Dedicated to Leonid Romanovich Volevich on the occasion of his seventieth birthday

Ergodicity for a Class of Markov Processes and Applications to Randomly Forced PDE's. I

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Abstract. The paper is devoted to studying the problem of ergodicity for dissipative PDE's perturbed by an external random force. We give a simple sufficient condition for uniqueness and stability of a stationary measure for a family of Markov processes. The result is applied to the 2D Navier–Stokes system.

0. INTRODUCTION

The problem of ergodicity for randomly forced PDE's of mathematical physics was in the focus of attention during the last ten years. The first result in this direction was obtained by Flandoli and Maslowski [FM95], who studied the 2D Navier–Stokes (NS) system perturbed by a random force that is white in time and sufficiently irregular in the space variables x. They proved that the Markov family associated with the above problem has a unique stationary measure, and all other solutions converge to it in the total variation norm. The method used in [FM95] is based on the strong Feller property, which does not hold for forcing smooth with respect to x. That case was first studied by Kuksin and the author [KS00]. We considered a large class of parabolic PDE's (including the 2D NS system and the complex Ginzburg–Landau equation) perturbed by a random force discrete in time and smooth in x. Under the assumption that the perturbation is sufficiently nondegenerate, it was proved that there is a unique stationary measure, which satisfies a mixing property. E, Mattingly, Sinai [EMS01] and Bricmont, Kupiainen, Lefevere [BKL02] studied later the NS system on the 2D torus. Assuming that the right-hand side is a white noise in time and a trigonometric polynomial in x of sufficiently high degree, they established the uniqueness of a stationary measure. Moreover, it was proved in [BKL02] that the model in question possesses the property of exponential mixing. Eckmann and Hairer [EH01] constructed an infinite-dimensional version of the Malliavin calculus and established the ergodicity for the real Ginzburg–Landau equation perturbed by a rough degenerate forcing. The above results were further developed in a number of works, and we refer the reader to the reviews [Bri02,Kuk02,Shi03] for more references and details.

The objective of this paper is to give a simple sufficient condition for ergodicity of Markov processes and to show that it applies to a class of randomly forced PDE's. Without going into details, we now outline the main idea, which is a variation of a classical coupling construction (for instance, see [Lin92,Ver87]). The scheme given below is not entirely accurate; however, it contains the essential points of the proof.

Let X be a separable Banach space with a norm $\|\cdot\|_X$, and let $(u_t, \mathbb{P}_u)$, $t \geqslant 0$, $u \in X$, be a Feller family of Markov processes in X. Here u_t is the trajectory of the process and $\mathbb{P}_u$ stands for the probability measure corresponding to the initial condition $u \in X$. Let $(u'_t, \mathbb{P}_{u'})$ be another (independent) copy of the above family. Suppose that we can prove the following assertions.

(1) The stopping time $\tau_\varepsilon = \min\{t \geqslant 0 : \|u_t - u'_t\|_X \leqslant \varepsilon\}$ is almost surely finite for any initial points $u, u' \in X$ and any $\varepsilon > 0$.

(2) There is a function $\delta(\varepsilon) \geqslant 0$ defined for $\varepsilon > 0$ and going to zero with ε such that, for any $u, u' \in X$ satisfying the inequality $\|u - u'\|_X \leqslant \varepsilon$, we have

$$\left\|\mathbb{P}_u\{u_t \in \cdot\} - \mathbb{P}_{u'}\{u'_t \in \cdot\}\right\|_{\mathcal{L}}^* \leqslant \delta(\varepsilon) \quad \text{for any } t \geqslant 0,$$

where $\|\cdot\|_{\mathcal{L}}^*$ stands for the dual Lipschitz norm on the space of probability measures on X

(see the notation below) and $\mathbb{P}\{\xi \in \cdot\}$ for the distribution of a random variable ξ under the law $\mathbb{P}$.

If these properties hold, then, using the strong Markov property, one can show that

$$\big\|\mathbb{P}_u\{u_t \in \cdot\} - \mathbb{P}_{u'}\{u'_t \in \cdot\}\big\|_{\mathcal{L}}^* \to 0 \quad \text{as} \quad t \to \infty \quad \text{for any} \quad u, u' \in X.$$

This implies the ergodicity of the Markov family under study.

In the present paper, we show that the above idea applies to the 2D NS system (see Section 1.2 for the exact formulation of the abstract result). Let us note that the idea of coupling was used earlier in [KS01, KPS02, Mat02, MY02, Hai02, KS02] and enabled one to prove the property of exponential mixing for various randomly forced PDE's, including the 2D NS system perturbed by white noise force. Our result gives no estimate for the rate of mixing. However, the proof presented here is much shorter, and we deal with a more general case in which the attractor of the unperturbed problem is nontrivial and the random force has a minimal regularity with respect to the space variables (see Remarks 2.4 and 2.5). In the second part of this work, we shall show that the same abstract result applies to other randomly forced PDE's.

The paper is organized as follows. Section 1 is devoted to an abstract theorem that gives a sufficient condition for ergodicity of Markov processes. In Section 2, we first recall a well-known result on the initial-boundary value problem for 2D NS equations and then formulate the main theorem on the uniqueness and mixing property. The proofs are given in Section 3.

Notation

Let X be a separable Banach space with a norm $\|\cdot\|_X$. We use the following notation:
$B_X(R)$ is the ball in X of radius R centered at zero;
$\mathcal{B}(X)$ is the Borel σ-algebra of X;
$\mathcal{P}(X)$ is the family of probability measures on $(X, \mathcal{B}(X))$;
$C_b(X)$ is the space of bounded continuous functions $f\colon X \to \mathbb{R}$; it is endowed with the norm

$$\|f\|_\infty := \sup_{u \in X} |f(u)|.$$

If $f \in C_b(X)$ and $\mu \in \mathcal{P}(X)$, then we denote by (f, μ) the integral of f over X with respect to μ.
$\mathcal{L}(X)$ is the space of functions $f \in C_b(X)$ such that

$$\|f\|_{\mathcal{L}} := \|f\|_\infty + \sup_{u \neq v} \frac{|f(u) - f(v)|}{\|u - v\|_X}.$$

The space of probability measures $\mathcal{P}(X)$ is endowed with the weak* topology, which is equivalent to the topology defined by the dual Lipschitz norm:

$$\|\mu_1 - \mu_2\|_{\mathcal{L}}^* := \sup_{\|f\|_{\mathcal{L}} \leqslant 1} \big|(f, \mu_1) - (f, \mu_2)\big|.$$

We shall also need the total variation norm, which is defined by the formula

$$\|\mu_1 - \mu_2\|_{\text{var}} := \sup_{\Gamma \in \mathcal{B}(X)} |\mu_1(\Gamma) - \mu_2(\Gamma)|.$$

If ξ is a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then we denote by $\mathcal{D}(\xi)$ or $\mathbb{P}\{\xi \in \cdot\}$ the distribution of ξ.

1. MAIN THEOREM

1.1. *Preliminaries*

Let $(u_t, \mathbb{P}_u)$, $t \geqslant 0$, be a Feller family of Markov processes defined on a measurable space $(\Omega, \mathcal{F})$ with range in a separable Banach space X. We always assume that the trajectories $u_t(\omega)$ are continuous for all $\omega \in \Omega$. Let $P(t, u, \Gamma)$ be the transition function associated with $(u_t, \mathbb{P}_u)$,

$$P(t, u, \Gamma) = \mathbb{P}_u\{u_t \in \Gamma\}, \quad u \in X, \quad \Gamma \in \mathcal{B}(X),$$

and let P_t and P_t^* be the corresponding Markov operators,

$$P_t f(u) = \int_X P(t, u, dv) f(v), \quad P_t^* \mu(\Gamma) = \int_X P(t, v, \Gamma) \mu(dv),$$

where $f \in C_b(X)$, $\mu \in \mathcal{P}(X)$, and $\Gamma \in \mathcal{B}(X)$. In what follows, we shall need a concept of extension for Markov processes.

