\text{ hold}\}\ge p:=p_0+p_1-1.\tag{4.35}$$

It is a matter of direct verification to show that if $L_0\ge 2(R_1^2L_1M^{-1}+R)$, then inequalities (4.33) and (4.34) imply that $\big(\binom{u_k}{\eta_k},k\le l_1\big)\in\mathbf{F}_N^s(2M,\rho_0)$. In view of (4.35), it follows that for any $\boldsymbol{U}\in\mathbf{F}^s(2M,R)$ we have

$$\mathbb{P}\big\{\big(\tbinom{u_k}{\eta_k},k\le l_1\big)\in\mathbf{F}^s(2M,\rho_0)\big\}\ge 1-p_1(R,R_1)-C_0\rho_0^{-1}.\tag{4.36}$$

It remains to note that if $\rho_0 \geq 1$ is so large that $C_0\rho_0^{-1} \leq 1/4$, then for any fixed $R \geq 0$ we can choose $R_1 \geq R$ such that the right-hand side of (4.36) is no less than 1/2. This completes the proof of (4.30).

Proof of (ii). We shall need the following elementary lemma.

Lemma 4.7. *Let* $(x_l, l \leq 0)$ *be e sequence of non-negative numbers such that*

$$\sum_{l=T}^{0} x_l \leq C(|T|+1) \quad \textit{for} \quad T \leq -\rho, \tag{4.37}$$

where $\rho \geq 0$ *is an integer and* $C > 0$ *is a constant not depending on* T*. Then every integer interval* $\Delta = [t_1, t_2]$ *such that* $t_1 \leq -\rho$ *and* $t_2 \leq 0$ *contains an integer point* p *such that*

$$\frac{1}{q-p+1}\sum_{l=p}^{q} x_l \leq C_0 := \frac{C|t_1|}{t_2-t_1+1} \quad \textit{for} \quad p \leq q \leq 0.$$

Proof. Assuming the contrary, for each $p \in \Delta$ we can find an integer $m(p)$, $p \leq m(p) \leq 0$, such that

$$\sum_{l=p}^{m(p)} x_l > C_0(m(p)-p+1). \tag{4.38}$$

Let us define a finite sequence of integers $p_1, p_2, \dots, p_n$ by the following rule: $p_1 = t_1$ and $p_j = m(p_{j-1}) + 1$ if $j \geq 2$ and $m(p_{j-1}) \leq t_2$. Setting $\Delta^j = [p_j, m(p_j)]$ and using inequality (4.38), we derive

$$\sum_{l=t_1}^{0} x_l \geq \sum_{j=1}^{n}\sum_{l\in\Delta^j} x_l > C_0\sum_{j=1}^{n}(m(p_j)-p_j+1) \geq C_0(t_2-t_1+1).$$

This contradicts inequality (4.37) with $T = t_1$. □

1) To establish (4.31), we regard (2.20) as an RDS in $\mathbf{X}$ (rather than a Markov chain), and using the isomorphism of (2.18) and (2.20), pass from a random trajectory $\{\boldsymbol{\Upsilon}^k\}$ to $\{\boldsymbol{u}^k = \Phi(\boldsymbol{\Upsilon}^k)\}$. More exactly, for $\boldsymbol{\Upsilon}$ and $\boldsymbol{\Upsilon}^0$ as in (4.31), let

$$\widehat{\boldsymbol{U}} = \begin{pmatrix}\hat{\boldsymbol{u}}\\ \hat{\boldsymbol{\eta}}\end{pmatrix} = \mathbf{\Pi}_N^{-1}\boldsymbol{\Upsilon} \in \mathbf{F}^s(2M,\rho), \quad \boldsymbol{U}^0 = \begin{pmatrix}\boldsymbol{u}^0\\ \boldsymbol{\eta}^0\end{pmatrix} = \mathbf{\Pi}_N^{-1}\boldsymbol{\Upsilon}^0 \in \mathbf{F}^s(2M,\rho_0).$$

We set $\mathcal{F}_L = \mathbf{F}_L(K,\rho)\cap\mathbf{F}_L^s$, where $L = N$ or $L = \infty$, and consider the restriction of $\Psi\colon \mathbf{F}_\infty^s \to \mathbf{F}_N^s$ to $\mathcal{F}_\infty$. In view of (2.16), inequality (2.22) implies that the mapping $\Psi\colon \mathcal{F}_\infty \to \mathcal{F}_N$ is uniformly Lipschitz with a Lipschitz constant d not depending on N. Therefore, inequality (4.31) will be proved if we show that

$$\mathbb{P}\bigl\{\boldsymbol{u}^{l_2} \in \mathcal{F}_\infty, \operatorname{dist}(\boldsymbol{u}^{l_2}, \hat{\boldsymbol{u}}) \leq r/d\bigr\} \geq \varepsilon, \tag{4.39}$$

where $\boldsymbol{u}^k$, $k \geq 0$, is the random trajectory of (2.18) starting from $\boldsymbol{u}^0$.

