Statistical hydrodynamics: the dynamical system approach

Armen Shirikyan

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Introduction

The aim of these lectures is to give a self-contained concise introduction to the ergodic theory of randomly forced partial differential equations (PDE's). We consider the 2D Navier–Stokes (NS) system perturbed by a bounded discrete force, namely,

$$\dot{u} + (u, \nabla)u - \Delta u + \nabla p = \eta(t, x), \quad \operatorname{div} u = 0. \tag{0.1}$$

Here η is a random process of the form

$$\eta(t, x) = \sum_{k=1}^{\infty} \eta_k(x)\delta(t - k), \tag{0.2}$$

where $\delta(t)$ is the Dirac measure concentrated at zero and $\eta_k(x)$ are independent identically distributed (i.i.d.) random variables. After studying the initial-boundary value problem for (0.1), (0.2), we show that the restriction of the corresponding random dynamical system to integer times generates a homogeneous family of Markov chains in an appropriate functional space. For this Markov chain, we investigate the following questions:

- existence of stationary measures;
- uniqueness of stationary measure;
- mixing properties.

We show that, under some non-degeneracy assumptions on the distribution of η_k, there is a unique stationary measure, which is exponentially mixing. The existence is a simple consequence of the Bogolyubov–Krylov argument and a smoothing property of the semigroup generated by the homogeneous 2D NS system. The problem of uniqueness and exponential mixing is much more delicate and was studied in [KS00, KS01, KPS02]. [1] The presentation here follows the papers [KS01, KPS02].

The lectures are organized as follows. In Section 1, we have compiled some basic facts from probability theory. In particular, we study different metrics on the space of probability measures and recall the concept of maximal coupling. Section 2 is devoted to the initial-boundary value problem for the NS system perturbed by a random force of the form (0.2). We construct solutions of this problem, derive some a priori estimates for them, and show that they form a homogeneous family of Markov chains. In Section 3, we first use the Bogolyubov–Krylov argument to construct a stationary measure and then formulate the main theorem on uniqueness and exponential mixing. The section is concluded by describing the scheme of the proof of that result. Section 4 is devoted to construction of the coupling operators associated to the family of Markov chains in question and to investigation of their properties. In the last, fifth section, we prove the main theorem.

[1] See also the papers [FM95, EMS01, BKL02, EH01, Mat02, MY02, Hai02, Kuk02b, KS02, Kuk02a, KS03, Shi04] and references therein for some further results on mixing properties of randomly forced PDE's.

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Notation

Given a Polish space (i.e., separable complete metric space), we always assume that it is endowed with its Borel σ-algebra and consider it as a measurable space (see Section 1.1 for details). For any random variable ξ, we denote by $\mathcal{D}(\xi)$ its distribution and by $\sigma(\xi)$ the σ-algebra generated by ξ. If Γ is a subset in a measurable space, then we denote by I_Γ the indicator function of Γ and by Γ^c its complement.

For any measurable space $(X, \mathcal{B})$ with measure m, we denote by $L^\infty(X, m)$ the space of Borel measurable functions $f: X \to \mathbb{R}$ such that

$$\|f\|_\infty = \operatorname*{ess\,sup}_{u \in X} |f(u)| < \infty.$$

For a Polish space X, we denote by $C_b(X)$ the space of continuous functions $f: X \to \mathbb{R}$ such that $\|f\|_\infty < \infty$. If X is a Banach space, then $B_X(R)$ stands for the ball in X of radius R centred at zero.

If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $B \in \mathcal{F}$ is an event of positive probability, then the probability of $A \in \mathcal{F}$ on condition B is defined by the formula $\mathbb{P}(A \,|\, B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}$.

For real numbers a and b, we denote $a \wedge b$ ($a \vee b$) their minimum (maximum).

1 Preliminaries

1.1 Probability spaces, random variables, distributions

Let Ω be a set with σ-algebra $\mathcal{F}$, i.e., a family of subsets of Ω that contains Ω and satisfies the following two properties:

- if $B_i \in \mathcal{F}$ for $i = 1, 2, \dots$, then $\bigcap_i B_i \in \mathcal{F}$;
- if $B \in \mathcal{F}$, then $B^c = \Omega \setminus B \in \mathcal{F}$.

Any pair $(\Omega, \mathcal{F})$ possessing the above properties will be called a *measurable space*.

Example 1.1. Let X be a Polish space and let $\mathcal{B}_X$ be the Borel σ-algebra on X, i.e., the minimal σ-algebra generated by the open subsets of X. Then $(X, \mathcal{B}_X)$ is a measurable space. In what follows, we assume that all Polish spaces are endowed with their Borel σ-algebra.

Let $\mathbb{P}$ be a probability measure on a measurable space $(\Omega, \mathcal{F})$, i.e., $\mathbb{P}$ is a countably additive function from $\mathcal{F}$ to $[0, 1]$ such that $\mathbb{P}(\Omega) = 1$. Any such triple $(\Omega, \mathcal{F}, \mathbb{P})$ will be called a *probability space*.