Let $\boldsymbol{X} = X \times X$ be the direct product of two copies of X. We write $\boldsymbol{u} = (u, u') \in \boldsymbol{X}$ and denote by

$$\Pi \colon \boldsymbol{u} \mapsto u, \quad \Pi' \colon \boldsymbol{u} \mapsto u'$$

the natural projections mapping the product $\boldsymbol{X}$ onto the components. Let $(\boldsymbol{u}_t, \mathbb{P}_{\boldsymbol{u}})$, $t \geqslant 0$, be a Feller family of Markov processes in $\boldsymbol{X}$ whose trajectories are continuous in time for all values of the random parameter. Denote by $\boldsymbol{P}(t, \boldsymbol{u}, \boldsymbol{\Gamma})$ the transition function of the family $(\boldsymbol{u}_t, \mathbb{P}_{\boldsymbol{u}})$ and by $\boldsymbol{P}_t$ and $\boldsymbol{P}_t^*$ the corresponding Markov operators.

Definition 1.1. *The Markov family $(\boldsymbol{u}_t, \mathbb{P}_{\boldsymbol{u}})$ is called an extension of $(u_t, \mathbb{P}_u)$ if*

$$\Pi_* \boldsymbol{P}(t, \boldsymbol{u}, \cdot) = P(t, u, \cdot), \quad \Pi'_* \boldsymbol{P}(t, \boldsymbol{u}, \cdot) = P(t, u', \cdot), \tag{1.1}$$

for any $\boldsymbol{u} = (u, u') \in \boldsymbol{X}$ and $t \geqslant 0$, where $\Pi_* \mu$ and $\Pi'_* \mu$ stand for the images of a measure μ under the projections Π and Π', respectively.

Let us consider a simple example of extension that will be used in Section 3. We denote by $(\Omega, \mathcal{F})$ the measurable space on which the family $(u_t, \mathbb{P}_u)$ is defined and by $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}})$ the direct product of two copies $(\Omega, \mathcal{F})$:

$$\boldsymbol{\Omega} = \Omega \times \Omega, \quad \boldsymbol{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}.$$

The points of $\boldsymbol{\Omega}$ will be denoted by $\boldsymbol{\omega} = (\omega, \omega')$. We now define a process $\boldsymbol{u}_t(\boldsymbol{\omega})$ in the space $\boldsymbol{X}$ by the formula

$$\boldsymbol{u}_t(\boldsymbol{\omega}) = (u_t(\omega), u_t(\omega'))$$

and, for any $\boldsymbol{u} = (u, u') \in \boldsymbol{X}$, denote by $\mathbb{P}_{\boldsymbol{u}}$ the unique probability measure on $\boldsymbol{X}$ such that

$$\mathbb{P}_{\boldsymbol{u}}(\Gamma \times \Gamma') = \mathbb{P}_u(\Gamma)\, \mathbb{P}_{u'}(\Gamma') \quad \text{for all } \Gamma, \Gamma' \in \mathcal{B}(X).$$

It is a matter of direct verification to show that the family $(\boldsymbol{u}_t, \mathbb{P}_{\boldsymbol{u}})$ is an extension of $(u_t, \mathbb{P}_u)$. Furthermore, the transition functions satisfy the relation

$$\boldsymbol{P}(t, \boldsymbol{u}, \Gamma \times \Gamma') = P(t, u, \Gamma)\, P(t, u', \Gamma') \tag{1.2}$$

for any $\boldsymbol{u} = (u, u') \in \boldsymbol{X}$ and $\Gamma, \Gamma' \in \mathcal{B}(X)$. In what follows, the extension constructed above is referred to as a *pair of independent copies of* $(u_t, \mathbb{P}_u)$.

1.2. *Formulation of the Result*

Let $(u_t, \mathbb{P}_u)$, $t \geqslant 0$, be a Feller family of Markov processes in a separable Banach space X and let $(\boldsymbol{u}_t, \mathbb{P}_{\boldsymbol{u}})$ be an arbitrary extension of $(u_t, \mathbb{P}_u)$ in $\boldsymbol{X} = X \times X$. As above, we assume that u_t and $\boldsymbol{u}_t$ are continuous with respect to t for all values of the random parameters. Denote by $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}})$ the measurable space on which the extension is defined and write $\boldsymbol{u}_t = (u_t, u_t')$. If $G \subset X$ is a closed subset, then we denote by $\boldsymbol{\tau}(G)$ the first hitting time of $G \times G$ for $\boldsymbol{u}_t$:

$$\boldsymbol{\tau}(G) = \min\{t \geqslant 0 : u_t \in G, u_t' \in G\}.$$

Suppose that, for any integer $m \geqslant 1$, there is a closed subset $G_m \subset X$ such that the following two properties are satisfied.

(P₁) For any $\boldsymbol{u} = (u, u') \in \boldsymbol{X}$ and $m \geqslant 1$, we have

$$\mathbb{P}_{\boldsymbol{u}}\{\boldsymbol{\tau}(G_m) < \infty\} = 1.$$

(P₂) There is a constant $T \geqslant 0$ and a sequence $\delta_m > 0$ tending to zero as $m \to \infty$ such that

$$\sup_{t \geqslant T} \big\|P(t, u, \cdot) - P(t, u', \cdot)\big\|_{\mathcal{L}}^{*} \leqslant \delta_m \quad \text{for any } \boldsymbol{u} \in G_m \times G_m.$$

Theorem 1.2. *Suppose that conditions* (P$_1$) *and* (P$_2$) *are satisfied for a sequence of closed subsets* $G_m \subset X$. *Then*

$$\big\|P(t, u, \cdot) - P(t, u', \cdot)\big\|_{\mathcal{L}}^{*} \to 0 \quad \textit{as} \quad t \to \infty \tag{1.3}$$

for any $u, u' \in X$.

A proof of this result is given in the next subsection. We now establish a simple corollary to this theorem. Recall that a measure $\mu \in \mathcal{P}(X)$ is said to be *stationary* for the family $(u_t, \mathbb{P}_u)$ if $P_t^* \mu = \mu$ for all $t \geqslant 0$.

Corollary 1.3. *Under the assumptions of Theorem* 1.2, *if* μ *is a stationary measure for the family* $(u_t, \mathbb{P}_u)$, *then this measure is unique, and*

$$\big\|P_t^* \lambda - \mu\big\|_{\mathcal{L}}^{*} \to 0 \quad \textit{as } t \to \infty \tag{1.4}$$

for any $\lambda \in \mathcal{P}(X)$.

Proof. The uniqueness immediately follows from (1.4). To prove (1.4), we take any function $f \in \mathcal{L}(H)$ and write

$$\begin{aligned} |(f, P_t^* \lambda - \mu)| &= |(f, P_t^* \lambda - P_t^* \mu)| \\ &= \left| \iint_{X \times X} \big(P_t f(u) - P_t f(u')\big)\, \lambda(du)\, \mu(du') \right| \leqslant \iint_{X \times X} \big|P_t f(u) - P_t f(u')\big|\, \lambda(du)\, \mu(du'). \end{aligned}$$

Taking into account (1.3) and using Lebesgue's dominated convergence theorem, we see that

$$(f, P_t^* \lambda - \mu) \to 0 \quad \text{as } t \to +\infty \text{ for any } f \in \mathcal{L}(H).$$

In view of Theorem 11.3.3 in [Dud02], this is equivalent to (1.4).