2) We fix arbitrary $\rho \geq \rho_0$ and $r > 0$. Let $B > 0$ be a sufficiently large constant which will be chosen later. Let an integer $T_1 = T_1(r, B) \geq 1$ and a positive constant $\delta_1 = \delta_1(r, B) \leq 1$ be such that $\operatorname{dist}(\boldsymbol{u'}, \boldsymbol{0}) \leq r/d$ for any element $\boldsymbol{u'} \in \mathbf{F}^s_\infty$ whose components satisfy the inequalities

$$|u'_j| \leq B\, e^{-(j+T_1-1)} + \delta_1, \quad 1 - T_1 \leq j \leq 0. \tag{4.40}$$

Since $\langle |\hat{\boldsymbol{u}}|^2\rangle_q^0 \leq 2M$ for $q \leq -\rho$, the sequence $x_l = |u_l|^2$, $l \leq 0$, satisfies the conditions of Lemma 4.7 with $C = 2M$. Let $T_2 = T_2(\rho, r, B)$ be the smallest even integer exceeding $(2T_1) \vee \rho$. Applying Lemma 4.7 with $t_1 = -T_2$ and $t_2 = -T_2/2$, we find an integer $T = T(\rho, r, B)$, $T_1 \leq T \leq T_2$, such that

$$\big\langle |\hat{\boldsymbol{u}}|^2 \big\rangle_{-T}^{-T+l} \leq 4M \quad \text{for} \quad 0 \leq l \leq T. \tag{4.41}$$

We claim that there is a deterministic trajectory $\tilde{\boldsymbol{u}}^l = (\boldsymbol{u}^0, \tilde{u}_1, \dots, \tilde{u}_l)$, $l = 1, \dots, T$, for (2.18) that corresponds to a control $\tilde{\eta}_l = \binom{\tilde{\varphi}_l}{\tilde{\psi}_l} \in \mathcal{H}^s$ and possesses the following properties:

$$|\tilde{u}_l - \hat{u}_{l-T}| \leq B\, e^{-l}, \quad l = 0, \dots, T, \tag{4.42}$$

$$\|\widetilde{\varphi}_l\|_p \leq 2B\alpha_N^{p/2}, \quad \widetilde{\psi}_l = \widehat{\psi}_{l-T}, \quad l = 1, \dots, T, \tag{4.43}$$

where $p \geq 0$. Taking this assertion for granted, let us show that (4.39) holds with $l_2 = T$. It follows from (4.43) and the inclusion $\widehat{\boldsymbol{U}} \in \mathbf{F}^s(2M, \rho)$ that

$$\|\tilde{\eta}_l\|_s \leq 2B\alpha_N^{s/2} + (2MT)^{1/2}, \quad l = 1, \dots, T.$$

Therefore, by Lemma 6.2, for any $\gamma > 0$ the probability of the event

$$\Omega_\gamma := \big\{|\eta_l - \tilde{\eta}_l| \leq \gamma, l = 1, \dots, T\big\}$$

can be estimated from below by a constant $\varepsilon > 0$ depending only on N, B, ρ, r, and γ (but not on $\boldsymbol{\Upsilon}$). In view of the continuous dependence of trajectories for (2.11) on the control η_l, for any $\omega \in \Omega_\gamma$ we have

$$|u_l - \tilde{u}_l| \leq \delta = \delta(\gamma), \quad l = 1, \dots, T,$$

where u_l, $l \geq 1$, is the trajectory of (2.11) corresponding to η_l, and $\delta(\gamma) \to 0$ as $\gamma \to 0$. Combining this with (4.42), we conclude that $u'_j := u_{j+T} - \hat{u}_j$, $j = 1 - T, \dots, 0$, satisfy (4.40) if $\delta(\gamma) \leq \delta_1$. Therefore,

$$d(\boldsymbol{u}^{l_2}, \hat{\boldsymbol{u}}) \leq r/d \quad \text{for} \quad \omega \in \Omega_\gamma, \quad \gamma \ll 1.$$

Moreover, it is a matter of direct verification to show that $\boldsymbol{u}^{l_2} \in \mathbf{F}_\infty(K, \rho)$, where $K = K(M, B)$ is sufficiently large. This completes the proof of (4.31).