Example 1.2. Let us consider the interval $I = [0, 1]$ endowed with the Borel σ-algebra $\mathcal{B}_I$, and let ℓ be the Lebesgue measure on I. Then $(I, \mathcal{B}_I, \ell)$ is a probability space.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(X, \mathcal{B})$ be a measurable space. Given an X-valued random variable ξ (i.e., a map from Ω to X such that $\xi^{-1}(\Gamma) \in \mathcal{F}$ for any $\Gamma \in \mathcal{B}$), we define its distribution $\mathcal{D}(\xi)$ as the image of $\mathbb{P}$ under ξ:

$$\mathcal{D}(\xi)(\Gamma) = \mathbb{P}\big(\xi^{-1}(\Gamma)\big) = \mathbb{P}\big(\{\omega \in \Omega : \xi(\omega) \in \Gamma\}\big).$$

Thus, the distribution of ξ is a probability measure on $(X, \mathcal{B})$. The space of all probability measures on a measurable space $(X, \mathcal{B})$ will be denoted by $\mathcal{P}(X)$.

Exercise 1.3. Let $\mu \in \mathcal{P}(\mathbb{R})$ be a probability measure on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$. Show that there is a real-valued random variable whose distribution coincides with μ. Formulate and prove a similar assertion for any measurable space $(X, \mathcal{B})$. *Hint:* Consider the probability space $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu)$ and the random variable $\xi(\omega) = \omega$.

If X is a separable Banach space, then an X-valued random variable ξ is said to be integrable if

$$\int_\Omega \|\xi(\omega)\|_X \,\mathbb{P}(d\omega) < \infty.$$

In this case, we denote by $\mathbb{E}\,\xi$ its mean value, that is,

$$\mathbb{E}\,\xi = \int_\Omega \xi(\omega)\,\mathbb{P}(d\omega).$$

1.2 Independence, product of probability spaces

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{A}$ be a set of indices, and let $\mathcal{F}_\alpha \subset \mathcal{F}$, $\alpha \in \mathcal{A}$, be a family of sub-σ-algebras in Ω.

Definition 1.4. The family $\{\mathcal{F}_\alpha, \alpha \in \mathcal{A}\}$ is said to be *independent* if for any finite set of indices $\alpha_1, \dots, \alpha_n \in \mathcal{A}$ and any $B_i \in \mathcal{F}_{\alpha_i}$, $i = 1, \dots, n$, we have

$$\mathbb{P}(B_1 \cdots B_n) = \mathbb{P}(B_1) \,\cdots\, \mathbb{P}(B_n).$$

Let $(X, \mathcal{B})$ be a measurable space. Consider a family of X-valued random variables ξ_α, $\alpha \in \mathcal{A}$, defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let us denote by $\mathcal{F}_\alpha = \sigma(\xi_\alpha)$ the σ-algebra generated by ξ_α, i.e., the family of sets $B \in \mathcal{F}$ that can be represented in the form $\xi_\alpha^{-1}(\Gamma)$ for some $\Gamma \in \mathcal{B}$.

Definition 1.5. The family $\{\xi_\alpha, \alpha \in \mathcal{A}\}$ is said to be *independent* if the corresponding family of σ-algebras $\{\mathcal{F}_\alpha, \alpha \in \mathcal{A}\}$ is independent, i.e., for any finite set of indices $\alpha_1, \dots, \alpha_n \in \mathcal{A}$ and any $\Gamma_i \in \mathcal{B}$, $i = 1, \dots, n$, we have

$$\mathbb{P}\{\xi_{\alpha_1} \in \Gamma_1, \dots, \xi_{\alpha_n} \in \Gamma_n\} = \prod_{i=1}^n \mathbb{P}\{\xi_{\alpha_i} \in \Gamma_i\}.$$

Exercise 1.6. Show that a family $\{\xi_\alpha, \alpha \in \mathcal{A}\}$ of X-valued random variables is independent iff for any finite set of indices $\alpha_1, \ldots, \alpha_n \in \mathcal{A}$ and any bounded measurable functions $f_i : X \to \mathbb{R}$, $i = 1, \ldots, n$, we have

$$\mathbb{E}\biggl\{\prod_{i=1}^n f_i(\xi_{\alpha_i})\biggr\} = \prod_{i=1}^n \mathbb{E}\, f_i(\xi_{\alpha_i}). \tag{1.1}$$

Hint: Begin with the case of simple functions, i.e., functions that take on finitely many different values.

We now describe a simple way for constructing independent random variables. Let $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha)$, $\alpha \in \mathcal{A}$, be a family of probability spaces. Define the product space

$$\Omega = \prod_{\alpha \in \mathcal{A}} \Omega_\alpha = \bigl\{\omega = (\omega_\alpha, \alpha \in \mathcal{A}) : \omega_\alpha \in \Omega_\alpha \text{ for any } \alpha\bigr\}$$

and denote by $\mathcal{F}$ the product σ-algebra, i.e., the minimal σ-algebra generated by the sets of the form

$$B_{\alpha_1,\ldots,\alpha_n} = \bigl\{(\omega_\alpha, \alpha \in \mathcal{A}) : \omega_{\alpha_1} \in B_1, \ldots, \omega_{\alpha_n} \in B_n\bigr\},$$

where n is a finite integer and $B_i \in \mathcal{F}_{\alpha_i}$ for $i = 1, \ldots, n$.

*Exercise** 1.7. Show that there is a unique probability measure on $(\Omega, \mathcal{F})$ such that

$$\mathbb{P}(B_{\alpha_1,\ldots,\alpha_n}) = \prod_{i=1}^n \mathbb{P}_{\alpha_i}(B_i) \quad \text{for any set } B_{\alpha_1,\ldots,\alpha_n}. \tag{1.2}$$

Hint: See Exercise 1.1.14 in [Str93].

