Euler equations are not exactly controllable by a finite-dimensional external force*

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Abstract

We show that the Euler system is not exactly controllable by a finite-dimensional external force. The proof is based on the comparison of the Kolmogorov ε-entropy for Hölder spaces and for the class of functions that can be obtained by solving the 2D Euler equations with various right-hand sides.

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0 Introduction

Let us consider the controlled Euler system on the 2D torus $\mathbb{T}^2$:

$$\dot{u} + \langle u, \nabla \rangle u + \nabla p = \eta(t, x), \quad \operatorname{div} u = 0. \tag{0.1}$$

Here u and p are unknown velocity field and pressure, and η stands for a control force taking values in a finite-dimensional space $E \subset L^2(\mathbb{T}^2, \mathbb{R}^2)$. Equations (0.1) are supplemented with the initial condition

$$u(0, x) = u_0(x). \tag{0.2}$$

It was proved by Agrachev and Sarychev [AS06] that Eqs. (0.1) are approximately controllable in L^2 and exactly controllable in observed projections. More precisely, they constructed a six-dimensional subspace $E \subset C^\infty(\mathbb{T}^2, \mathbb{R}^2)$ such that the following properties hold for any $T > 0$:

Approximate controllability: For any divergence-free vector fields u_0 and $\hat{u}$ that belong to the Sobolev space $H^2(\mathbb{T}^2, \mathbb{R}^2)$ and any $\varepsilon > 0$ there is a smooth E-valued control $\eta(t)$ such that the solution u of problem (0.1), (0.2) satisfies the inequality $\|u(T) - \hat{u}\|_{L^2} < \varepsilon$.

Exact controllability in projections: For any subspace $F \subset H^2(\mathbb{T}^2, \mathbb{R}^2)$ of finite dimension, any divergence-free vector field $u_0 \in H^2(\mathbb{T}^2, \mathbb{R}^2)$, and any function $\hat{u} \in F$ there is a smooth E-valued control $\eta(t)$ such that $\mathsf{P}_F u(T) = \hat{u}$, where P_F denotes the orthogonal projection in L^2 onto the space F.

In view of the above results, an important question arises here: is it possible to prove the exact controllability for (0.1), or more generally, given an initial state u_0 and a control space E, what is the set of attainability at a time T, i. e., the family of functions $\mathcal{A}_T(u_0, E)$ that can be obtained at the time T by solving problem (0.1), (0.2)? Since the Euler system is time-reversible, a natural class of final states $\hat{u}$ for which one may wish to prove the exact controllability is dictated by the regularity of the initial state u_0 and the control η. Namely, let us denote by C^s the Hölder space of order s on the torus and by C^s_σ the space of divergence-free vector fields $u \in C^s$; see Notations below for the exact definition. Assume that the initial state u_0 and the control η are C^s-smooth with respect to the space variables. In this case, if $s > 1$, then the solution $u(t)$ belongs to C^s for any $t \geq 0$. Conversely, for any divergence-free vector field $\hat{u} \in C^s$ we can find $u_0 \in C^s$ such that the solution of (0.1), (0.2) with $\eta \equiv 0$ issued from u_0 coincides with $\hat{u}$ at $t = T$. Thus, it is reasonable to study the problem of exact controllability for the class of final states that are as regular as the initial function and the control. The following theorem, which is a simplified version of the main result of this paper, shows that the set of attainability is much smaller than the above-mentioned class of functions.

Main Theorem. *Let u_0 be an arbitrary divergence-free vector field belonging to the Hölder space C^s with a non-integer $s > 2$ and let $E \subset C^s$ be any finite-dimensional subspace. Then, for any $T > 0$, the complement in C^s_σ of the set of attainability $\mathcal{A}_T(u_0, E)$ is everywhere dense in C^s_σ.*

The proof of this theorem is based on two key observations. The first of them is the Lipschitz continuity of the resolving operator for (0.1), (0.2) with respect to the controls η endowed with the relaxation norm[1] (see Theorem 6 in [AS06] and Proposition 1.3 below). It is curious that this property is also crucial for proving the approximate controllability and exact controllability in projections [AS06]. The second key ingredient of the proof is an upper bound for the ε-entropy of the space of controls. Roughly speaking, we combine these two properties to establish an upper bound for the ε-entropy for set of attainability $\mathcal{A}_T(u_0, E)$ with given initial function $u_0 \in C^s_\sigma$ and control space $E \subset C^s$. It turns out that this upper bound is much smaller than the ε-entropy of C^s_σ, and the required property follows.

It should be mentioned that the above theorem is false in the case when E is the space of functions supported by a given domain $D \subset \mathbb{T}^2$. In this situation, it is well known that the Euler system is exactly controllable (see [Cor96] and [Gla00] for the 2D and 3D cases, respectively).