3) Thus, it remains to establish the existence of a deterministic trajectory $\tilde{\boldsymbol{u}}^l$ satisfying (4.42) and (4.43).

Let us set

$$\widetilde{\varphi}_l = \mathsf{P}_N\big(\tilde{u}_{l-T} - S(\tilde{u}_{l-1})\big), \quad \widetilde{\psi}_l = \widehat{\psi}_{l-T}, \quad l = 1, \dots, T, \tag{4.44}$$

where $\tilde{u}_0$ is the zeroth component of $\boldsymbol{u}^0$. Note that the first relation in (4.44) implies that $\tilde{v}_l = \hat{v}_{l-T}$ for $l = 1, \dots, T$. We claim that (4.42) and (4.43) hold with an appropriate constant $B > 0$. Indeed, (4.43) is a simple consequence of inequality (4.42) whose proof is by induction on l. In view of (4.41) with $l = 0$ and the inclusion $\boldsymbol{u}^0 \in \mathbf{F}_\infty(2M, \rho_0)$, we have

$$|\tilde{u}_0 - \hat{u}_{-T}| \le |\tilde{u}_0| + |\hat{u}_{-T}| \le \big(2M(\rho_0 + 1)\big)^{1/2} + 2M^{1/2} := B.$$

Let us assume that (4.42) is proved for $0 \le l \le k-1$, $k \ge 1$. It follows from (4.41) and inequality (2.24) in which $m = 0$, $l = k$, $u_r^1 = \tilde{u}_r$, and $u_r^2 = \hat{u}_{r-T}$ that

$$\begin{aligned}|\tilde{u}_k - \hat{u}_{k-T}| &\le (C\alpha_N^{-1/2})^k \exp\Big\{Ck\big(\langle|\tilde{u}_j|^2\rangle_0^{k-1} + \langle|\hat{u}_{j-T}|^2\rangle_0^{k-1}\big)\Big\}|\tilde{u}_0 - \hat{u}_{-T}| \\ &\le (C\alpha_N^{-1/2})^k \exp\big\{2C(6M + B^2)k\big\}B \le e^{-k}B,\end{aligned}$$

where the integer $N \ge 1$ is so large that

$$\log \alpha_N \ge 4C(6M + B^2) + 2(1 + \log C).$$

This completes the induction and the proof of the proposition. □

5 Uniqueness and mixing for the original system

We recall that the Markov semigroups P_k and P_k^* associated with Equation (1.1) and the space $C(H^s, \beta)$ of continuous functions with exponential growth at infinity and the corresponding norm $\|f\|_{s,\beta}$ were introduced in Section 1.3. Also recall that we set $\beta_d(r) = (1 + r)^d$, $r \ge 0$.

As before, we assume that $\nu = 1$. For any integer $R \ge 0$, we denote by $H(R)$ the set of those $u \in H$ for which there is $\boldsymbol{u} \in \mathbf{F}_\infty^s(2M, R)$ such that $u_0 = u$, where u_0 is the zeroth component of $\boldsymbol{u}$.

Theorem 5.1. *Suppose that condition* (1.4) *is satisfied for some* $s > 0$. *Then there is an integer* $N \ge 1$ *such that if*

$$b_j \ne 0 \quad \textit{for} \quad j = 1, \dots, N, \tag{5.1}$$

then the Markov semigroup P_k^* *has a unique stationary measure* $\lambda \in \mathcal{P}(H)$ *satisfying condition* (1.34). *Moreover, the measure* λ *is concentrated on* H^s, *and if* $f \in C(H^s, \beta_m)$ *for some* $m \ge 1$, *then for any integer* $R \ge 0$, *we have*

$$P_k f(u) \to (\lambda, f) \quad \textit{as} \quad k \to \infty \quad \textit{uniformly in} \quad u \in H(R). \tag{5.2}$$

In particular, convergence (5.2) *holds for* λ*-almost all* $u \in H$. *Finally, if all the constants* b_j *in* (1.3) *are non-zero, then* (5.2) *holds uniformly with respect to* $u \in H^s$, $\|u\|_s \le R$, *for any* $R \ge 0$.

Remark 5.2. The existence and uniqueness of a stationary measure and convergence (5.2) can be established under a weaker assumption. Namely, instead of (0.9), it suffices to assume that

$$\int_{-\infty}^{\infty} |r|^{20} p_j(r)\, dr \le C \quad \text{for all} \quad j \ge 1. \tag{5.3}$$

This assertion can be derived by analysing the arguments in Sections 1 – 5. We do not dwell on it and only show where the exponent 20 in (5.3) comes from.