1.3. *Proof of Theorem* 1.2

For $t \geqslant T$, write

$$\boldsymbol{\tau}(m,t) = \boldsymbol{\tau}(G_m) \wedge (t-T), \quad p(\boldsymbol{u},m,t) = \mathbb{P}_{\boldsymbol{u}}\{\boldsymbol{\tau}(G_m) > t\}.$$

Applying the strong Markov property (SMP) to the family $(\boldsymbol{u}_t, \mathbb{P}_{\boldsymbol{u}})$ and taking into account the first relation in (1.1), we obtain

$$P(t,u,\Gamma) = \boldsymbol{P}(t,\boldsymbol{u},\Gamma\times X) = \mathbb{E}_{\boldsymbol{u}} \boldsymbol{P}(t-\boldsymbol{\tau}(m,t), \boldsymbol{u}_{\tau(m,t)}, \Gamma \times X) = \mathbb{E}_{\boldsymbol{u}} P(t-\boldsymbol{\tau}(m,t), u_{\tau(m,t)}, \Gamma)$$

for any integer $m \geqslant 1$. A similar relation holds for $P(t,u',\Gamma)$. It follows that

$$\big\|P(t,u,\cdot) - P(t,u',\cdot)\big\|_{\mathcal{L}}^{*} \leqslant \mathbb{E}_{\boldsymbol{u}} g_m(t-\boldsymbol{\tau}(m,t), \boldsymbol{u}_{\tau(m,t)}) \quad \text{for } t \geqslant T, \tag{1.5}$$

where

$$g_m(s,\boldsymbol{z}) = \big\|P(s,z,\cdot) - P(s,z',\cdot)\big\|_{\mathcal{L}}^{*}$$

for $s \geqslant 0$ and $\boldsymbol{z} = (z,z') \in \boldsymbol{X}$. Note that, by assumption (P_2), we have

$$\sup_{s \geqslant T} g_m(s,\boldsymbol{z}) \leqslant \delta_m \quad \text{for } \boldsymbol{z} \in G_m \times G_m. \tag{1.6}$$

On the other hand, the definition of $\boldsymbol{\tau}(m,t)$ and $p(\boldsymbol{u},m,t)$ implies that

$$\mathbb{P}_{\boldsymbol{u}}\big\{\boldsymbol{u}_{\boldsymbol{\tau}(m,t)} \notin G_m \times G_m\big\} \leqslant p(\boldsymbol{u},m,t). \tag{1.7}$$

Combining (1.5)–(1.7) and taking into account the inequality $t - \tau(m,t) \geqslant T$, we see that

$$\big\|P(t,u,\cdot) - P(t,u',\cdot)\big\|_{\mathcal{L}}^{*} \leqslant \delta_m + p(\boldsymbol{u},m,t) \quad \text{for } t \geqslant T.$$

In view of hypotheses (P_1) and (P_2), the right-hand side of this inequality can be made arbitrarily small by choosing m first and then taking t to be sufficiently large. This completes the proof of Theorem 1.2.

2. MIXING FOR THE TWO-DIMENSIONAL NAVIER–STOKES SYSTEM

2.1. *Initial-Boundary Value Problem*

Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary ∂D. Denote by $\boldsymbol{n}$ the unit outward pointing normal to ∂D. For any $s \in \mathbb{R}$, let $H^s(D,\mathbb{R}^2)$ be the space of vector functions (u_1,u_2) on D whose components belong to the Sobolev space of order s. If $s=0$, we write $L^2(D,\mathbb{R}^2)$.

Let us consider the 2D Navier–Stokes system in D. After projecting to the space of divergence-free square-integrable vector fields on D whose normal components vanish on ∂D, we obtain the following evolution equation:

$$\dot{u} + \nu L u + B(u,u) = h + \eta(t). \tag{2.1}$$

Here $\nu > 0$ stands for the viscosity, $L = -\Pi\Delta$, and $B(u,v) = \Pi(u,\nabla)v$, where Π means the orthogonal projection from $L^2(D,\mathbb{R}^2)$ to the subspace

$$H = \Big\{u \in L^2(D,\mathbb{R}^2) : \operatorname{div} u = 0 \text{ in } D, (u,\boldsymbol{n})\big|_{\partial D} = 0\Big\}.$$

Concerning the right-hand side of (2.1), we assume that $h \in H$ and η is a random process of the form

$$\eta(t,x) = \frac{\partial}{\partial t}\zeta(t,x), \quad \zeta(t,x) = \sum_{j=1}^{\infty} b_j \beta_j(t) e_j(x), \tag{2.2}$$

where $\{e_j\}$ is a complete set of normalized eigenfunctions of the operator L supplemented by the Dirichlet boundary condition, $\{\beta_j\}$ is a sequence of independent standard Brownian motions defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $b_j \geqslant 0$ are some constants such that

$$B_0 := \sum_{j=1}^{\infty} b_j^2 < \infty. \tag{2.3}$$

Roughly speaking, this means that the function $\zeta(t)$ belongs to H for almost all values of the random parameter.

Let us consider the Cauchy problem for (2.1), (2.2):

$$u(0) = u_0, \tag{2.4}$$

where $u_0 = u_0(x)$ is an H-valued random variable on Ω independent of $\{\beta_j\}$. We first recall the concept of solution for (2.1), (2.4).

For a closed interval $I \subset \mathbb{R}$ and a Banach space X, we denote by $C(I, X)$ the space of continuous functions $f\colon I \to X$ and by $L^2_{\mathrm{loc}}(I, X)$ the space of Bochner-measurable functions $f\colon I \to X$ such that $\int_J \|f(t)\|_X^2 dt < \infty$ for any compact interval $J \subset I$.

Definition 2.1. An H-valued random process $u(t, x)$, $t \geqslant 0$, which is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a *strong solution* of problem (2.1), (2.2), (2.4) if it is progressively measurable with respect to the filtration generated by u_0 and $\{\beta_j\}$ and satisfies the following two properties:

(i) almost every realization of $u(t, x)$ belongs to the space

$$\mathcal{X} := C(\mathbb{R}_+, H) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+, V),$$

where $\mathbb{R}_+ = [0, +\infty)$ and $V = H \cap H_0^1(D, \mathbb{R}^2)$;

(ii) almost every realization of $u(t, x)$ satisfies the relation

$$u(t) + \int_0^t (\nu L u + B(u, u) - h)\, ds = u_0 + \zeta(t), \qquad t \geqslant 0,$$

where the left- and right-hand sides are regarded as elements of $H^{-1}(D, \mathbb{R}^2)$.

The following result on the existence and uniqueness of a solution for the Cauchy problem for the 2D Navier–Stokes system is well known (see [VF88, Fla94]).

Proposition 2.2. *Suppose that condition* (2.3) *holds and* $h \in H$*. Then, for any* H*-valued random variable* u_0 *independent of* $\{\beta_j\}$*, there is a unique solution of problem* (2.1), (2.2), (2.4).

2.2. *Uniqueness of a Stationary Measure And the Mixing Property*

It is well known that the solutions of (2.1), (2.2), (2.4) corresponding to all possible deterministic initial functions $u_0 \in H$ form a Feller family of Markov processes in H (see [VF88, Fla94]). We denote this family by $(u_t, \mathbb{P}_u)$, where u_t stands for the trajectory of the process and $\mathbb{P}_u$ is the probability measure corresponding to the initial condition $u \in H$. Let $P_t(u, \Gamma) = \mathbb{P}_u\{u_t \in \Gamma\}$ be the transition function associated with the family $(u_t, \mathbb{P}_u)$, and let $P_t\colon C_b(H) \to C_b(H)$ and $P_t^*\colon \mathcal{P}(H) \to \mathcal{P}(H)$ be the corresponding Markov operators. The following theorem is the main result of this paper.

Theorem 2.3. *Suppose that* $h \in H$*, condition* (2.3) *is satisfied, and*

$$b_j \neq 0 \quad \textit{for all} \quad j \geqslant 1. \tag{2.5}$$

Then, for any $\nu > 0$, the family $(u_t, \mathbb{P}_u)$ associated with problem (2.1), (2.2) *has a unique stationary measure $\mu \in \mathcal{P}(H)$. Moreover, the measure μ is mixing in the following sense: the convergence relation* (1.4) *holds for any initial measure $\lambda \in \mathcal{P}(H)$.*

A proof of this theorem is given in the next section. Here we make two remarks.