In conclusion, let us note that the Kolmogorov ε-entropy has proved to be an effective tool for studying various problems in analysis. For instance, we refer the reader to [Mit61, Lor66, Lor86, KH95, VC98, CE99, Zel01, CV02] for a number of applications of the ε-entropy in approximation theory, dynamical systems, and theory of attractors. This paper shows that it can also be used in the control theory for PDE's.

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Notations

Let X be a Banach space with a norm $\|\cdot\|_X$, let $J \subset \mathbb{R}$ be a finite closed interval, let $s > 0$ be a non-integer, and let $\mathbb{T}^d$ be the d-dimensional torus. We shall use the following function spaces.

$L^p(J, X)$ is the space of Bochner-measurable functions $f : J \to X$ such that

$$\|f\|_{L^p(J,X)} := \left(\int_J \|f(t)\|^p_X dt \right)^{1/p} < \infty.$$

In the case $p = \infty$, the above norm should be replaced by

$$\|f\|_{L^\infty(J,X)} := \operatorname*{ess\,sup}_{t \in J} \|f(t)\|_X.$$

[1]The relaxation norm of η is defined as the least upper bound of the norm for the integral of η with respect to time.

$W^{1,p}(J,X)$ stands the space of functions $f \in L^p(J,X)$ such that $\partial_t f \in L^p(J,X)$. It is endowed with the natural norm. In the case $X = \mathbb{R}$, we shall write $L^p(J)$ and $W^{1,p}(J)$.
$C(\mathbb{T}^d)$ is the space of continuous functions $u : \mathbb{T}^d \to \mathbb{R}^d$ with the norm

$$\|u\| := \sup_{x \in \mathbb{T}^d} |u(x)|.$$

$C^s(\mathbb{T}^d)$ is the Hölder class of order s with the norm

$$\|u\|_s := \max_{|\alpha| \le [s]} \|\partial^\alpha u\| + \max_{|\alpha|=[s]} \sup_{x \ne y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\gamma},$$

where ∂^α is a standard notation for derivatives, $[s]$ stands for the integer part of s, and $\gamma = s - [s]$.
$C^s_\sigma(\mathbb{T}^d)$ denotes the space of functions $u \in C^s(\mathbb{T}^d)$ such that $\operatorname{div} u \equiv 0$. In the case $d = 2$, we shall drop $\mathbb{T}^d$ from the notation and write C^s and C^s_σ.
We denote by $\langle a, b\rangle$ or $a \cdot b$ the usual scalar product of the vectors $a, b \in \mathbb{R}^2$ and by $C_1, C_2, \dots$ unessential positive constants.

1 Cauchy problem for Euler equations on the 2D torus

1.1 Existence and uniqueness of solution

Consider the Cauchy problem for the following Euler type system on the 2D torus $\mathbb{T}^2$:

$$\dot u + \langle u + z, \nabla\rangle(u+z) + \nabla p = f(t,x), \quad \operatorname{div} u = 0, \tag{1.1}$$

$$u(0,x) = u_0(x), \tag{1.2}$$

where z, f, and u_0 are given functions, and $\nabla = (\partial_1, \partial_2)$. Let us recall the concept of strong solution for (1.1), (1.2). We fix a time interval $J = [0,T]$ and a non-integer $s > 1$ and introduce the spaces

$$\mathcal{D}_T := C^s_\sigma \times W^{1,1}(J, C^s_\sigma) \times L^1(J, C^s), \quad \mathcal{X}_T := L^\infty(J, C^s) \cap W^{1,1}(J, C^{s-1}),$$

where the spaces C^s, C^s_σ and $W^{1,p}$ are defined in the Introduction (see Notations). The spaces $\mathcal{D}_T$ and $\mathcal{X}_T$ are endowed with natural norms.

Definition 1.1. Let $(u_0, z, f) \in \mathcal{D}_T$ be an arbitrary triple. A pair of functions (u,p) is called a *strong solution of the Cauchy problem for the Euler type system* (1.1) if u and p belong to the spaces $\mathcal{X}_T$ and $L^1(J, C^s)$, respectively, and Eqs. (1.1), (1.2) are satisfied in the sense of distributions.

Note that the function z is assumed to be more regular than the solution u. Our choice of the functional spaces is dictated by the class of problems we are interested in; see Step 2 in Section 3.2.

In what follows, when dealing with solutions of Eq. (1.1), we shall sometimes omit the function $p(t,x)$ and write simply $u(t,x)$. This will not lead to a confusion because p can be found, up to an additive function depending only on time, from the relation

$$\Delta p = \operatorname{div}\bigl(f - \langle u+z, \nabla\rangle(u+z)\bigr),$$

which is obtained by taking the divergence of the first equation in (1.1).

The following existence and uniqueness result is essentially due to Wolibner [Wol33] and Kato [Kat67] (see also [Gér92] for a concise presentation of the proofs).