When verifying condition (H_1), we used (see (4.26)) inequality (4.11) with $m = 3$, which, in turn, is based on the fact that the third moment of the random variable $T_\nu(\omega)$ (see Proposition 1.5) is finite. The mth moment of T_ν can be estimated by a constant depending only on N_{2m} and $\mathbb{E}|\eta_1|^{4(m+2)}$ (see (1.16) and (1.18)), and N_{2m} admits an estimate in terms of $\mathbb{E}\,|\eta_k|^{2m}$ (see (1.11)). For $m = 3$ we obtain the expression $\mathbb{E}\,|\eta_k|^{20}$, which can be estimated by the constant C in (5.3).

Proof of Theorem 5.1. The existence of a stationary measure satisfying (1.34) and the fact that $\lambda(H^s) = 1$ are established in Proposition 1.11. The uniqueness of such a measure follows from Theorem 2.2 and Corollary 4.3. Let us prove (5.2).

1) We begin with the case $f \in C_b(H)$. Let us define a function $\boldsymbol{f} \in C_b(\mathbf{H})$, $\mathbf{H} = H^{\mathbb{Z}_0}$, by the formula

$$\boldsymbol{f}(\boldsymbol{u}) = f(u_0), \quad \boldsymbol{u} = (u_l, l \le 0).$$

We recall that $\mathbf{P}_k$ and $\mathbf{P}_k^*$ stand for the Markov semigroups associated with the family (2.17), (2.18). It is clear that $\mathbf{P}_k \boldsymbol{f}(\boldsymbol{u}) = P_k f(u_0)$ for any $\boldsymbol{u} \in \mathbf{F}_\infty^s$. Let $\boldsymbol{\lambda} \in \mathcal{P}(\mathbf{F}_\infty^s)$ be the unique stationary measure for $\mathbf{P}_k^*$. By Corollary 4.3, we have

$$\mathbf{P}_k \boldsymbol{f}(u) \to (\boldsymbol{\lambda}, \boldsymbol{f}) \quad \text{as} \quad k \to \infty \quad \text{uniformly in} \quad \boldsymbol{u} \in \mathbf{F}_\infty^s(2M, R).$$

Since the projection $\boldsymbol{u} = (u_l, l \le 0) \mapsto u_0$ maps $\boldsymbol{\lambda}$ to λ, we conclude that $(\boldsymbol{\lambda}, \boldsymbol{f}) = (\lambda, f)$. Therefore (5.2) holds uniformly in $u \in H(R)$ for any $R \ge 0$.

2) To show that (5.2) remains valid for $f \in C(H, \beta_m)$, we use Lemma 1.8. Namely, for $L > 0$ let $h_L(r)$ denote a continuous function that is equal to 1 and 0 for $r \le L$ and $r \ge L + 1$, respectively. We take an arbitrary function $f \in C(H, \beta_m)$ and represent it in the form

$$f(u) = f_L(u) + g_L(u), \quad f_L(u) = h_L(|u|) f(u).$$

Since $f_L \in C_b(H)$, we conclude that

$$P_k f_L(u) \to (\lambda, f_L) \quad \text{as} \quad k \to \infty \quad \text{uniformly in} \quad u \in H(R).$$

It is easy to see that $(\lambda, f_L) \to (\lambda, f)$ as $L \to \infty$. Furthermore, we note that

$$\|g_L\|_{0,\beta_{m'}} \to 0 \quad \text{as} \quad L \to \infty,$$

where for any $m' > m$. By Lemma 1.8, the norm of the operators

$$P_k : C(H, \beta_m) \to C(H, \beta_{m'})$$

is bounded uniformly in $k \geq 1$. It follows that

$$\big|P_k g_L(u)\big| \to 0 \quad \text{as} \quad L \to \infty$$

uniformly in $k \geq 0$ and $u \in H(R) \subset B_H(R_1)$, $R_1 = (2M(R+1))^{1/2}$.

We now write

$$\big|P_k f(u) - (\lambda, f)\big| \leq \big|P_k f_L(u) - (\lambda, f_L)\big| + \big|P_k g_L(u)\big| + \big|(\lambda, f_L - f)\big|. \tag{5.4}$$

What has been said above implies that the right-hand side of (5.4) tends to zero as $k \to \infty$.

The fact that (5.2) holds also for $f \in C(H^s, \beta_m)$ follows from Lemma 1.8.