Remark 2.4. In the case of periodic boundary conditions, it is possible to establish the same result under more general assumptions. Namely, one can replace (2.5) with the requirement that sufficiently many coefficients b_j are nonzero. Indeed, the fact that all coefficients b_j are nonzero is used to show that the probability of transition from any initial point to an arbitrary open ball is positive. The proof of this assertion is based on the approximate controllability of Eq. (2.1) (see Lemmas 3.1 and 3.2). As was recently established by Agrachev and Sarychev [AS05], the NS system on the 2D torus possesses the property of approximate controllability for any positive value of viscosity, provided that sufficiently many Fourier modes are controlled.

Remark 2.5. As was mentioned in the introduction, there are many papers devoted to the problem of ergodicity for the NS system for the case in which the right-hand side is white noise in time and smooth in x (see [EMS01, BKL02, Mat02, KS02]). In all these papers, it was assumed that

$$h \equiv 0, \qquad \sum_{j=1}^{\infty} \alpha_j b_j^2 < \infty,$$

where α_j is the eigenvalue of L corresponding to the eigenfunction e_j (cf. (2.3)). The first of the above conditions implies that the dynamics of the unperturbed equation is trivial (all solutions converge to zero exponentially fast), and the other condition ensures that the random force $\eta(t)$ belongs to the space V almost surely. Thus, Theorem 2.3 gives more general conditions for ergodicity.

3. PROOFS

3.1. *Proof of Theorem 2.3*

Existence of a stationary measure is well known (see [VF88, Fla94]). To prove the uniqueness and the mixing property, we first note that the assertion of the theorem concerns only the Markov semigroups corresponding to the family $(u_t, \mathbb{P}_u)$. The properties of these semigroups do not depend on the probability space on which the Markov family is defined. Therefore, if we prove Theorem 2.3 for some particular choice of the probability space, then this theorem will be proved in the general case as well.

From now on, we make the following assumptions on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

(1) Ω coincides with the Fréchet[1] space $C_0(\mathbb{R}_+, H)$ of continuous functions from $\mathbb{R}_+$ to H vanishing at $t = 0$;
(2) $\mathcal{F}$ is the Borel σ-algebra of Ω;
(3) $\mathbb{P}$ is a colored Wiener process on Ω with the covariance operator $K\colon H \to H$ defined by the relation $Ke_j = b_j^2 e_j$ for all $j \geqslant 1$; in other words, $\mathbb{P}$ is the distribution in $C_0(\mathbb{R}_+, H)$ of the process $\zeta(t, x)$ defined by (2.2).

Assume that the random process $\zeta(t) = \zeta_\omega(t)$ entering the right-hand side of (2.1) is defined on the above probability space and has the form

$$\zeta_\omega(t) = \omega_t \quad \text{for} \quad \omega \in \Omega.$$

To prove the theorem, it suffices to show that the family $(u_t, \mathbb{P}_u)$ associated with problem (2.1), (2.2) possesses an extension $(\boldsymbol{u}_t, \mathbb{P}_{\boldsymbol{u}})$ satisfying properties (P_1) and (P_2) introduced in Section 1.2 (see Theorem 1.2 and Corollary 1.3). We claim that a pair of independent copies of $(u_t, \mathbb{P}_u)$ (see the end of Section 1.1) satisfies (P_1) and (P_2) with $G_m = B_H(1/m)$. The proof of this fact is divided into several steps. In what follows, we assume that $\nu = 1$; the proof in the general case is similar.

[1]The space $C_0(\mathbb{R}_+, H)$ is endowed with the topology of uniform convergence on the compact intervals of $\mathbb{R}_+$.

Step 1. We first show that condition (P_1) is satisfied. Namely, we claim that, for any $r > 0$, there are positive constants C and γ such that

$$\mathbb{E}_{\boldsymbol{u}} \exp(\gamma \boldsymbol{\tau}_r) \leqslant C(1 + \|\boldsymbol{u}\|^2) \quad \text{for any} \quad \boldsymbol{u} = (u, u') \in \boldsymbol{H}, \tag{3.1}$$

where $\boldsymbol{\tau}_r = \boldsymbol{\tau}(B_H(r))$ denotes the first hitting time of $\boldsymbol{B}_H(r) := B_H(r) \times B_H(r)$ for the process $\boldsymbol{u}_t$ and $\|\boldsymbol{u}\|^2 = \|u\|^2 + \|u'\|^2$.

A well-known argument based on stopping time techniques[2] shows that (3.1) will be established if we prove the following two assertions:

(i) there are positive constants R, K, and α such that

$$\mathbb{E}_{\boldsymbol{u}} \exp(\alpha \boldsymbol{\sigma}_R) \leqslant 1 + K\|\boldsymbol{u}\|^2 \quad \text{for any} \quad \boldsymbol{u} = (u, u') \in \boldsymbol{H}, \tag{3.2}$$

where $\boldsymbol{\sigma}_R$ denotes the first hitting time of $\boldsymbol{B}_V(R) := B_V(R) \times B_V(R)$ for the process $\boldsymbol{u}_t$;

(ii) for any $T > 0$ and $R > r > 0$, there is a $p > 0$ such that

$$\boldsymbol{P}(T, \boldsymbol{u}, \boldsymbol{B}_H(r)) \geqslant p \quad \text{for any} \quad \boldsymbol{u} \in \boldsymbol{B}_V(R). \tag{3.3}$$

We shall see that the particular choice of the underlying probability space plays no role in the proof of these assertions.

Step 2. Let us prove (i). The definition of the process $\boldsymbol{u}_t = (u_t, u_t')$ implies that its components satisfy the stochastic equations

$$\partial_t u_t + Lu_t + B(u_t, u_t) = h + \eta, \qquad \partial_t u_t' + Lu_t' + B(u_t', u_t') = h + \eta', \tag{3.4}$$

where $\eta = \partial_t \zeta$ and $\eta' = \partial_t \zeta'$ are (independent) processes of the form (2.2) defined on the same probability space. Let us apply the Itô formula to the functional $V(t, \boldsymbol{u}) = e^{\alpha_1 t}\|\boldsymbol{u}\|^2$, where α_1 is the first eigenvalue of L. Setting $V_1(t, \boldsymbol{u}) = e^{\alpha_1 t}\|\boldsymbol{u}\|_1^2$, where $\|\boldsymbol{u}\|_1^2 = \|u\|_1^2 + \|u'\|_1^2$, and making simple manipulations, we obtain[3]

$$dV(t, \boldsymbol{u}_t) = \bigl\{\alpha_1 V(t, \boldsymbol{u}_t) - 2V_1(t, \boldsymbol{u}_t) + 2e^{\alpha_1 t}\bigl(B_0 + (u_t, h) + (u_t', h)\bigr)\bigr\}\, dt \\ + 2e^{\alpha_1 t}\bigl\{(u, d\zeta) + (u', d\zeta')\bigr\}. \tag{3.5}$$

Choosing any constants $T > 0$ and $R > 0$, integrating (3.5) with respect to $t \in (0, T \wedge \boldsymbol{\sigma}_R)$, and taking the mean value, we see that

$$\mathbb{E}_{\boldsymbol{u}} V(T \wedge \boldsymbol{\sigma}_R, \boldsymbol{u}(T \wedge \boldsymbol{\sigma}_R)) \\ = \|\boldsymbol{u}\|^2 + \mathbb{E}_{\boldsymbol{u}} \int_0^{T \wedge \boldsymbol{\sigma}_R} \bigl\{\alpha_1 V(t, \boldsymbol{u}_t) - 2V_1(t, \boldsymbol{u}_t) + 2e^{\alpha_1 t}\bigl(B_0 + (u_t, h) + (u_t', h)\bigr)\bigr\}\, dt. \tag{3.6}$$

To estimate the integral on the right-hand side of (3.6), recall the inequalities

$$V_1(t, \boldsymbol{u}_t) \geqslant \alpha_1 V(t, \boldsymbol{u}_t), \qquad \bigl|(u_t, h) + (u_t', h)\bigr| \leqslant \tfrac{\alpha_1}{4}\|\boldsymbol{u}_t\|^2 + \tfrac{2}{\alpha_1}\|h\|^2.$$

Substituting them into (3.6) gives

$$\mathbb{E}_{\boldsymbol{u}} V(T \wedge \boldsymbol{\sigma}_R, \boldsymbol{u}(T \wedge \boldsymbol{\sigma}_R)) \leqslant \|\boldsymbol{u}\|^2 + \mathbb{E}_{\boldsymbol{u}} \int_0^{T \wedge \boldsymbol{\sigma}_R} \bigl\{-\tfrac{1}{2}V_1(t, \boldsymbol{u}_t) + 2e^{\alpha_1 t}\bigl(B_0 + 2\alpha_1^{-1}\|h\|^2\bigr)\bigr\}\, dt. \tag{3.7}$$

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[2] For instance, see Sections III.7 and IV.2 in [Has80], Section 13 in [Ver00], or Proposition 2.3 in [Shi04].