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called the *product space* of $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha)$, $\alpha \in \mathcal{A}$. It follows from (1.2) that if ξ_α are some random variables defined on Ω_α, then their natural extensions[2] to Ω are independent.

*Exercise** 1.8. Let $\xi_1, \ldots, \xi_n$ be independent X-valued random variables and let $f\colon X \times \cdots \times X \to \mathbb{R}$ be a bounded measurable function of n variables. Then

$$\mathbb{E}\, f(\xi_1, \ldots, \xi_n) = \mathbb{E}_{\omega_1} \ldots \mathbb{E}_{\omega_n} f(\xi_1(\omega_1), \ldots, \xi_n(\omega_n)). \tag{1.3}$$

Hint: Use the technique of π- and λ-systems (cf. Theorem 4.4.2 in [Dud02] or Section I.5 in [Kry95]).

1.3 Conditional expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let ξ be a real-valued integrable random variable.

[2] By the natural extension of ξ_α to Ω we mean the random variable defined by the relation $\tilde{\xi}_\alpha(\omega) = \xi_\alpha(\omega_\alpha)$.

Proposition 1.9. *For any sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$ there is a $\mathcal{G}$-measurable random variable η such that*

$$\int_B \xi(\omega)\mathbb{P}(d\omega) = \int_B \eta(\omega)\mathbb{P}(d\omega) \quad \textit{for any } B \in \mathcal{G}. \tag{1.4}$$

If $\tilde{\eta}$ is another $\mathcal{G}$-measurable random variable satisfying (1.4), then $\eta(\omega) = \tilde{\eta}(\omega)$ for a.e. ω.

Proof. Let us consider a signed measure on $(\Omega, \mathcal{G})$ defined by the formula

$$\mu(B) = \int_B \xi(\omega)\,\mathbb{P}(d\omega), \quad B \in \mathcal{G}. \tag{1.5}$$

The measure μ is absolutely continuous with respect to $\mathbb{P}$. Hence, by the Radon–Nikodym theorem (see Theorem 5.5.4 in [Dud02]), there is a $\mathcal{G}$-measurable function $\eta(\omega)$ such that

$$\mu(B) = \int_B \eta(\omega)\,\mathbb{P}(d\omega) \quad \text{for any } B \in \mathcal{G}.$$

Comparing this relation with (1.5), we arrive at (1.4).

If $\tilde{\eta}$ is another $\mathcal{G}$-measurable random variable satisfying (1.4), then

$$\int_B \big(\eta(\omega) - \tilde{\eta}(\omega)\big)\,\mathbb{P}(d\omega) = 0 \quad \text{for any } B \in \mathcal{G},$$

whence it follows that $\eta = \tilde{\eta}$ almost surely. □

Definition 1.10. The random variable η constructed in Proposition 1.9 is called the *conditional expectation of ξ given $\mathcal{G}$* and is denoted by $\mathbb{E}\,(\xi \,|\, \mathcal{G})$.

Exercise 1.11. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Suppose that Ω is represented as a countable union of disjoint subsets Ω_i, $i \geq 1$, and let $\mathcal{G}$ be the sub-σ-algebra generated by $\{\Omega_i,\, i \geq 1\}$. Construct the conditional expectation of a real-valued random variable ξ given $\mathcal{G}$.

Exercise 1.12. Show that, if ξ is $\mathcal{G}$-measurable, then $\mathbb{E}\,(\xi \,|\, \mathcal{G}) = \xi$, and if $\sigma(\xi)$ and $\mathcal{G}$ are independent, then $\mathbb{E}\,(\xi \,|\, \mathcal{G}) = \mathbb{E}\,\xi$. Furthermore, if $\mathcal{G} \subset \mathcal{G}'$, then

$$\mathbb{E}\,\big(\mathbb{E}\,(\xi \,|\, \mathcal{G}') \,|\, \mathcal{G}\big) = \mathbb{E}\,(\xi \,|\, \mathcal{G}). \tag{1.6}$$

*Exercise** 1.13. Let ξ and η be random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and valued in a Polish space X and let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra such that ξ is $\mathcal{G}$-measurable and η is independent of $\mathcal{G}$. Show that for any bounded measurable function $f\colon X \times X \to \mathbb{R}$ we have

$$\mathbb{E}\,\big(f(\xi, \eta) \,|\, \mathcal{G}\big) = \big(\mathbb{E}\, f(x, \eta)\big)\big|_{x=\xi}.$$

Hint: Use the technique of π- and λ-systems.

1.4 Metrics on the space of probability measures

Let X be a Polish space endowed with its Borel σ-algebra $\mathcal{B}_X$. We denote by $C_b(X)$ the space of continuous functions $f\colon X \to \mathbb{R}$ with finite norm

$$\|f\|_\infty := \sup_{u\in X} |f(u)|.$$

Since the family $\mathcal{P}(X)$ of probability measures on $(X, \mathcal{B}_X)$ is a subset in the dual space of $C_b(X)$, we can endow it with the dual metric

$$\|\mu_1 - \mu_2\|_\infty^* := \sup\bigl\{|(f,\mu_1) - (f,\mu_2)| : f \in C_b(X), \|f\|_\infty \le 1\bigr\},$$

where, for any $f \in C_b(X)$ and $\mu \in \mathcal{P}(X)$, we set

$$(f,\mu) := \int_X f(u)\mu(du) = \int_X f(u)\, d\mu.$$

Let us introduce another metric on $\mathcal{P}(X)$.