3) We now assume that $b_j \neq 0$ for all $j \geq 1$. To prove that (5.2) holds uniformly in $u \in H^s$, $\|u\|_s \leq R$, it suffices to show that the ball $B_{H^s}(R)$ is contained in $H_{R'}$ for some $R' \gg 1$. This assertion follows immediately from the definition of $\mathbf{F}^s_\infty(2M, R)$. □

Remark 5.3. If in Theorem 5.1 we assume that condition (B) is also satisfied, then convergence (5.2) holds for functions with exponential growth at infinity (see Main Theorem in the Introduction). Namely, it suffices to assume that $f(u)$ is a continuous function on H^s such that $|f(u)| \leq \text{const}\exp(\sigma\|u\|_s^{2\kappa_l})$, where l is the smallest integer no less than s, κ_l is the constant in Theorem 1.4 with $\rho = \infty$, and $\sigma > 0$ is sufficiently small. This assertion can easily be proved by repeating the above arguments and using Remark 1.9, and we shall not dwell on it.

As we saw above, convergence (5.2) is a simple consequence of Theorem 4.1. The following assertion shows that under the same conditions we have a much stronger result. Its proof requires some new ideas and will be presented in a subsequent publication.

Theorem 5.4. *Under the conditions of Theorem 5.1, if* (5.1) *is satisfied for a sufficiently large $N \geq 1$, then for any $f \in C(H^s, \beta_m)$, $m \geq 1$, and $R > 0$, we have*

$$P_k f(u) \to (\lambda, f) \quad \textit{as} \quad k \to \infty \quad \textit{uniformly in} \quad u \in B_H(R). \tag{5.5}$$

Moreover, if condition (B) *is also satisfied, then* (5.5) *holds for any function described in Remark 5.3.*

6 Appendix

6.1 Proof of Theorem 1.3

The existence and uniqueness of a solution are obvious, so that we confine ourselves to the proof of (1.11) and (1.12).

1) We begin with the case $s=0$. Taking the scalar product of (1.8) and $u(t)$ in H, we obtain

$$|S_t(u)| \le e^{-\alpha_1 \nu t}|u^0|, \quad t \ge 0. \tag{6.1}$$

Since

$$u_k = S(u_{k-1}) + \eta_k, \quad k \ge 1, \tag{6.2}$$

we conclude from inequality (6.1) with $t=1$ that, for any $\delta > 0$,

$$|u_k|^m \le (1+\delta)e^{-\nu\alpha_1 m}|u_{k-1}|^m + C_m \delta^{-(m-1)}|\eta_k|^m,$$

where the constant $C_m > 0$ depends on m solely. Choosing $q = e^{-\alpha_1 \nu}$ and $\delta = e^{(m-1)\alpha_1\nu} - 1$, we derive

$$|u_k|^m \le q|u_{k-1}|^m + C(m)\nu^{-(m-1)}|\eta_k|^m.$$

Taking the average and iterating the resulting inequality, we obtain (1.11).

2) We now consider the case $s > 0$. We shall need the following lemma, which is proved in Subsection 6.3.

Lemma 6.1. *The resolving operator S_t of the free NS system* (1.8) *is continuous from H to H^s for any $t > 0$ and $s \ge 0$. Moreover, for any integer $l \ge 2$ there is a constant $C_l \ge 1$ such that if $u(t,x)$ is a solution of* (1.8) *for $t \ge 0$, then*

$$t^l \|u(t)\|_l^2 + 2\nu \int_0^t \theta^l \|u(\theta)\|_{l+1}^2 d\theta \le \begin{cases} \nu^{-l}|u^0|^2, & l = 0, 1, \\ C_l\big(\nu^{-l}|u^0|^2 + \nu^{-5l}|u^0|^{2/\kappa_l}\big), & l \ge 2, \end{cases} \tag{6.3}$$

where $t \ge 0$, and the constant κ_l is defined in Theorem 1.4. Furthermore, for $l = 0$ the inequality sign in (6.3) *can be replaced by equality.*

To simplify notation, we confine ourselves to the case $s > 1$. Let us fix an arbitrary $k \ge 1$. In view of relation (6.2) and inequality (6.3) with $t = 1$, we have (the integer $l = l(s)$ is defined in Theorem 1.3)

$$\|u_k\|_s^m \le 2^{m-1}\big(\|S(u_{k-1})\|_l^m + \|\eta_k\|_s^m\big) \le C_{ml}\big(1 + \nu^{-5lm/2}|u_{k-1}|^{m(2l+1)} + \|\eta_k\|_s^m\big),$$

which implies (1.12).