[3] We confine ourselves to formal calculations. The justification of these calculations can be obtained by standard arguments (see, e.g., [Par79, KR77, Shi02]).

We now assume that $R^2 \geqslant 8(B_0 + 2\alpha_1^{-1}\|h\|^2)$. Then

$$V_1(t, \boldsymbol{u}_t) \geqslant R^2 e^{\alpha_1 t} \geqslant 8e^{\alpha_1 t}(B_0 + 2\alpha_1^{-1}\|h\|^2) \quad \text{for} \quad 0 \leqslant t < \boldsymbol{\sigma}_R \tag{3.8}$$

by the definition of $\boldsymbol{\sigma}_R$. Combining (3.7) and (3.8), we obtain the inequality

$$\mathbb{E}_{\boldsymbol{u}} \left(\frac{R^2}{4} \int_0^{T \wedge \boldsymbol{\sigma}_R} e^{\alpha_1 t} dt \right) \leqslant \|\boldsymbol{u}\|^2,$$

and hence

$$\mathbb{E}_{\boldsymbol{u}} \exp\{\alpha_1 (T \wedge \boldsymbol{\sigma}_R)\} \leqslant 1 + 4\alpha_1 R^{-2} \|\boldsymbol{u}\|^2.$$

Passing to the limit as $T \to \infty$ and using the Fatou lemma, we arrive at (3.2) with $\alpha = \alpha_1$ and $K = 4\alpha_1 R^{-2}$.

Step 3. We now prove assertion (ii) (see Step 1). Let us choose arbitrary values T, R, and r such that $T > 0$ and $R > r > 0$. In view of (1.2), it suffices to show that

$$P(T, u, B_H(r)) \geqslant \sqrt{p} \quad \text{for any} \quad u \in B_V(R).$$

In other words, we must show that

$$\mathbb{P}\{\|u(T; u_0)\| \leqslant r\} \geqslant \sqrt{p} \quad \text{for any} \quad u_0 \in B_V(R), \tag{3.9}$$

where $u(t; u_0)$ stands for the solution of problem (2.1), (2.4).

To prove (3.9), we use a modification of an argument in Section 5 of [FM95]. We seek $u(t; u_0)$ in the form $u = z + v$, where z is the solution of the linear problem

$$\dot{z} + Lz = \eta(t), \qquad z(0) = 0. \tag{3.10}$$

Thus, z is the Ornstein–Uhlenbeck process defined by the formula

$$z(t) = \int_0^t e^{-(t-s)L} d\zeta(s). \tag{3.11}$$

It is a matter of simple calculation to show that the unknown function v must be a solution of the problem

$$\dot{v} + Lv + B(v + z, v + z) = h, \qquad v(0) = u_0. \tag{3.12}$$

Let us introduce the Banach spaces

$$\begin{aligned} \mathcal{X}_T &:= C(0, T; H) \cap L^2(0, T; V), \\ \dot{\mathcal{X}}_T &:= \{u \in \mathcal{X}_T : u(0) = 0\}, \\ \mathcal{Y}_T &:= \{u \in \mathcal{X}_T : \partial_t u \in L^2(0, T; V^*)\} \end{aligned}$$

and endow them with the natural norms. Standard methods similar to those used in the theory of 2D Navier–Stokes equations (see [Lio69, CF88]) enable one to show that, for any $u_0 \in H$ and $z \in \mathcal{X}_T$, problem (3.12) has a unique solution $v \in \mathcal{Y}_T$. Moreover, the operator $F \colon H \times \mathcal{X}_T \to \mathcal{Y}_T$ taking (u_0, z) to v is uniformly Lipschitz continuous on bounded subsets of $H \times \mathcal{X}_T$. Denote by $F_t(u_0, z)$ the restriction of $F(u_0, z)$ to the time t. We need the following lemma, whose proof is given in the next subsection.

Lemma 3.1. *For any $T > 0$, there is a continuous operator $\mathcal{Z}\colon H \to \dot{\mathcal{X}}_T$ such that the unique solution $v \equiv F(u_0, z) \in \mathcal{Y}_T$ of problem* (3.12) *with $z = \mathcal{Z}(u_0)$ satisfies the relation*

$$\bigl(v + \mathcal{Z}(u_0)\bigr)\big|_{t=T} = 0. \tag{3.13}$$

Moreover, the operator $\mathcal{Z}$ is uniformly Lipschitzian on bounded subsets of H.

Let us choose a large constant $\rho > 0$ for which the image of $B_V(R)$ under $\mathcal{Z}$ is contained in the ball $B_{\dot{\mathcal{X}}_T}(\rho)$. Since F_T is uniformly Lipschitzian on bounded subsets, there is a $C \geqslant 1$ such that

$$\|F_T(u_0, z_0) - F_T(u_0, z_1)\| \leqslant C\,\|z_0 - z_1\|_{\dot{\mathcal{X}}_T}$$

for any $u_0 \in B_V(R)$ and $z_0, z_1 \in B_{\dot{\mathcal{X}}_T}(\rho)$. In particular, taking $z_1 = \mathcal{Z}(u_0)$ with $u_0 \in B_V(R)$, we see that

$$\|F_T(u_0, z_0) - F_T(u_0, \mathcal{Z}(u_0))\| \leqslant \tfrac{r}{2} \quad \text{for} \quad \|z_0 - \mathcal{Z}(u_0)\|_{\dot{\mathcal{X}}_T} \leqslant \tfrac{r}{2C}. \tag{3.14}$$

Recalling that $u = z + v$, where z is the Ornstein–Uhlenbeck process defined by (3.11), and using (3.13), we see that

$$\|u(T, u_0)\| \leqslant \|z(T) + F_T(u_0, z)\| \leqslant \|z - \mathcal{Z}(u_0)\|_{\dot{\mathcal{X}}_T} + \|F_T(u_0, z) - F_T(u_0, \mathcal{Z}(u_0))\|.$$

Combining this inequality with (3.14), we conclude that

$$\mathbb{P}\{\|u(T, u_0)\| \leqslant r\} \geqslant \mathbb{P}\{\|z - \mathcal{Z}(u_0)\|_{\dot{\mathcal{X}}_T} < \tfrac{r}{2C}\} \quad \text{for any} \quad u_0 \in B_V(R). \tag{3.15}$$

Let us find a lower bound of the right-hand side of (3.15). To this end, we need the following lemma whose proof is given in the next subsection.

Lemma 3.2. *Let $\mathcal{O}$ and $\mathcal{K}$ be subsets of of $\dot{\mathcal{X}}_T$ such that $\mathcal{O}$ is open and $\mathcal{K}$ is compact. If conditions* (2.3) *and* (2.5) *are fulfilled, then there is a $\beta > 0$ such that process* (3.11) *satisfies the inequality*

$$\inf_{\hat{z} \in \mathcal{K}} \mathbb{P}\{z - \hat{z} \in \mathcal{O}\} \geqslant \beta.$$

Write $\mathcal{K} = \mathcal{Z}(B_V(R))$ and denote by $\mathcal{O} \subset \dot{\mathcal{X}}_T$ the open ball of radius $r/(2C)$ centered at zero. Since $B_V(R)$ is compact in H, its image under the continuous operator $\mathcal{Z}$ is also compact, and we can apply Lemma 3.2. We conclude that

$$\mathbb{P}\{\|z - \hat{z}\|_{\dot{\mathcal{X}}_T} < r/(2C)\} \geqslant \beta \quad \text{for any } \hat{z} \in \mathcal{Z}(B_V(R)).$$

Combining this with (3.15), we arrive at (3.9).