Definition 1.14. The *variational distance* between two probability measures μ_1 and μ_2 is defined by the formula

$$\|\mu_1 - \mu_2\|_{\mathrm{var}} := \sup\bigl\{|\mu_1(\Gamma) - \mu_2(\Gamma)| : \Gamma \in \mathcal{B}_X\bigr\}.$$

Theorem 1.15. *For any $\mu_1, \mu_2 \in \mathcal{P}(X)$, we have*

$$\|\mu_1 - \mu_2\|_\infty^* = 2\,\|\mu_1 - \mu_2\|_{\mathrm{var}}. \tag{1.7}$$

Proof. We shall need the following auxiliary assertion, which is of independent interest.

Proposition 1.16. *Let m be a positive Borel measure on X. Suppose that $\mu_1, \mu_2 \in \mathcal{P}(X)$ are absolutely continuous with respect to m. Then*

$$\|\mu_1 - \mu_2\|_{\mathrm{var}} = \frac{1}{2}\int_X \bigl|\rho_1(u) - \rho_2(u)\bigr|\, dm = 1 - \int_X (\rho_1 \wedge \rho_2)(u)\, dm, \tag{1.8}$$

where $\rho_i(u)$, $i = 1, 2$, is the density of μ_i with respect to m.

Taking this proposition for granted, let us prove the theorem. Let m be a measure satisfying the conditions of Proposition 1.16. For instance, we can take $m = \mu_1 + \mu_2$. Using the first relation in (1.8), for any $f \in C_b(X)$ with $\|f\|_\infty \le 1$ we derive

$$\bigl|(f,\mu_1) - (f,\mu_2)\bigr| \le \int_X \bigl|f(u)\bigl(\rho_1(u) - \rho_2(u)\bigr)\bigr|\, dm \le 2\,\|\mu_1 - \mu_2\|_{\mathrm{var}},$$

which implies that

$$\|\mu_1 - \mu_2\|_\infty^* \le 2\,\|\mu_1 - \mu_2\|_{\mathrm{var}}.$$

To establish the converse inequality, we set

$$Y = \{u \in X : \rho_1(u) > \rho_2(u)\}. \tag{1.9}$$

Let us consider a function $f(u)$ that is equal to 1 on Y and to -1 on the complement of Γ. We have

$$\begin{aligned}(f,\mu_1) - (f,\mu_2) &= \int_X f(u)\big(\rho_1(u) - \rho_2(u)\big)\,dm \\ &= \int_X \big|\rho_1(u) - \rho_2(u)\big|\,dm = 2\,\|\mu_1 - \mu_2\|_{\mathrm{var}},\end{aligned} \tag{1.10}$$

where we used the first relation in (1.8). To complete the proof of (1.7), it suffices to choose a sequence $f_n \in C_b(X)$ such that $\|f_n\|_\infty \le 1$ for all n and $f_n(u) \to f(u)$ for m-a.e. $u \in X$ and note that $(f_n,\mu_1) - (f_n,\mu_2)$ tends to the left-hand side of (1.10) as $n \to \infty$. □

*Exercise** 1.17. Let X be a Polish space endowed with its Borel σ-algebra and let m be a positive measure on X. Show that for any $f \in L^\infty(X,m)$ there is a sequence of continuous functions uniformly bounded by $\|f\|_\infty$ that converges to f almost surely. *Hint:* Any bounded measurable function can be approximated uniformly by bounded simple functions; the indicator function of any measurable set can be approximated (in the sense of a.s. convergence) by bounded continuous functions.

Proof of Proposition 1.16. A direct verification shows that

$$\tfrac{1}{2}\,|\rho_1 - \rho_2| = \tfrac{1}{2}\,(\rho_1 + \rho_2) - \rho,$$

where $\rho = \rho_1 \wedge \rho_2$. Integrating the above relation over X with respect to m, we obtain the second equality in (1.8).

We now show that

$$\|\mu_1 - \mu_2\|_{\mathrm{var}} \le 1 - \int_X \rho(u)\,dm. \tag{1.11}$$

Let Y be the set defined by (1.9). Then, for any $\Gamma \in \mathcal{B}_X$, we have

$$\begin{aligned}\mu_1(\Gamma) - \mu_2(\Gamma) &= \int_\Gamma (\rho_1 - \rho_2)\,dm \le \int_{\Gamma\cap Y} (\rho_1 - \rho_2)\,dm \\ &= \int_{\Gamma\cap Y} (\rho_1 - \rho)\,dm \le \int_X (\rho_1 - \rho)\,dm = 1 - \int_X \rho(u)\,dm.\end{aligned}$$

In view of the symmetry, this inequality implies (1.11).