Theorem 1.2. *For any non-integer $s > 1$, any time interval $J = [0,T]$ and an arbitrary triple $(u_0, z, f) \in \mathcal{D}_T$, problem (1.1), (1.2) has a unique solution $u \in \mathcal{X}_T$. Moreover, the resolving operator*

$$\mathcal{R} : \mathcal{D}_T \to \mathcal{X}_T, \quad (u_0, z, f) \mapsto u(t,x),$$

is bounded, that is, it maps bounded sets in $\mathcal{D}_T$ to bounded sets in $\mathcal{X}_T$.

Proof. In the case $z \equiv 0$, existence and uniqueness of solutions for (1.1), (1.2) is proved in [Kat67]. The general case can be reduced to the former by the change of unknown function $u = v - z$. Boundedness of the resolving operator follows easily from the proof of existence given in [Kat67]. □

1.2 Lipschitz continuity of the resolving operator

We now study continuity properties of the operator $\mathcal{R}$ constructed in Theorem 1.2. Let $B_{\mathcal{D}_T}(R)$ be the closed ball in $\mathcal{D}_T$ of radius R centred at origin. The following proposition is one of the two key points in the proof of non-controllability for the Euler equations. A similar result in the case of L^2-norm in the target space was established earlier by Agrachev and Sarychev [AS06].

Proposition 1.3. *For any positive constants T and R and any non-integer $s > 2$ there is $C = C(T,R,s) > 0$ such that*

$$\begin{aligned}\|\mathcal{R}(u_{01}, z_1, f_1) &- \mathcal{R}(u_{02}, z_2, f_2)\|_{L^\infty(J,C^{s-1})} \\ &\le C\left(\|u_{01}-u_{02}\|_{C^{s-1}} + \|z_1 - z_2\|_{L^1(J,C^s)} + \|f_1 - f_2\|_{L^1(J,C^{s-1})}\right), \quad (1.3)\end{aligned}$$

where (u_{0i}, z_i, f_i), $i = 1, 2$, are arbitrary triples belonging to the ball $B_{\mathcal{D}_T}(R)$.

Proof. Derivation of (1.3) is based on a well-known idea of reduction of the 2D Euler system to a nonlinear transport equation for the vorticity (e.g., see [Gér92]). For the reader's convenience, we give a detailed proof of the proposition. We shall confine ourselves to derivation of (1.3) for smooth solutions. The proof in the general case can be carried out by a standard approximation argument.

Let $u(t, x)$ be a smooth solution for (1.1), (1.2). Applying the operator $\nabla^\perp = (-\partial_2, \partial_1)$ to the first relation in (1.1) and to (1.2), we obtain

$$\dot v + \langle u + z, \nabla\rangle (v + \zeta) = g, \quad v(0, x) = v_0(x),$$

where $v = \nabla^\perp \cdot u$, $\zeta = \nabla^\perp \cdot z$, $g = \nabla^\perp \cdot f$, and $v_0 = \nabla^\perp \cdot u_0$. It follows that if u_i, $i = 1, 2$, are two smooth solutions associated with data (u_{0i}, z_i, f_i), then the function $v = \nabla^\perp (u_1 - u_2)$ is a solution of the problem

$$\dot v + \langle u_2 + z_2, \nabla\rangle v = g - \langle u_2 + z_2, \nabla\rangle \zeta - \langle u + z, \nabla\rangle (v_1 + \zeta_1), \tag{1.4}$$

$$v(0, x) = v_0(x), \tag{1.5}$$

where $u = u_1 - u_2$, $z = z_1 - z_2$, $\zeta = \nabla^\perp \cdot z$, $\zeta_i = \nabla^\perp \cdot z_i$, $g = \nabla^\perp \cdot (f_1 - f_2)$, and $v_0 = \nabla^\perp \cdot (u_{01} - u_{02})$. Thus, v is a solution of an inhomogeneous transport equation associated with the divergence-free vector field $u_2 + z_2$. It follows that

$$v(t, x) = v_0(U_{0,t}(x)) + \int_0^t h(\tau, U_{\tau,t}(x))\, d\tau, \tag{1.6}$$

where $U_{t,\tau}(x)$ denotes the flow defined by the vector field $u_2 + z_2$, and h stands for the right-hand side in (1.4). Let us denote by Δ^{-1} the inverse of the Laplace operator in the space of functions on $\mathbb{T}^2$ with zero mean value. Recalling that the functions u and v are connected by the relations $v = \nabla^\perp \cdot u$ and $u = Gv$, where $G = \nabla^\perp \Delta^{-1}$, from (1.6) we derive

$$u(t, x) = G\big[(\nabla^\perp u_0)(U_{0,t}(x))\big] + \int_0^t G\big[h(\tau, U_{\tau,t}(x))\big]\, d\tau. \tag{1.7}$$