Step 4. Let us now show that condition (P_2) is also fulfilled. To this end, we first explain the main idea. Let us fix a small constant $\varepsilon > 0$ and two initial conditions $u, u' \in B_H(\varepsilon)$ and denote by $u_t(\boldsymbol{\omega})$ and $u_t'(\boldsymbol{\omega})$ the corresponding solutions of (3.4). Suppose that we have constructed two H-valued random processes $\hat{u}_t(\boldsymbol{\omega})$ and $\hat{u}_t'(\boldsymbol{\omega})$ with the same distribution and two measurable transformations

$$\Phi\colon \boldsymbol{\Omega} \to \boldsymbol{\Omega}, \quad \Phi'\colon \boldsymbol{\Omega} \to \boldsymbol{\Omega}$$

such that

$$\sup_{t \geqslant 0} \mathbb{P}\bigl\{\|u_t(\boldsymbol{\omega}) - \hat{u}_t(\Phi(\boldsymbol{\omega}))\| \geqslant \delta\bigr\} \leqslant \delta,$$
$$\sup_{t \geqslant 0} \mathbb{P}\bigl\{\|u_t'(\boldsymbol{\omega}) - \hat{u}_t'(\Phi(\boldsymbol{\omega}))\| \geqslant \delta\bigr\} \leqslant \delta,$$

for some constant $\delta > 0$, where $\boldsymbol{\mathbb{P}}$ stands for the direct product of two copies of $\mathbb{P}$. Then it can be shown (see Lemma 3.4 below) that, for any $t \geqslant 0$, we have

$$\|\mathcal{D}(u_t) - \mathcal{D}(u_t')\|_{\mathcal{L}}^* \leqslant 6\delta + 2\,\|\boldsymbol{\mathbb{P}} - \Phi_*(\boldsymbol{\mathbb{P}})\|_{\mathrm{var}} + 2\,\|\boldsymbol{\mathbb{P}} - \Phi'_*(\boldsymbol{\mathbb{P}})\|_{\mathrm{var}}. \tag{3.16}$$

Thus, condition (P_2) will be verified if we show that the constant δ and the transformations Φ and Φ' can be chosen in such a way that the right-hand side of (3.16) tends to zero (as ε tends) uniformly with respect to $u, u' \in B_H(\varepsilon)$.

To construct Φ and Φ', we first consider the original problem (2.1), (2.4) and denote by $\tilde{u}_t(\omega)$ its solution corresponding to the initial function $u_0 = 0$. For any $u \in B_H(\varepsilon)$ with sufficiently small $\varepsilon > 0$, we shall construct a measurable transformation $\Psi_u : \Omega \to \Omega$ such that

$$\sup_{t\geqslant 0} \mathbb{P}\big\{\|u_t(\omega) - \tilde{u}_t(\Psi_u(\omega))\| \geqslant \delta\big\} \leqslant \delta, \tag{3.17}$$

$$\|\mathbb{P} - \Psi_{u*}(\mathbb{P})\|_{\mathrm{var}} \leqslant \delta, \tag{3.18}$$

where $u_t(\omega)$ stands for the solution of (2.1), (2.4) with $u_0 = u$ and $\delta > 0$ for a function of $\varepsilon > 0$ vanishing as $\varepsilon \to 0$. If such a family of transformations is constructed, then, for any $\boldsymbol{\omega} = (\omega, \omega') \in \boldsymbol{\Omega}$, we can set

$$\hat{u}_t(\boldsymbol{\omega}) = \tilde{u}_t(\omega), \qquad \hat{u}_t'(\boldsymbol{\omega}) = \tilde{u}_t(\omega'), \tag{3.19}$$

$$\Phi(\boldsymbol{\omega}) = (\Psi_u(\omega), \omega'), \qquad \Phi'(\boldsymbol{\omega}) = (\omega, \Psi_{u'}(\omega')). \tag{3.20}$$

The desired properties of Φ and Φ' follow from the similar properties of Ψ_u.

Step 5. Let us turn to the rigorous proof. We need the following proposition.

Proposition 3.3. *Let $\tilde{u}_t(\omega)$ be the solution of the problem* (2.1), (2.4) *with $u_0 = 0$. Then, for any $\delta > 0$, there is an $\varepsilon > 0$ such that, for any $u \in B_H(\varepsilon)$, one can find a measurable transformation $\Psi_u : \Omega \to \Omega$ for which* (3.17) *and* (3.18) *hold.*

Taking this proposition for granted, let us show that condition (P_2) is fulfilled. Choose an integer $m \geqslant 1$ and two initial functions $u, u' \in B_H(1/m)$. Let $u_t(\boldsymbol{\omega})$ and $u_t'(\boldsymbol{\omega})$ be the solutions of (3.4) corresponding to the initial functions u and u', respectively. We claim that

$$\sup_{u,u'\in B_H(1/m)} \sup_{t\geqslant 0} \|\mathcal{D}(u_t) - \mathcal{D}(u_t')\|_{\mathcal{L}}^* \to 0 \quad \text{as} \quad m \to \infty. \tag{3.21}$$

To show this, we need the following lemma whose proof is given in the next subsection.

Lemma 3.4. *Let X be a separable Banach space, and let $u(\omega)$ and $\hat{u}(\omega)$ be two X-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that there is a measurable transformation $\Psi\colon \Omega \to \Omega$ for which*

$$\mathbb{P}\big\{\|u(\omega) - \hat{u}(\Psi(\omega))\| \geqslant \delta\big\} \leqslant \delta, \tag{3.22}$$

where $\delta > 0$ is a constant. Then

$$\|\mathcal{D}(u) - \mathcal{D}(\hat{u})\|_{\mathcal{L}}^* \leqslant 3\delta + 2\,\|\mathbb{P} - \Psi_*(\mathbb{P})\|_{\mathrm{var}}. \tag{3.23}$$

The random variables $\hat{u}_t$ and $\hat{u}_t'$ defined by (3.19) have the same distribution. Therefore,

$$\|\mathcal{D}(u_t) - \mathcal{D}(u_t')\|_{\mathcal{L}}^* \leqslant \|\mathcal{D}(u_t) - \mathcal{D}(\hat{u}_t)\|_{\mathcal{L}}^* + \|\mathcal{D}(u_t') - \mathcal{D}(\hat{u}_t')\|_{\mathcal{L}}^*. \tag{3.24}$$

Furthermore, it follows from (3.17), (3.18) and from the very definition of $\hat{u}_t$ and Φ that

$$\sup_{t\geqslant 0} \boldsymbol{\mathbb{P}}\big\{\|u_t(\boldsymbol{\omega}) - \hat{u}_t(\Phi(\boldsymbol{\omega}))\| \geqslant \delta_m\big\} \leqslant \delta_m,$$

$$\|\boldsymbol{\mathbb{P}} - \Phi_*(\boldsymbol{\mathbb{P}})\|_{\mathrm{var}} \leqslant \delta_m,$$

where $\delta_m \to 0$ as $m \to \infty$. Applying Lemma 3.4, we see that

$$\sup_{t\geqslant 0} \|\mathcal{D}(u_t) - \mathcal{D}(\hat{u}_t)\|_{\mathcal{L}}^* \leqslant 3\delta_m + 2\,\|\mathbb{P} - \Phi_*(\mathbb{P})\|_{\mathrm{var}} \leqslant 5\delta_m.$$

A similar inequality is valid for u_t', $\hat{u}_t'$, and Φ'. Combining these inequalities with (3.24), we obtain (3.21).

Step 6. To complete the proof, it remains to establish Proposition 3.3. It is the very point at which the particular choice of the underlying probability space plays the role (see the beginning of the proof).