To prove the converse inequality, we denote by Y^c the complement of Y and note that $\rho = \rho_1$ on Y^c and $\rho = \rho_2$ on Y. It follows that

$$\begin{aligned}
\mu_1(Y) - \mu_2(Y) &= \int_Y (\rho_1 - \rho_2)\, dm \\
&= \left(\int_Y \rho_1 dm + \int_{Y^c} \rho\, dm\right) - \left(\int_Y \rho_2 dm + \int_{Y^c} \rho\, dm\right) \\
&= \left(\int_Y \rho_1 dm + \int_{Y^c} \rho_1 dm\right) - \left(\int_Y \rho\, dm + \int_{Y^c} \rho\, dm\right) \\
&= 1 - \int_X \rho\, dm.
\end{aligned}$$

This completes the proof of the proposition. $\square$

In what follows, we shall need a weaker topology on $\mathcal{P}(X)$. Let $\mathcal{L}(X)$ be the space of functions $f \in C_b(X)$ such that

$$\|f\|_{\mathcal{L}} := \|f\|_\infty + \sup_{u \neq v} \frac{|f(u) - f(v)|}{d_X(u, v)} < \infty,$$

where d_X is the metric on X. For any $\mu_1, \mu_2 \in \mathcal{P}(X)$, we set

$$\|\mu_1 - \mu_2\|_{\mathcal{L}}^* := \sup\bigl\{|(f, \mu_1) - (f, \mu_2)| : f \in \mathcal{L}(X), \|f\|_{\mathcal{L}} \le 1\bigr\}. \tag{1.12}$$

*Exercise** 1.18. Show that $\|\mu_1 - \mu_2\|_{\mathcal{L}}^*$ defines a metric on $\mathcal{P}(X)$. *Hint:* The triangle inequality is obvious; to prove that $\mu_1 = \mu_2$ if $\|\mu_1 - \mu_2\|_{\mathcal{L}}^* = 0$, it suffices to show that $\mu_1(F) = \mu_2(F)$ for any closed set $F \subset X$; to this end, find a sequence $f_k \in \mathcal{L}(X)$ converging to the indicator function of F.

The following theorem is of fundamental importance. Its proof can be found in [Dud02, Corollary 11.5.5].

Theorem 1.19. *The set $\mathcal{P}(X)$ is a complete metric space with respect to $\|\cdot\|_{\mathcal{L}}^*$.*

1.5 Maximal coupling of measures

Let X be a Polish space and let $\mu_1, \mu_2 \in \mathcal{P}(X)$.

Definition 1.20. A pair of random variables (ξ_1, ξ_2) defined on the same probability space is called a *coupling for* (μ_1, μ_2) if

$$\mathcal{D}(\xi_1) = \mu_1, \quad \mathcal{D}(\xi_2) = \mu_2.$$

Let (ξ_1, ξ_2) be a coupling for (μ_1, μ_2). Then for any $\Gamma \in \mathcal{B}_X$ we have

$$\begin{aligned}
\mu_1(\Gamma) - \mu_2(\Gamma) &= \mathbb{P}\{\xi_1 \in \Gamma\} - \mathbb{P}\{\xi_2 \in \Gamma\} \\
&= \mathbb{E}\bigl\{I_{\{\xi_1 \neq \xi_2\}}\bigl(I_\Gamma(\xi_1) - I_\Gamma(\xi_2)\bigr)\bigr\} \le \mathbb{P}\{\xi_1 \neq \xi_2\},
\end{aligned}$$

whence it follows that

$$\mathbb{P}\{\xi_1 \neq \xi_2\} \ge \|\mu_1 - \mu_2\|_{\text{var}}.$$

Definition 1.21. A coupling (ξ_1, ξ_2) for (μ_1, μ_2) is said to be *maximal* if

$$\mathbb{P}\{\xi_1 \neq \xi_2\} = \|\mu_1 - \mu_2\|_{\mathrm{var}},$$

and the random variables ξ_1 and ξ_2 conditioned on the event [3] $N = \{\xi_1 \neq \xi_2\}$ are independent, that is, for any $\Gamma_1, \Gamma_2 \in \mathcal{B}_X$,

$$\mathbb{P}\{\xi_1 \in \Gamma_1, \xi_2 \in \Gamma_2 \,|\, N\} = \mathbb{P}\{\xi_1 \in \Gamma_1 \,|\, N\}\, \mathbb{P}\{\xi_1 \in \Gamma_1 \,|\, N\},$$

where, for any $B \in \mathcal{F}$, we set $\mathbb{P}(B \,|\, N) = \frac{\mathbb{P}(BN)}{\mathbb{P}(N)}$.

Theorem 1.22. *For any pair of measures $\mu_1, \mu_2 \in \mathcal{P}(X)$, there is a maximal coupling.*

Proof. If $\delta := \|\mu_1 - \mu_2\|_{\mathrm{var}} = 1$, then any pair (ξ_1, ξ_2) of independent random variables with $\mathcal{D}(\xi_i) = \mu_i$, $i = 1, 2$, is a maximal coupling for (μ_1, μ_2). If $\delta = 0$, then $\mu_1 = \mu_2$, and for any random variable ξ with distribution μ_1 the pair (ξ, ξ) is a maximal coupling. Hence, we can assume that $0 < \delta < 1$.