Now note that $U_{t,\tau}(x)$, $t, \tau \in J$, are diffeomorphisms of the torus with uniformly bounded C^s-norms, and the function h can be written as

$$h = \nabla^\perp \cdot f - \operatorname{div}\big(\zeta(u_2 + z_2) - (v_1 + \zeta_1)(u + z)\big),$$

where $f = f_1 - f_2$. Since the operator $G : C^{s-1} \to C^s$ is bounded (see [GT01, Section 4.3], taking the C^{s-1}-norm of both sides in (1.7), we see that

$$\|u(t)\|_{s-1} \le C_1 \|u_0\|_{s-1} + C_1 \int_0^t \big(\|f\|_{s-1} + \|\zeta(u_2 + z_2) - (v_1 + \zeta_1)(u + z)\|_{s-1}\big)\, d\tau, \tag{1.8}$$

where $C_1 > 0$ depends only on R. The second term under the integral in (1.8) can be estimated by

$$\|\zeta\|_{s-1}\|u_2 + z_2\|_{s-1} + \|v_1 + \zeta_1\|_{s-1}\|u + z\|_{s-1} \le C_2\big(\|z\|_s + \|u\|_{s-1}\big).$$

Substituting this expression into (1.8), we obtain

$$\|u(t)\|_{s-1} \le C_1 \|u_0\|_{s-1} + C_3 \int_0^t \big(\|f\|_{s-1} + \|z\|_s + \|u\|_{s-1}\big)\, d\tau,$$

where C_3 is a constant depending only on R. Application of the Gronwall inequality gives the required estimate (1.3). $\square$

2 Kolmogorov ε-entropy

2.1 Definition and an elementary property

Let X be a Banach space and let $K \subset X$ be a compact subset. Let us recall the concept of ε-entropy, which characterises the "massiveness" of K (e.g., see [Lor86]). For any $\varepsilon > 0$, we denote by $N_\varepsilon(K)$ the minimal number of sets of diameters $\leq 2\varepsilon$ that are needed to cover K. The *Kolmogorov ε-entropy* (or simply ε-entropy) of K is defined as $H_\varepsilon(K) = \ln N_\varepsilon(K)$. Thus, the ε-entropy of a compact set $K \subset X$ is a non-increasing function of $\varepsilon > 0$, and it is easy to see that $H_\varepsilon(K)$ depends only on the metric on K (and not on the ambient space X). If we wish to emphasise that K is endowed with the norm of X, then we shall write $H_\varepsilon(K, X)$.

Now let Y be another Banach space and let $f : K \to Y$ be a Lipschitz-continuous function:

$$\|f(u_1) - f(u_2)\|_Y \leq L\|u_1 - u_2\|_X \quad \text{for } u_1, u_2 \in K, \tag{2.1}$$

where $L > 0$ is a constant. The following lemma is a straightforward consequence of the definition.

Lemma 2.1. *For any compact set $K \subset X$ and any function $f : K \to Y$ satisfying inequality* (2.1)*, we have*

$$H_\varepsilon(f(K)) \leq H_{\varepsilon/L}(K) \quad \textit{for all } \varepsilon > 0. \tag{2.2}$$

2.2 Estimates for the ε-entropy of some compact sets

Let φ_1 and φ_2 be two non-increasing functions of $\varepsilon > 0$. We shall write $\varphi_1 \prec \varphi_2$ if there are positive constants C and ε_0 such that

$$\varphi_1(\varepsilon) \leq C\varphi_2(\varepsilon) \quad \text{for } 0 < \varepsilon \leq \varepsilon_0.$$

If $\varphi_1 \prec \varphi_2$ and $\varphi_2 \prec \varphi_1$, then we write $\varphi_1 \sim \varphi_2$. The second key ingredient of the proof of non-controllability for the Euler system is given by the following two propositions.

Proposition 2.2. *Let $r < s$ be positive non-integers such that $s - r \notin \mathbb{Z}$ and let $B \subset C^s_\sigma(\mathbb{T}^d)$ be an arbitrary closed ball. Then, for any $\delta > 0$, we have*

$$H_\varepsilon(B, C^r(\mathbb{T}^d)) \succ \Big(\frac{1}{\varepsilon}\Big)^{\frac{d}{s-r} - \delta}. \tag{2.3}$$

Proof. Step 1. Suppose we have shown that if Q is a closed ball in $C^q_\sigma(\mathbb{T}^d)$ with a non-integer $q > 0$, then

$$H_\varepsilon(Q, C(\mathbb{T}^d)) \sim \Big(\frac{1}{\varepsilon}\Big)^{\frac{d}{q}}. \tag{2.4}$$

Since $C^\nu(\mathbb{T}^d)$ is continuously embedded in $C(\mathbb{T}^d)$ for any $\nu > 0$, it follows from (2.4) that if $A \subset C_\sigma^{s-r+\nu}(\mathbb{T}^d)$ is any closed subset with non-empty interior and $s - r + \nu \notin \mathbb{Z}$, then

$$H_\varepsilon(A, C^\nu(\mathbb{T}^d)) \succ \left(\frac{1}{\varepsilon}\right)^{\frac{d}{s-r+\nu}}. \tag{2.5}$$