6.2 Proof of Theorem 1.4

We confine ourselves to the case $s = 0$. It follows from (6.1) and (6.2) that, for any $\delta > 0$,

$$|u_k|^2 \le (1+\delta)e^{-2\alpha_1\nu}|u_{k-1}|^2 + (1+\delta^{-1})|\eta_k|^2. \tag{6.4}$$

We set $q = e^{-\alpha_1\nu}$, $\delta = e^{\alpha_1\nu} - 1$, and $\sigma_0 = \rho \wedge (a\alpha_1 e^{-\alpha_1})$. Inequality (6.4) implies that

$$\sigma_0\nu|u_k|^2 \le \sigma_0\nu q|u_{k-1}|^2 + a|\eta_k|^2,$$

and therefore, in view of independence of u_{k-1} and η_k, we have

$$\mathbb{E}\, e^{\sigma_0\nu|u_k|^2} \le \mathbb{E}\, e^{a|\eta_k|^2}\mathbb{E}\big(e^{\sigma_0\nu|u_{k-1}|^2}\big)^q \le \mathbb{E}\, e^{a|\eta_k|^2}\big(\mathbb{E}\, e^{\sigma_0|u_{k-1}|^2}\big)^q.$$

Arguing by induction on k, we derive (1.14).

6.3 Proof of Lemma 6.1

Inequality (6.3) is proved by induction on l. For $l = 0$, it is well known (see [CF]). We now fix an arbitrary $l = m \geq 1$ and assume that inequality (6.3) is established for $l < m$. Let us take the scalar product in H of equation (1.8) and the function $L^m u$. Performing some simple transformations, we derive

$$\partial_t\bigl(t^m\|u\|_m^2\bigr) - mt^{m-1}\|u\|_m^2 + 2\nu t^m\|u\|_{m+1}^2 + 2t^m\bigl(L^{\frac{m+1}{2}}u, L^{\frac{m-1}{2}}B(u,u)\bigr) = 0. \tag{6.5}$$

If $m = 1$, then the last term on the left-hand side of (6.5) vanishes, and the required inequality can be established by integration with respect to time. Therefore we assume that $m \geq 2$. In this case, we have the following estimate, which follows easily from Hölder's and interpolation inequalities:

$$\begin{aligned}\bigl|\bigl(L^{\frac{m+1}{2}}u, L^{\frac{m-1}{2}}B(u,u)\bigr)\bigr| &\leq c_m\bigl\|u\bigr\|_{m+1}^{\frac{4m-1}{2m}}\bigl\|u\bigr\|^{\frac{m+1}{2m}}\bigl|u\bigr|^{\frac{1}{2}} \\ &\leq \frac{\nu}{2}\bigl\|u\bigr\|_{m+1}^2 + c_m'\nu^{1-4m}\bigl\|u\bigr\|^{2(m+1)}|u|^{2m},\end{aligned} \tag{6.6}$$

where c_m and c_m' are positive constants. Substituting (6.6) into (6.5) and integrating in time, we obtain

$$\begin{aligned}&t^m\|u(t)\|_m^2 + \nu\int_0^t \theta^m\|u(\theta)\|_{m+1}^2 d\theta \leq \\ &\qquad \leq m\int_0^t \theta^{m-1}\|u(\theta)\|_m^2 d\theta + c_m'\nu^{1-4m}\int_0^t \theta^m\|u(\theta)\|^{2(m+1)}|u(\theta)|^{2m}d\theta.\end{aligned}$$

The required inequality follows now from the induction hypothesis.

6.4 Proof of Lemma 2.4

1) Let $u_i(t)$, $i = 1, 2$, be two solutions of the free NS system (1.8) with initial functions u_i^0. Then the difference $u = u_1 - u_2$ satisfies the equation (recall that $\nu = 1$)

$$\dot{u} + Lu + B(u, u_1) + B(u_2, u) = 0. \tag{6.7}$$

Let us take the scalar product of this equation with $2u(t)$ in H. Since

$$\bigl|(B(u,u_1),u)\bigr| \leq c_1|u|\,\|u\|\,\|u_1\| \leq \frac{1}{2}\|u\|^2 + \frac{c_1^2}{2}\|u_1\|^2|u|^2$$

and $(B(u_2,u),u) = 0$, we derive the differential inequality

$$\partial_t|u|^2 + \|u\|^2 \leq c_1^2\|u_1\|^2|u|^2 \tag{6.8}$$

Applying the Gronwall inequality, we obtain

$$|u(t)|^2 \leq \exp\biggl\{c_1^2\int_0^t \bigl\|u_1(\theta)\bigr\|^2 d\theta\biggr\}|u^0|^2, \quad u^0 = u_1^0 - u_2^0, \tag{6.9}$$