Let $m(du)$ be a measure satisfying the conditions of Proposition 1.16 and let

$$\rho_i = \frac{d\mu_i}{dm}, \quad \rho = \rho_1 \wedge \rho_2, \quad \hat{\rho}_i = \delta^{-1}(\rho_i - \rho).$$

Direct verification shows that $\hat{\mu}_i = \hat{\rho}_i dm$ and $\mu = (1-\delta)^{-1}\rho\, dm$ are probability measures on X. Let ζ_1, ζ_2, ζ, and α be independent random variables defined on the same probability space such that

$$\mathcal{D}(\zeta_i) = \hat{\mu}_i, \quad \mathcal{D}(\zeta) = \mu, \quad \mathbb{P}\{\alpha = 1\} = 1 - \delta, \quad \mathbb{P}\{\alpha = 0\} = \delta.$$

We claim that the random variables $\xi_i = \alpha\zeta + (1-\alpha)\zeta_i$, $i = 1, 2$, form a maximal coupling for (μ_1, μ_2). Indeed, for any $\Gamma \in \mathcal{B}_X$, we have

$$\begin{aligned}
\mathbb{P}\{\xi_i \in \Gamma\} &= \mathbb{P}\{\xi_i \in \Gamma, \alpha = 0\} + \mathbb{P}\{\xi_i \in \Gamma, \alpha = 1\} \\
&= \mathbb{P}\{\alpha = 0\}\mathbb{P}\{\zeta_i \in \Gamma\} + \mathbb{P}\{\alpha = 1\}\mathbb{P}\{\zeta \in \Gamma\} \\
&= \delta \int_\Gamma \hat{\rho}_i(u)\, dm + \int_\Gamma \rho(u)\, dm = \mu_i(\Gamma),
\end{aligned}$$

where we used the independence of $(\zeta_1, \zeta_2, \zeta, \alpha)$ and the relation $\rho_i = \rho + \delta\hat{\rho}_i$. Furthermore,

$$\begin{aligned}
\mathbb{P}\{\xi_1 \neq \xi_2\} &= \mathbb{P}\{\xi_1 \neq \xi_2, \alpha = 0\} + \mathbb{P}\{\xi_1 \neq \xi_2, \alpha = 1\} \\
&= \mathbb{P}\{\alpha = 0\}\mathbb{P}\{\zeta_1 \neq \zeta_2\} = \delta,
\end{aligned}$$

where we used again the independence of $(\zeta_1, \zeta_2, \zeta, \alpha)$ and also the relation

$$\mathbb{P}\{\zeta_1 = \zeta_2\} = \delta^{-2} \iint\limits_{\{u_1 = u_2\}} \big(\rho_1(u_1) - \rho(u_1)\big)\big(\rho_2(u_2) - \rho(u_2)\big)\, m(du_1)m(du_2) = 0.$$

A similar argument shows that the random variables ξ_1 and ξ_2 conditioned on $\{\xi_1 \neq \xi_2\}$ are independent. This completes the proof of Theorem 1.22. □

[3] In the case $\mathbb{P}\{\xi_1 \neq \xi_2\} = 0$, this condition should be omitted.

In what follows, we deal with pairs of measures depending on a parameter and we shall need a maximal coupling for them that depends on the parameter on a measurable manner.

Theorem 1.23. *Let $\mu_i(z,dx)$, $i=1,2$, be two probability measures on $\mathbb{R}^N$ that depend on a parameter $z\in\mathbb{R}^n$. Suppose that*

$$\mu_i(z,dx)=p_i(z,x)\,dx \quad \text{for any } z\in\mathbb{R}^n,$$

where $p_i(z,x)$ is a measurable function of $(z,x)\in\mathbb{R}^n\times\mathbb{R}^N$. Then there are measurable functions $\xi_i(z,\omega)\colon\mathbb{R}^n\times\Omega\to\mathbb{R}^N$, $i=1,2$, defined on the same probability space such that $(\xi_1(z,\cdot),\xi_2(z,\cdot))$ is a maximal coupling for $(\mu_1(z,dx),\mu_2(z,dx))$ for any $z\in\mathbb{R}^n$.

*Exercise** 1.24. Prove Theorem 1.23. *Hint:* Repeat the construction of Theorem 1.22 choosing for ζ_1, ζ_2, ζ, α measurable functions of (z,ω); begin with the case $N=1$ (see [KS01, Section 4] for details).

2 Randomly forced Navier–Stokes equations

2.1 Cauchy problem

Let us consider the 2D Navier–Stokes (NS) system in a bounded domain $D\subset\mathbb{R}^2$ with smooth boundary ∂D:

$$\dot u+(u,\nabla)u-\Delta u+\nabla p=\eta(t,x),\quad \operatorname{div}u=0,\quad x\in D. \tag{2.1}$$

Here $u=(u_1,u_2)$ is the velocity field of the fluid, $p(t,x)$ is the pressure, and $\eta(t,x)$ is an external force. Equations (2.1) are supplemented with the Dirichlet boundary condition for u:

$$u\big|_{\partial D}=0. \tag{2.2}$$

Let us set

$$\mathcal{V}=\left\{u\in C_0^\infty(D,\mathbb{R}^2):\operatorname{div}u=0\right\}$$

and denote by H and V the closure of $\mathcal{V}$ in $L^2(D,\mathbb{R}^2)$ and $H^1(D,\mathbb{R}^2)$, respectively. Denoting by $\Pi\colon L^2(D,\mathbb{R}^2)\to H$ the orthogonal projection onto H and applying it (formally) to Eqs. (2.1), we obtain

$$\dot u+Lu+B(u,u)=\eta(t), \tag{2.3}$$

where $L=-\Pi\Delta$, $B(u,v)=\Pi(u,\nabla)v$, and we retained the notation for the right-hand side.