Choose an integer $N \geqslant 1$. Denote by H_N the vector space spanned by $e_1, \dots, e_N$ and by $H_N^\perp$ the orthogonal complement of H_N in H. Let $\mathsf{P}_N : H \to H_N$ and $\mathsf{Q}_N : H \to H_N^\perp$ be the corresponding projections. Consider the following auxiliary problem in $H_N^\perp$:

$$\dot{w} + Lw + \mathsf{Q}_N B(v+w) = \mathsf{Q}_N h + \mathsf{Q}_N \eta, \tag{3.25}$$

$$w(0) = w_0. \tag{3.26}$$

Here $w_0 \in H_N^\perp$ is an initial condition and $v \in C(\mathbb{R}_+, H_N)$ is a given function, and we set $B(u) = B(u,u)$. The following proposition can be established by using standard methods of the theory of NS equations (see [VF88, CF88, Fla94]).

Proposition 3.5. *Let η be defined by* (2.2)*, where the constants $b_j \geqslant 0$ satisfy* (2.3)*. Then there is a full-measure Borel subset $\Omega_0 \subset \Omega$ such that the following assertions hold for any $\omega \in \Omega_0$ and $h \in H$.*

- **(i)** *For any $v \in C(\mathbb{R}_+, H_N)$ and $w_0 \in H_N^\perp$, problem* (3.25), (3.26) *has a unique solution $w \in \mathcal{X} := C(\mathbb{R}_+, H_N^\perp) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+, H_N^\perp \cap V)$.*
- **(ii)** *The function $w(t)$ depends only on the restrictions of v and $\mathsf{Q}_N\omega$ to the interval $[0,t]$.*
- **(iii)** *For any $\omega \in \Omega$, let $\mathcal{W}(\cdot,\cdot,\omega) \colon C(\mathbb{R}_+, H_N) \times H_N^\perp \to \mathcal{X}$ be the operator defined by the formula*

$$\mathcal{W}(v, w_0, \omega) = \begin{cases} w & \text{if } \ \omega \in \Omega_0, \\ 0 & \text{if } \ \omega \in \Omega \setminus \Omega_0, \end{cases}$$

where $w \in \mathcal{X}$ is the solution of (3.25), (3.26)*. Then $\mathcal{W}$ is uniformly Lipschitz continuous with respect to (v, w_0) on the bounded subsets for any fixed $\omega \in \Omega$ and is measurable*[4] *with respect to (v, w_0, ω).*

Denote by $\mathcal{W}_t(v, w_0, \omega)$ the value of the function $\mathcal{W}(v, w_0, \omega) \in \mathcal{X}$ at the time t. We shall sometimes write $\mathcal{W}_t(v., w_0, \omega)$ to stress that v is a function of t.

We can now define the transformation $\Psi_u : \Omega \to \Omega$. Let $\theta \in C^\infty(\mathbb{R})$ be an arbitrary function such that $\theta_t = 1$ for $t \leqslant 0$ and $\theta_t = 0$ for $t \geqslant 1$, and let $\Theta_t = \int_0^t \theta_s ds$. Choose any function $u \in H$ and write

$$v_t(\omega) = \mathsf{P}_N u_t(\omega), \qquad w_t(\omega) = \mathsf{Q}_N u_t(\omega),$$

where $u_t(\omega)$ stands for the solution of (2.1), (2.4) with $u_0 = u$. Let $\Psi_u(\omega)$ be defined by the relations

$$\mathsf{P}_N \Psi_u(\omega) = \mathsf{P}_N\bigl(\omega - \theta u - \Theta L u\bigr) + \int_0^t D_s(u,\omega)\,ds, \qquad \mathsf{Q}_N\Psi_u(\omega) = \mathsf{Q}_N\omega,$$

where we set

$$D_s(u,\omega) = \mathsf{P}_N\bigl\{B\bigl(v_s - \theta_s \mathsf{P}_N u + \mathcal{W}_s(v. - \theta.\mathsf{P}_N u, 0, \omega)\bigr) - B\bigl(v_s + \mathcal{W}_s(v., \mathsf{Q}_N u, \omega)\bigr)\bigr\}.$$

[4]Every Polish space is endowed with their Borel σ-algebra.

It is clear that $\Psi_u(\omega)$ is measurable with respect to $(u,\omega)\in H\times\Omega$. We claim that, if $N\geqslant 1$ is sufficiently large, then, for every $\delta>0$, there is an $\varepsilon>0$ such that inequalities (3.17) and (3.18) hold for any $u\in B_H(\varepsilon)$.

Step 7. Let us prove (3.17). To this end, we first show that

$$\mathsf{P}_N\tilde u_t(\Psi_u(\omega)) = v_t(\omega)-\theta_t\mathsf{P}_N u, \qquad t\geqslant 0, \tag{3.27}$$

for a.a. $\omega\in\Omega$. Indeed, the definition of the operator $\mathcal{W}$ implies that problem (2.1), (2.4) is equivalent to the following system for the components (v_t,w_t) of u_t:

$$\dot v + Lv + \mathsf{P}_N B\big(v+\mathcal{W}_t(v_\cdot,\mathsf{Q}_N u_0,\omega)\big) = \mathsf{P}_N h + \mathsf{P}_N\dot\omega, \tag{3.28}$$

$$\dot w + Lw + \mathsf{Q}_N B(v+w) = \mathsf{Q}_N h + \mathsf{Q}_N\dot\omega, \tag{3.29}$$

$$v_0=\mathsf{P}_N u_0, \qquad w_0 = \mathsf{Q}_N u_0. \tag{3.30}$$

It follows from (3.28) and (3.30) that the left-hand side of (3.27), which we denote by $p_t(\omega)$, is a solution of the problem

$$\dot p + Lp + \mathsf{P}_N B(p+\mathcal{W}_t(p_\cdot,0,\omega)) = \mathsf{P}_N h + \varphi_t, \tag{3.31}$$

$$p_0 = 0. \tag{3.32}$$

where φ_t is given by the relation

$$\varphi_t = \mathsf{P}_N(\dot\omega - \dot\theta u - \theta L u) + D_t(u,\omega).$$

It is a matter of direct verification to show that the function $v_t(\omega)-\theta_t\mathsf{P}_N u$ also satisfies (3.31) and (3.32). Thus, relation (3.27) will be established if we show that the finite-dimensional problem (3.31), (3.32) admits at most one solution. This is a straightforward consequence of the Gronwall lemma and the local Lipschitz property of the operators entering (3.31).

Relation (3.27) implies that, if $u\in B_H(\varepsilon)$, then for a.a. $\omega\in\Omega$ we have

$$\sup_{t\geqslant 0}\|\mathsf{P}_N\big(u_t(\omega)-\tilde u_t(\Psi_u(\omega))\big)\|\leqslant \|u\|\leqslant\varepsilon. \tag{3.33}$$

Let us now estimate the norm of the projection of $u_t(\omega)-\tilde u_t(\Psi_u(\omega))$ to the subspace $H_N^\perp$.