Furthermore, if $\nu \notin \mathbb{Z}$, then the operator $(1-\Delta)^{-(r-\nu)/2}$ (where Δ is the Laplacian) defines an isomorphism from $C^\nu(\mathbb{T}^d)$ to $C^r(\mathbb{T}^d)$ and from $C_\sigma^{s-r+\nu}(\mathbb{T}^d)$ to $C_\sigma^s(\mathbb{T}^d)$ (see [GT01, Section 4.3]). Combining this with relation (2.5) and Lemma 2.1, we see that

$$H_\varepsilon(B, C^r(\mathbb{T}^d)) \succ \left(\frac{1}{\varepsilon}\right)^{\frac{d}{s-r+\nu}}, \tag{2.6}$$

where B is an arbitrary closed ball in $C_\sigma^s(\mathbb{T}^d)$. It remains to note that the left-hand side of (2.6) does not depend on the parameter $\nu > 0$, which can be chosen arbitrarily small.

Step 2. We now prove (2.4). Since $C_\sigma^q(\mathbb{T}^d)$ is a closed subspace in $C^q(\mathbb{T}^d)$, it follows from Theorem 3 in Section 10.2 of [Lor86] that

$$H_\varepsilon(Q, C(\mathbb{T}^d)) \prec \left(\frac{1}{\varepsilon}\right)^{\frac{d}{q}}.$$

Let us prove the converse inequality

$$H_\varepsilon(Q, C(\mathbb{T}^d)) \succ \left(\frac{1}{\varepsilon}\right)^{\frac{d}{q}}. \tag{2.7}$$

Abusing slightly the notation, we shall denote by the same symbol the spaces of scalar and vector functions.

Let $\Sigma^q(\mathbb{T}^d)$ be the closure of the space of vector functions $u \in C^q(\mathbb{T}^d)$ that are representable in the form $u = (\partial_2 v, -\partial_1 v, 0, \dots, 0)$ for some scalar function $u \in C^{q+1}(\mathbb{T}^d)$. Suppose we have shown that

$$H_\varepsilon(Q, C(\mathbb{T}^d)) \succ \left(\frac{1}{\varepsilon}\right)^{\frac{d}{q}} \quad \text{for any closed ball } Q \subset \Sigma^q(\mathbb{T}^d). \tag{2.8}$$

Since $\Sigma^q(\mathbb{T}^d)$ is a closed subspace in $C_\sigma^q(\mathbb{T}^d)$, we conclude that, for any closed ball $Q \subset C_\sigma^q(\mathbb{T}^d)$ centred at origin,

$$H_\varepsilon(Q, C(\mathbb{T}^d)) \geq H_\varepsilon(Q \cap \Sigma^q(\mathbb{T}^d), C(\mathbb{T}^d)) \succ \left(\frac{1}{\varepsilon}\right)^{\frac{d}{q}}.$$

Recalling that the ε-entropy is invariant with respect to translations, we conclude that (2.7) is valid for any closed ball $Q \subset C_\sigma^q(\mathbb{T}^d)$.

Step 3. To prove (2.8), write $x = (x_1, \dots, x_d) = (x_1, x')$ and denote by $\dot{C}^q(\mathbb{T}^d)$ the space of scalar functions $u \in C^q(\mathbb{T}^d)$ such that

$$\int_0^{2\pi} u(x_1, x')\, dx_1 = 0 \quad \text{for any } x' \in \mathbb{T}^{d-1}.$$

We claim that $C^q(\mathbb{T}^d)$ is representable as the direct sum

$$C^q(\mathbb{T}^d) = \dot{C}^q(\mathbb{T}^d) \dotplus C^q(\mathbb{T}^{d-1}), \tag{2.9}$$

where the space $C^q(\mathbb{T}^{d-1})$ is taken with respect to the variables x'. Indeed, any scalar function $u \in C^q(\mathbb{T}^d)$ can be written as

$$u(x) = \tilde{u}(x) + v(x'), \quad \tilde{u}(x) = u(x) - \frac{1}{2\pi} \int_0^{2\pi} u(y, x')\, dy.$$

Furthermore, it is straightforward to see that the above representation is unique.