Let us assume that η has the form

$$\eta(t)=\sum_{k=1}^{\infty}\eta_k\delta(t-k), \tag{2.4}$$

where $\{\eta_k\}$ is a sequence in H and $\delta(t)$ is the Dirac measure concentrated at zero.

Definition 2.1. A function $u_t : \mathbb{R}_+ \to H$ is called a *solution of* (2.3), (2.4) if the following two properties hold for any integer $k \geq 1$.

(i) The restriction of u_t to interval $I_k := [k-1, k)$ belongs to the space $C(I_k, H) \cap L^2(I_k, V)$ and satisfies the homogeneous NS system

$$\dot{u} + Lu + B(u, u) = 0. \tag{2.5}$$

(ii) There is a limit $\lim_{t \to k^-} u_t = u_k^-$, and $u_k = u_k^- + \eta_k$ (see Figure 1).

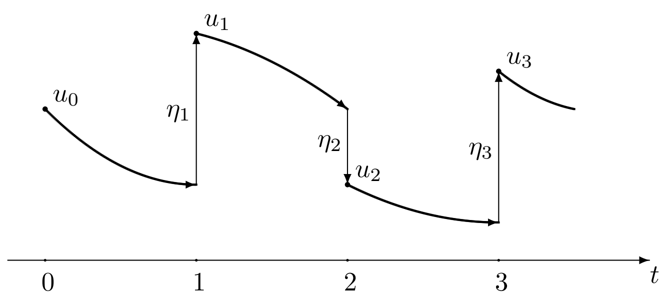

Figure 1: Evolution defined by Equations (2.3), (2.4)

Let us fix an arbitrary function $v \in H$ and consider the Cauchy problem for (2.3), (2.4):

$$u_0 = v. \tag{2.6}$$

We denote by $S_t : \mathbb{R}_+ \to H$ the resolving semigroup for Eq. (2.5), i.e., we set $S_t(v) = u_t$, where $u_t \in C(\mathbb{R}_+, H) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+, V)$ is the unique solution of (2.5), (2.6) defined on the half-line $\mathbb{R}_+$.

Theorem 2.2. *For any $v \in H$, the problem (2.3), (2.4) has a unique solution u_t satisfying the initial condition (2.6). Moreover, for any integer $k \geq 1$, we have*

$$u_k = S(u_{k-1}) + \eta_k, \tag{2.7}$$

where $S = S_1$.

Proof. For integer values of t, we define u_t inductively by relation (2.7), and for $t \in [k, k+1)$, we set $u_t = S_{t-k}(u_k)$. The resulting function is the unique solution of the problem in question. $\square$

2.2 A priori estimates

The operator $L = -\Pi\Delta$ with domain $D(L) = H^2(D, \mathbb{R}^2) \cap V$ is self-adjoint, and its inverse is compact. It follows that the set $\{e_j\}$ of normalised eigenfunctions of L form an orthonormal basis in H. We shall denote by $\alpha_1 \le \alpha_2 \le \cdots$ the corresponding eigenvalues. Furthermore, for $\gamma \in \mathbb{R}$, we set

$$D(L^\gamma) = \left\{ u(x) = \sum_{j=1}^{\infty} f_j e_j(x) : \sum_{j=1}^{\infty} \alpha_j^{2\gamma} f_j^2 < \infty \right\}$$

and define powers of L by the formula

$$(L^\gamma u)(x) = \sum_{j=1}^{\infty} \alpha_j^\gamma f_j e_j(x).$$

Finally, let us introduce the Sobolev norm $\|u\|_\gamma = \big(L^{\frac{\gamma}{2}} u, L^{\frac{\gamma}{2}} u\big)^{\frac12}$ for $u \in D(L^{\frac{\gamma}{2}})$. We shall write $|\cdot|$ and $\|\cdot\|$ instead of $\|\cdot\|_0$ and $\|\cdot\|_1$ for the norms in H and V, respectively.

Theorem 2.3. **(i)** *Let u_t be a solution of (2.5). Then*

$$|u_t|^2 + 2\int_0^t \|u_s\|^2 ds = |u_0|^2, \tag{2.8}$$

$$t\,\|u_t\|_{1/2}^2 + \int_0^t s\,\|u_s\|_{3/2}^2 ds \le C\,|u_0|^2 \exp\biggl(C\int_0^t \|u_s\|^2 ds\biggr), \tag{2.9}$$

where $t \ge 0$, and $C > 0$ is a constant not depending on u_t.

(ii) *Let u_t and u_t' be two solutions of (2.5). Then*

$$\|u_t - u_t'\|_{1/2} \le C\,t^{-\frac12}\big(1 + |u_0| + |u_0'|\big)^{\frac32} \exp\biggl(C\int_0^t \big(\|u_s\|^2 + \|u_s'\|^2\big)ds\biggr)|u_0 - u_0'|, \tag{2.10}$$

where the constant $C > 0$ does not depend on solutions.

Proof. Step 1. Let $(\cdot,\cdot)$ be the natural scalar product in H. Taking the scalar product of (2.5) with $2u_t$, we derive

$$\partial_t |u_t|^2 + 2(Lu_t, u_t) = 0, \tag{2.11}$$

where we used the relation

$$(B(v,w), w) = 0, \quad v, w \in V. \tag{2.12}$$

Integration of (2.11) with respect to time results in (2.8).