Let $q_t(\omega)=\mathsf{Q}_N u_t(\Psi_u(\omega))$ and $z_t=w_t-q_t$. It follows from (3.27), (3.29), and (3.30) that z is the solution of the problem

$$\dot z_t + Lz_t + \mathsf{Q}_N\big(B(u_t)-B(u_t-\theta_t\mathsf{P}_N u - z_t)\big) = 0, \qquad z_0=\mathsf{Q}_N u. \tag{3.34}$$

Taking the inner product in H of the function $2z_t$ and the first equation in (3.34) and using the fact that $(B(g,z_t),z_t)=0$ for any $g\in H$, we obtain

$$\partial_t\|z_t\|^2 + 2\|z_t\|_1^2 = \theta_t\chi_t(u,\omega)+\rho_t(u,\omega), \tag{3.35}$$

where

$$\chi_t(u,\omega) = 2\left(B(u_t,\mathsf{P}_N u) + B(\mathsf{P}_N u, u_t-\theta_t\mathsf{P}_N u), z_t\right),$$
$$\rho_t(u,\omega) = 2\big(B(z_t, u_t-\theta_t\mathsf{P}_N u), z_t\big).$$

Standard estimates for the trilinear form $(B(u,v),w)$ imply that

$$|\chi_t(u,\omega)|\leqslant \frac12\|z_t\|_1^2 + \varepsilon^2 C_N\big(\|u_t\|^2+1\big),$$
$$|\rho_t(u,\omega)|\leqslant \frac12\|z_t\|_1^2 + \big(C\|u_t\|_1^2 + \varepsilon^2 C_N\big),$$

for any $u\in B_H(\varepsilon)$, where C and C_N are some positive constants. Substituting these estimates into (3.35) and using the inequality $\|z_t\|_1^2\geqslant \alpha_{N+1}\|z_t\|^2$, we see that

$$\partial_t\|z_t\|^2 + \big(\alpha_{N+1} - C\,\|u_t\|^2 - \varepsilon^2 C_N\big)\,\|z_t\|^2 \leqslant C_N\varepsilon^2\theta_t\big(\|u_t\|^2+1\big). \tag{3.36}$$

We now need the following lemma whose proof can be obtained by an unessential modification of the argument in [KS02, Lemma 2.3].

Lemma 3.6. *For any* $R > 0$ *and any initial condition* $u \in H$*, let*

$$\Omega_R(u) := \Big\{\omega \in \Omega : \int_0^t \|u_s(\omega)\|_1^2 ds \leqslant B_0 t + \|u\|^2 + R\Big\},$$

where $u_t(\omega)$ *is the solution of* (2.1) *such that* $u_0 = u$ *and* B_0 *is defined by* (2.3)*. Then*

$$\sup_{u \in H} \mathbb{P}\big\{\Omega_R(u)\big\} \to 0 \quad as \quad R \to +\infty. \tag{3.37}$$

Let us choose an integer $N \geqslant 1$ and a constant $\varepsilon > 0$ such that

$$\alpha_{N+1} \geqslant CB_0 + 1, \qquad \varepsilon^2 C_N \leqslant 1.$$

Applying the Gronwall inequality to (3.36), we obtain

$$\|z_t\|^2 \leqslant \|z_0\|^2 \exp\Big\{\int_0^t \big(-\alpha_{N+1} + C\|u_t\|^2 + C_N\varepsilon^2\big)\, ds\Big\}$$
$$+ C_N\varepsilon^2 \int_0^1 \exp\Big\{\int_s^t \big(-\alpha_{N+1} + C\|u_\tau\|^2 + C_N\varepsilon^2\big)\, d\tau\Big\}\theta_s\big(\|u_s\|^2 + 1\big)\, ds \leqslant K_R\varepsilon^2$$

for any $u \in B_H(\varepsilon)$ and $\omega \in \Omega_R(u)$, where $K_R > 0$ is a constant not depending on ε and u. Choosing $R > 0$ so large that $\mathbb{P}\big\{\Omega_R(u)\big\} \leqslant \delta$ for $u \in H$ (cf. (3.37)), we see that

$$\mathbb{P}\Big\{\sup_{t \geqslant 0} \|\mathsf{Q}_N\big(u_t(\omega) - \tilde{u}_t(\Psi_u(\omega))\big)\| \geqslant K_R\varepsilon^2\Big\} \leqslant \delta.$$

Combining this inequality with (3.33), we obtain the desired estimate (3.17) for a sufficiently small $\varepsilon > 0$.

To complete the proof of Proposition 3.3, it remains to establish inequality (3.18). Its proof is based on the Girsanov theorem. In the context of randomly forced PDE's, this type of arguments was first used in [EMS01] for the case of finite-dimensional perturbations. The proof of (3.18) almost literally repeats that of Lemma 4.3 in [KS02], and therefore we omit it. In the second part of this work, we shall discuss the corresponding arguments in detail for the complex Ginzburg–Landau equation, which is technically more complicated.

3.2. *Proof of Auxiliary Assertions*

Proof of Lemma 3.1. Let us choose a constant $T > 0$ and set

$$\hat{u}(t) = \chi(t)e^{-tL}u_0, \quad 0 \leqslant t \leqslant T, \tag{3.38}$$

where $\chi \in C^\infty(\mathbb{R})$ is a function such that $0 \leqslant \chi \leqslant 1$, $\chi(t) = 1$ for $t \leqslant 0$ and $\chi(t) = 1$ for $t \geqslant T$. It is a matter of direct verification to show that $\hat{u} \in \mathcal{Y}_T$, and hence

$$\hat{\eta} := \partial_t \hat{u} + L\hat{u} + B(\hat{u}, \hat{u}) - h \in L^2(0, T; V^*).$$

We now define $\mathcal{Z}(u_0) \in \dot{\mathcal{X}}_T$ as a unique solution of problem (3.10) with the deterministic right-hand side $\eta = \hat{\eta}$. Such a solution can readily be constructed with the help of decomposition in the eigenbasis $\{e_j\}$ (see also [Hen81, Section 3.2] or [CF88, Sects. 2, 3]). Moreover, it immediately follows from the definition that $\mathcal{Z}(u_0)$ is uniformly Lipschitzian on the bounded subsets of H. Finally, the construction of $\mathcal{Z}(u_0)$ implies that $v + \mathcal{Z}(u_0) = \hat{u}$, and therefore (3.13) follows from (3.38).

Proof of Lemma 3.2. It follows from conditions (2.3) and (2.5) that the distribution of z is a nondegenerate Gaussian measure on $\dot{\mathcal{X}}_T$. In particular, the measure of any open subset is positive (see Section 7.4 in [DZ96]). It follows that, for any $\hat{z} \in \dot{\mathcal{X}}_T$ and any open subset $O \subset \dot{\mathcal{X}}_T$, the probability $p(\hat{z}) = \mathbb{P}\{z - \hat{z} \in O\}$ is positive. Note that the distribution of the random variable $z - \hat{z}$ is continuous with respect to $\hat{z}$ in the weak* topology. Therefore, the function $p(\hat{z}) > 0$ is lower semicontinuous, and hence is separated from zero on any compact subset.

Proof of Lemma 3.4. We have

$$\|\mathcal{D}(u) - \mathcal{D}(\hat{u})\|_{\mathcal{L}}^* \leqslant \|\mathcal{D}(u) - \mathcal{D}(\hat{u} \circ \Psi)\|_{\mathcal{L}}^* + \|\mathcal{D}(\hat{u} \circ \Psi) - \mathcal{D}(\hat{u})\|_{\mathcal{L}}^*. \tag{3.39}$$

Inequality (3.22) readily implies that

$$\|\mathcal{D}(u) - \mathcal{D}(\hat{u} \circ \Psi)\|_{\mathcal{L}}^* \leqslant 3\delta. \tag{3.40}$$

Moreover, since

$$\|\mu_1 - \mu_2\|_{\mathcal{L}}^* \leqslant 2\,\|\mu_1 - \mu_2\|_{\mathrm{var}} \quad \text{for any} \quad \mu_1, \mu_2 \in \mathcal{P}(X),$$

we conclude that

$$\begin{aligned} \|\mathcal{D}(\hat{u} \circ \Psi) - \mathcal{D}(\hat{u})\|_{\mathcal{L}}^* &\leqslant 2 \sup_{\Gamma \in \mathcal{B}(X)} \bigl|\mathbb{P}\{\hat{u} \circ \Psi \in \Gamma\} - \mathbb{P}\{\hat{u} \in \Gamma\}\bigr| \\ &= 2 \sup_{\Gamma \in \mathcal{B}(X)} \bigl|\Psi_*(\mathbb{P})\{\hat{u} \in \Gamma\} - \mathbb{P}\{\hat{u} \in \Gamma\}\bigr| \leqslant 2\,\|\Psi_*(\mathbb{P}) - \mathbb{P}\|_{\mathrm{var}}. \end{aligned} \tag{3.41}$$

Combining (3.39)–(3.41), we arrive at (3.23).

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