Now let Q_1 and Q_2 be closed balls in the spaces $\dot{C}^q(\mathbb{T}^d)$ and $C^q(\mathbb{T}^{d-1})$, respectively. By Theorem 3 in Section 10.2 of [Lor86], we have

$$H_\varepsilon(Q_2, C(\mathbb{T}^d)) \sim \Big(\frac{1}{\varepsilon}\Big)^{\frac{d-1}{q}}, \quad H_\varepsilon(Q_1 \times Q_2, C(\mathbb{T}^d)) \sim \Big(\frac{1}{\varepsilon}\Big)^{\frac{d}{q}}. \tag{2.10}$$

On the other hand, it follows from representation (2.9) and inequality (7) in Section 10.1 of [Lor86] that

$$H_\varepsilon(Q_1 \times Q_2, C(\mathbb{T}^d)) \le H_\varepsilon(Q_1, C(\mathbb{T}^d)) + H_\varepsilon(Q_2, C(\mathbb{T}^d)). \tag{2.11}$$

Comparing (2.10) and (2.11), we see that

$$H_\varepsilon(Q_1, C(\mathbb{T}^d)) \sim \Big(\frac{1}{\varepsilon}\Big)^{\frac{d}{q}} \quad \text{for any closed ball } Q_1 \subset \dot{C}^q(\mathbb{T}^d). \tag{2.12}$$

Step 4. We can now easily complete the proof of (2.8). Let us denote by $\Pi : (u_1, \dots, u_d) \mapsto u_2$ the projection to the second component. Using a standard approximation argument, it is easy to prove that $\Pi\Sigma^q(\mathbb{T}^d) = \dot{C}^q(\mathbb{T})$. It follows that the projection ΠQ of any closed ball $Q \subset \Sigma^q(\mathbb{T}^d)$ contains a closed ball $Q_1 \subset \dot{C}^q(\mathbb{T})$, and therefore, by inequality (2.12) and Lemma 2.1, we have

$$H_\varepsilon(Q, C(\mathbb{T}^d)) \succ H_\varepsilon(Q_1, C(\mathbb{T}^d)) \succ \Big(\frac{1}{\varepsilon}\Big)^{\frac{d}{q}}.$$

This completes the proof of the proposition. $\square$

The following proposition is a particular case of more general results established in [BS67]. For the reader's convenience, we give a complete proof of the estimate we need.

Proposition 2.3. *Let $J = [0, T]$ and let E be a finite-dimensional vector space. Then, for any closed ball $B \subset W^{1,1}(J, E)$, we have*

$$H_\varepsilon(B, L^1(J, E)) \prec \frac{1}{\varepsilon} \ln \frac{1}{\varepsilon}. \tag{2.13}$$

Proof. We first note that it suffices to prove (2.13) for scalar functions. Indeed, if E is an n-dimensional vector space, then B is a subset of the direct product of n balls $B_1 \subset W^{1,1}(J)$. If (2.13) is established in the case $\dim E = 1$, then

$$H_{n\varepsilon}(B, W^{1,1}(J,E)) \leq nH_\varepsilon(B_1, W^{1,1}(J)) \leq \frac{Cn}{\varepsilon} \ln \frac{1}{\varepsilon};$$

see inequality (7) in Section 10.1 of [Lor86]. Replacing $n\varepsilon$ by ε in the above estimate, we obtain (2.13).

We now prove (2.13) for scalar functions. Without loss of generality, we can assume that $J = [0,1]$ and $B \subset W^{1,1}(J)$ is a closed ball of radius R centred at zero. Let us fix $\varepsilon > 0$ and describe a finite family of functions $\mathcal{F} \subset L^1(J)$ that form an ε-net for B. To this end, we choose sufficiently large integers L and M and denote by I_k the interval $[t_{k-1}, t_k)$, where $t_k = k/L$. The family $\mathcal{F}$ consists of all functions $f \in L^1(J)$ that are constant on every interval I_k, $k = 1, \dots, L$, and take one of the values $2jR/M$, $j = -M, \dots, M$, on each interval of constancy. It is clear that $\mathcal{F}$ consists of $N(L,M) := (2M+1)^L$ elements. Let us show that, for an appropriate choice of L and M, the family $\mathcal{F}$ is an ε-net for B.

We first note that

$$\|u\|_{L^\infty(J)} \leq 2R \quad \text{for any } u \in B. \tag{2.14}$$

Furthermore,

$$|u(t) - u(t_{k-1})| \leq \int_{t_{k-1}}^t |\dot{u}(\tau)|\, d\tau \quad \text{for } t \in I_k,$$

whence it follows that

$$\begin{aligned} \sum_{k=1}^L \int_{I_k} |u(t) - u(t_{k-1})|\, dt &\leq \sum_{k=1}^L \int_{I_k} \int_{t_{k-1}}^t |\dot{u}(\tau)|\, d\tau\, dt \\ &\leq \sum_{k=1}^L \int_{I_k} |\dot{u}(\tau)|(t_k - \tau)\, d\tau \\ &\leq L^{-1} \|\dot{u}\|_{L^1(J)} \leq RL^{-1}. \end{aligned} \tag{2.15}$$

In view of (2.14), for any L-tuple $(u_0, \dots, u_{L-1})$ there is $f \in \mathcal{F}$ such that

$$|f(t) - u_{k-1}| \leq 2RM^{-1} \quad \text{for } t \in I_k,\ k = 1, \dots, L. \tag{2.16}$$