which coincides with (2.22). Integration of (6.8) now results in

$$\int_0^t \big\|u(\theta)\big\|^2 d\theta \le |u^0|^2 + c_1^2 \int_0^t \big\|u_1(\theta)\big\|^2 |u(\theta)|^2 d\theta$$
$$\le |u^0|^2 \bigg(1 + \int_0^t c_1^2 \|u_1(\theta)\big\|^2 e^{c_1^2 \int_0^\theta \|u_1(\sigma)\|^2 d\sigma} d\theta\bigg)$$
$$\le \exp\bigg\{c_1^2 \int_0^t \big\|u_1(\theta)\big\|^2 d\theta\bigg\} |u^0|^2. \tag{6.10}$$

2) We now take the scalar product of (6.7) with $2tLu(t)$ in H:

$$\partial_t\big(t\|u\|^2\big) + 2t|Lu|^2 = \|u\|^2 - 2t\big(B(u,u_1),Lu\big) - 2t\big(B(u_2,u),Lu\big). \tag{6.11}$$

Let us use the inequalities

$$\|v\|_\infty^2 \le c_2^2 |v|\,|Lv|, \quad \|v\|^2 \le |v|\,|Lv|$$

to estimate the second and third terms on the right-hand side of (6.11):

$$\big|(B(u,u_1),Lu)\big| \le \|u\|_\infty \|u_1\|\,|Lu| \le c_2 |u|^{1/2} |Lu|^{3/2} |u_1|^{1/2} |Lu_1|^{1/2}$$
$$\le \frac{1}{2}|Lu|^2 + c_2^4 |u|^2 |u_1|^2 |Lu_1|^2, \tag{6.12}$$
$$\big|(B(u_2,u),Lu)\big| \le \|u_2\|_\infty \|u\|\,|Lu| \le c_2 |u_2|^{1/2} |Lu_2|^{1/2} |u|^{1/2} |Lu|^{3/2}$$
$$\le \frac{1}{2}|Lu|^2 + c_2^4 |u|^2 |u_2|^2 |Lu_2|^2. \tag{6.13}$$

We now note that (see (6.1))

$$|u_i(t)| \le |u_i^0|, \quad t \ge 0, \quad i = 1,2. \tag{6.14}$$

Substituting (6.12) – (6.14) into (6.11) and integrating with respect to t, we derive

$$t\|u\|^2 \le \int_0^t \|u(\theta)\|^2 d\theta + 2c_2^4 \bigg\{ |u_1^0|^2 \int_0^t \theta |u|^2 |Lu_1|^2 d\theta + |u_2^0|^2 \int_0^t \theta |u|^2 |Lu_2|^2 d\theta \bigg\}. \tag{6.15}$$

To estimate the expression in the brackets on the right-hand side of (6.15), we apply inequalities (6.9) and (6.3) with $l = 1$:

$$|u_i^0|^2 \int_0^t \theta |u|^2 |Lu_i|^2 d\theta \le |u^0|^2 |u_i^0|^2 \exp\bigg\{c_1^2 \int_0^t \big\|u_1(\theta)\big\|^2 d\theta\bigg\} \int_0^t \theta |Lu_i|^2 d\theta$$
$$\le |u^0|^2 |u_i^0|^4 \exp\bigg\{c_1^2 \int_0^t \big\|u_1(\theta)\big\|^2 d\theta\bigg\}. \tag{6.16}$$

Furthermore, it follows from (6.1) and (6.3) with $l = 0$ that

$$|u_i^0|^2 \le 2\big(1 - e^{-2\alpha_1 t}\big)^{-1} \int_0^t \|u_i\|^2 d\theta \le c_3 (t^{-1} \vee 1) \int_0^t \big\|u_i(\theta)\big\|^2 d\theta. \tag{6.17}$$

Substitution of (6.17) into (6.16) results in

$$
\begin{aligned}
|u_i^0|^2 \int_0^t \theta |u|^2 |Lu_i|^2 d\theta \leq \\
&\leq c_3^2 (t^{-2} \vee 1) \biggl(\int_0^t \bigl\| u_i(\theta) \bigr\|^2 d\theta \biggr)^2 \exp\biggl\{ c_1^2 \int_0^t \|u_1(\theta)\|^2 d\theta \biggr\} |u^0|^2 \\
&\leq c_3^2 (t^{-2} \vee 1) \exp\biggl\{ \int_0^t \bigl(c_1^2 \|u_1(\theta)\|^2 + \|u_2(\theta)\|^2 \bigr) d\theta \biggr\} |u^0|^2. \qquad (6.18)
\end{aligned}
$$

The required inequality (2.23) follows now from (6.15), (6.10), and (6.18).