Combining inequalities (2.15) and (2.16), in which $u_k = u(t_k)$, we obtain

$$\begin{aligned} \int_0^1 |u(t) - f(t)|\, dt &= \sum_{k=1}^L \int_{I_k} |u(t) - f(t)|\, dt \\ &\leq \sum_{k=1}^L \int_{I_k} \big(|u(t) - u(t_{k-1})| + |u(t_{k-1}) - f(t)|\big)\, dt \\ &\leq RL^{-1} + 2RLM^{-1}. \end{aligned} \tag{2.17}$$

Let us set

$$L = \big[2R/\varepsilon\big] + 1, \quad M = \big[4RL/\varepsilon\big] + 1, \tag{2.18}$$

where $[a]$ stands for the integer part of $a \geq 0$. In this case, it follows from (2.17) that

$$\|u - f\|_{L^1(J)} \leq \varepsilon.$$

Thus, the family $\mathcal{F}$ is an ε-net for B.

Let us estimate the number of elements in $\mathcal{F}$. Relations (2.18) imply that

$$N(L, M) = (2M+1)^L \leq \big(C_1\varepsilon^{-2}\big)^{C_2\varepsilon^{-1}} \leq \exp\big(C_3\varepsilon^{-1}\ln\varepsilon^{-1}\big).$$

Taking the logarithm, we arrive at the required estimate (2.13). $\square$

3 Main result

3.1 Formulation

Let us fix a time interval $J = [0, T]$ and consider the controlled 2D Euler system on the domain $J \times \mathbb{T}^2$:

$$\dot{u} + \langle u, \nabla\rangle u + \nabla p = h(t, x) + \eta(t, x), \quad \operatorname{div} u = 0, \tag{3.1}$$

$$u(0, x) = u_0(x). \tag{3.2}$$

Here h and u_0 are given functions, and η is a control. In what follows, we fix a non-integer $s > 2$ and assume that $h \in L^1(J, C^s)$ and $u_0 \in C^s$. Let $E \subset C^s$ be a closed subspace and let $\mathcal{K} \subset C^s_\sigma$ be any subset.

Definition 3.1. We shall say that the 2D Euler system with given external force $h \in L^1(J, C^s)$ and initial function $u_0 \in C^s_\sigma$ is *exactly controllable in time T for the class $\mathcal{K}$* if for any $\hat{u} \in \mathcal{K}$ there is $\eta \in L^1(J, E)$ such that

$$u(T, x) = \hat{u}(x),$$

where $u \in \mathcal{X}_T$ stands for the solution of (3.1), (3.2).

Let us give an equivalent definition of exact controllability in terms of the set of attainability. Let us denote by $\mathcal{R}_t(u_0, f)$ the operator that takes the pair $(u_0, f) \in C^s_\sigma \times L^1(J, C^s)$ to $u(t) \in C^s$, where $u \in \mathcal{X}_T$ stands for the solution of problem (1.1), (1.2) with $z \equiv 0$. For given u_0 and h, let $\mathcal{A}_T(u_0, h, E)$ be the image of $L^1(J, E)$ under the mapping $\mathcal{R}_T(u_0, h + \cdot)$. It is clear that the Euler system is exactly controllable in time T for a class $\mathcal{K} \subset C^s_\sigma$ if and only if $\mathcal{A}_T(u_0, h, E) \supset \mathcal{K}$.

Let $\mathcal{A}^c_T(u_0, h, E)$ be the complement of $\mathcal{A}_T(u_0, h, E)$ in the space C^s_σ. The following theorem is the main result of this paper.

Theorem 3.2. *Let $s > 2$ be any non-integer, let $E \subset C^s$ be an arbitrary finite-dimensional subspace, and let $u_0 \in C^s_\sigma$ and $h \in L^1(J, C^s)$ be given functions. Then, for any non-negative $\gamma < 1$ and any ball $Q \subset C^{s+\gamma}_\sigma$, we have*

$$\mathcal{A}^c_T(u_0, h, E) \cap Q \neq \varnothing. \tag{3.3}$$

In particular, the 2D Euler system is not exactly controllable in any time T for the class $C^{s+\gamma}_\sigma$.

Note that the above result is stronger than the Main Theorem announced in the Introduction, because Eqs. (3.1) contain an extra term $h(t,x)$, and the complement of the set of attainability is proved to be dense in the more regular space $C^{s+\gamma}_\sigma$.

3.2 Proof of Theorem 3.2

Step 1. We first show that it suffices to consider the case $E \subset C^s_\sigma$. Indeed, let us denote by Π the Leray projection, that is,

$$\Pi u = u - \nabla\bigl(\Delta^{-1}(\operatorname{div} u)\bigr).$$

The above relation and the continuity of Δ^{-1} in Hölder spaces (see [GT01]) imply that Π is a continuous operator from C^s to C^s_σ. It is well known that

$$\mathcal{R}_T(u_0, f) = \mathcal{R}_T(u_0, \Pi f),$$

whence it follows that $\mathcal{A}_T(u_0, h, E) = \mathcal{A}_T(u_0, h, \Pi E)$. Thus, if relation (3.3) is established for any finite-dimensional subspace $E \subset C^s_\sigma$, then it remains true in the general case.

Step 2. We now assume that E is a finite-dimensional subspace in C^s_σ. Let us write solutions of (3.1), (3.2) in the form

$$u(t,x) = v(t,x) + z(t,x), \quad z(t,x) = \int_0^t \eta(\tau, x)\, d\tau. \tag{3.4}$$

In this case, the function v belongs to the space $\mathcal{X}_T$ and satisfies the equations

$$\dot v + \langle v + z, \nabla\rangle(v+z) + \nabla p = h(t,x), \quad \operatorname{div} v = 0, \quad v(0,x) = u_0(x). \tag{3.5}$$

In view of Theorem 1.2, for any $z \in W^{1,1}(J, E)$, problem (3.5) has a unique solution $v \in \mathcal{X}_T$. Let us denote by $\mathcal{S} : W^{1,1}(J,E) \to \mathcal{X}_T$ the operator that takes z to v and by $\mathcal{S}_T$ its restriction to the time T. It follows from (3.4) that we can write the solution of (3.1), (3.2) at the time $t = T$ in the form

$$\mathcal{R}_T(u_0, h + \eta) = z(T) + \mathcal{S}_T(z), \tag{3.6}$$

where z is given by the second relation in (3.4).

To prove (3.3), we argue by contradiction. Suppose that $\mathcal{A}_T(u_0, h, E)$ contains a closed ball $Q \subset C^{s+\gamma}_\sigma$. In this case, it follows from (3.6) that the image of the space $E \times W^{1,1}(J,E)$ under the mapping $K(y,z) := y + \mathcal{S}_T(z)$ contains Q. Let us write

$$E \times W^{1,1}(J,E) = \bigcup_{n=1}^{\infty} B_n, \tag{3.7}$$

where B_n denotes the closed ball in $E \times W^{1,1}(J,E)$ of radius n centred at zero. Since the union of $K(B_n)$ covers Q, by the Baire theorem, there is an integer $m \geq 1$ such that $K(B_m)$ is dense in a ball $\widehat{Q} \subset Q$ with respect to the metric of $C^{s+\gamma}$. Furthermore, Proposition 1.3 implies that the mapping K is continuous from $E \times L^1(J,E)$ to C^{s-1}. Now note B_m is compact in $E \times L^1(J,E)$. It follows that $K(B_m)$ is closed in C^{s-1} and, hence, $K(B_m) \cap C^{s+\gamma}$ is closed in $C^{s+\gamma}$. Thus, $K(B_m)$ contains $\widehat{Q}$. On the other hand, we shall show in the next step that

$$H_\varepsilon(K(B_m), C^{s-1}) \prec \varepsilon^\nu H_\varepsilon(\widehat{Q}, C^{s-1}), \tag{3.8}$$

where $\nu > 0$. This contradicts the inclusion $\widehat{Q} \subset K(B_m)$.

Step 3. Without loss of generality, we can assume that $s + \gamma \notin \mathbb{Z}$. By Proposition 2.2, for any $\delta > 0$, we have

$$H_\varepsilon(\widehat{Q}, C^{s-1}) \succ \left(\frac{1}{\varepsilon}\right)^{\frac{2}{1+\gamma} - \delta}.$$

Let us choose $\delta > 0$ so small that the exponent in the right-hand side of the above relation is bigger than 1. Thus, we can find $\alpha > 1$ such that

$$H_\varepsilon(\widehat{Q}, C^{s-1}) \succ \left(\frac{1}{\varepsilon}\right)^{\alpha}. \tag{3.9}$$

On the other hand, let us endow B_m with the metric of $E \times L^1(J,E)$. Since E is finite-dimensional, for any ball $B \subset E$, we have (see Theorem 2 in Section 10.1 of [Lor86])

$$H_\varepsilon(B, E) \sim \ln \frac{1}{\varepsilon}.$$

Combining this with Proposition 2.3, we see that

$$H_\varepsilon(B_m, E \times L^1(J,E)) \prec \frac{1}{\varepsilon} \ln \frac{1}{\varepsilon}. \tag{3.10}$$

It follows from Proposition 1.3 that the mapping K is Lipschitz-continuous from B_m to C^{s-1}. Relation (3.10) and Lemma 2.1 now imply that

$$H_\varepsilon(K(B_m), C^{s-1}) \prec \frac{1}{\varepsilon} \ln \frac{1}{\varepsilon}. \tag{3.11}$$

The required inequality (3.8) is a consequence of (3.9) and (3.11). The proof of the theorem is complete.

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