6.5 Lower bound for measures with positive density

Lemma 6.2. *Let γ be the distribution in H of the random variable*

$$\eta(x) = \sum_{j=1}^{\infty} b_j \xi_j e_j(x),$$

where b_j are real numbers satisfying condition (1.4) and ξ_j are independent scalar random variables whose distributions have strictly positive, continuous densities $p_j(r)$ with respect to the Lebesgue measure such that $\int_{\mathbb{R}} r^2 p_j(r)\, dr \leq C$ for all $j \geq 1$ and some constant $C > 0$ not depending on j. Then $\gamma(B) > 0$ for any open ball $B \subset \mathcal{H}^s$. Moreover, for any $p > s$, $R > 0$, and $r > 0$ there is $\varepsilon = \varepsilon(p, R, r) > 0$ such that $\gamma(B) \geq \varepsilon$ for any open ball $B \subset \mathcal{H}^s$ of radius r centred at a point $u^0 \in \mathcal{H}^p$, $\|u^0\|_p \leq R$.

Proof. We recall that $\mathcal{H}_L^s$ and $\mathcal{H}_L^{s\perp}$ denote the closed subspaces in $\mathcal{H}^s$ spanned by the vectors e_j, $j = 1, \dots, L-1$, and e_j, $j \geq L$, respectively, and that P_L and Q_L are the orthogonal projections in[5] $\mathcal{H}$ onto $\mathcal{H}_L$ and $\mathcal{H}_L^{\perp}$. It is clear that for any $u^0 \in \mathcal{H}^s$ and $r > 0$ we have

$$B_{\mathcal{H}^s}(u^0, r) \supset B_{\mathcal{H}_L^s}(v^0, r/\sqrt{2}) \times B_{\mathcal{H}_L^{s\perp}}(w^0, r/\sqrt{2}),$$

where $v^0 = \mathsf{P}_L u^0$, $w^0 = \mathsf{Q}_L u^0$, $\varphi = \mathsf{P}_L \eta$, $\psi = \mathsf{Q}_L \eta$, and $L \geq 2$ is an arbitrary integer. Since φ and ψ are independent, we conclude that

$$\mathbb{P}\bigl\{\eta \in B_{\mathcal{H}^s}(u^0, r)\bigr\} \geq \mathbb{P}\bigl\{\varphi \in B_{\mathcal{H}_L^s}(v^0, r/\sqrt{2})\bigr\} \mathbb{P}\bigl\{\psi \in B_{\mathcal{H}_L^{s\perp}}(w^0, r/\sqrt{2})\bigr\}. \quad (6.19)$$

Let us choose an integer $L \geq 2$ so large that

$$\bigl\|w^0\bigr\|_s \leq \frac{r}{2\sqrt{2}}, \qquad \sum_{j=L}^{\infty} b_j^2 \alpha_j^s < \frac{r^2}{8C}. \qquad (6.20)$$

Since $\mathcal{D}(\varphi)$ has a strictly positive continuous density with respect to Lebesgue measure, we conclude that the first factor on the right-hand side of (6.19) is

[5]In the case $s = 0$, we drop the index s from the notation of the spaces $\mathcal{H}^s$, $\mathcal{H}_L^s$, and $\mathcal{H}_L^{s\perp}$.

positive. To estimate the second factor, note that, in view of the first inequality in (6.20), we have

$$B_{\mathcal{H}_L^{s\perp}}(w^0, r/\sqrt{2}) \supset B_{\mathcal{H}_L^{s\perp}}(r/2\sqrt{2}).$$

Therefore,

$$\mathbb{P}\{\psi \in B_{\mathcal{H}_L^{s\perp}}(w^0, r/\sqrt{2})\} \geq \mathbb{P}\{\|\psi\|_s \leq r/2\sqrt{2}\} = 1 - \mathbb{P}\{\|\psi\|_s^2 \geq r^2/8\}. \quad (6.21)$$

By the second inequality in (6.20), we have

$$\mathbb{E}\,\|\psi\|_s^2 \leq C \sum_{j=L}^{\infty} b_j^2 \alpha_j^s < \frac{r^2}{8}.$$

The Chebyshev inequality now implies that the right-hand side of (6.21) is also positive.

To prove the second assertion, it suffices to note that the integer $L \geq 2$ satisfying (6.20) can be chosen uniformly with respect to the set of balls described in the statement of the lemma. □

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