Step 2. Let us take the scalar product of (2.5) with $2tL^{\frac12}u$. Performing some simple transformations, we derive

$$\partial\big(t\|u\|_{1/2}^2\big) - \|u\|_{1/2}^2 + 2t\,\|u\|_{3/2}^2 + 2t\,(B(u,u), L^{\frac12}u) = 0.$$

Taking into account the inequality (see Exercise 2.4 below)

$$\big|(B(u,u), L^{\frac12}u)\big| \le C_1 \|u\|_{3/2}\|u\|\,\|u\|_{1/2} \le \tfrac12\big(\|u\|_{3/2}^2 + C_1^2\,\|u\|^2\|u\|_{1/2}^2\big), \tag{2.13}$$

we obtain

$$\partial\big(t\|u\|_{1/2}^2+1\big)+t\|u\|_{3/2}^2\le C_2\|u\|^2\big(t\|u\|_{1/2}^2+1\big). \tag{2.14}$$

Ignoring the second term on the left-hand side of (2.14) and applying the Gronwall inequality, we derive

$$t\,\|u_t\|_{1/2}^2+1\le \exp\biggl(C_2\int_0^t\|u_s\|^2ds\biggr). \tag{2.15}$$

Integrating (2.14) with respect to time and using (2.15) and (2.8) to estimate the right-hand side, we arrive at (2.9).

Step 3. The difference $w=u-u'$ of two solutions satisfies the equation

$$\dot w+Lw+B(w,u)+B(u',w)=0. \tag{2.16}$$

Taking the scalar product of (2.16) with $2w$ and recalling relation (2.12) and also the inequality

$$|(B(w,u),w)|\le C_3|w|\,\|w\|\,\|u\|\le \tfrac12\left(\|w\|^2+C_3^2\,\|u\|^2|w|^2\right),$$

we derive

$$\partial_t|w|^2+\|w\|^2\le C_3^2\,\|u\|^2|w|^2.$$

Repeating the argument applied in Step 2, we see that

$$|u_t-u_t'|^2+\int_0^t\|u_s-u_s'\|^2ds\le \exp\biggl(C_4\int_0^t\|u_s\|^2ds\biggr)|u_0-u_0'|^2,\quad t\ge0. \tag{2.17}$$

We now take the scalar product of (2.16) with $2tL^{\frac12}w$:

$$\partial\big(t\|w\|_{1/2}^2\big)+2t\,\|w\|_{3/2}^2=\|w\|_{1/2}^2-2t\big\{(B(w,u),L^{\frac12}w)+(B(u',w),L^{\frac12}w)\big\}. \tag{2.18}$$

We have (see Exercise 2.4)

$$\begin{aligned}\big|(B(w,u),L^{\frac12}w)\big|&\le C_5\big(\|w\|_{3/2}^4\|u\|_{3/2}^2|u|\,|w|^2\big)^{\frac13}\\&\le\frac18\,\|w\|_{3/2}^2+2\,C_5^2\,\|u\|_{3/2}^2|u|\,|w|^2, \qquad (2.19)\\ \big|(B(u',w),L^{\frac12}w)\big|&\le C_6\big(\|u'\|_{3/2}^2|u'|\|w\|_{3/2}^4|w|^2\big)^{\frac13}\\&\le\frac18\,\|w\|_{3/2}^2+2\,C_6^2\,\|u'\|_{3/2}^2|u'|\,|w|^2. \qquad (2.20)\end{aligned}$$

Substituting these estimates into (2.18), we obtain

$$\partial\big(t\|w\|_{1/2}^2\big)+t\,\|w\|_{3/2}^2\le\|w\|_{1/2}^2+C_7t\left(\|u\|_{3/2}^2|u|+\|u'\|_{3/2}^2|u'|\right)|w|^2.$$

Integrating this inequality with respect to time and using (2.8), (2.9), and (2.17), we arrive at (2.10). $\square$

Exercise 2.4. Prove the first inequalities in (2.13), (2.19), and (2.20). *Hint:* Use the estimate

$$\|u\|_{L^\infty} + \|u\| \le \text{const}\,\|u\|_{1+\alpha}^{\frac{\beta}{\alpha+\beta}}\|u\|_{1-\beta}^{\frac{\alpha}{\alpha+\beta}}, \tag{2.21}$$

where $\|u\|_{L^\infty}$ is the essential supremum of $u(x)$ and $0 < \alpha, \beta \le 1$.

In what follows, we shall need the following three estimates for S. They are consequences of inequalities (2.8), (2.10), and (2.17).

Corollary 2.5. *For any $v \in H$, we have*

$$|S(v)| \le q\,|v|, \quad q = e^{-\alpha_1}. \tag{2.22}$$

Furthermore, there is a constant $C > 0$ such that, for any $v, v' \in H$, we have

$$\big|S(v) - S(v')\big| \le \exp\Big\{C\int_0^1 \|u_s\|^2 ds\Big\}|v - v'|, \tag{2.23}$$

$$\big\|S(v) - S(v')\big\|_{1/2} \le C\,\exp\Big\{C\int_0^1 \big(\|u_s\|^2 + \|u_s'\|^2\big)\,ds\Big\}|v - v'|, \tag{2.24}$$

where u_s and u_s' are the solutions of (2.5) that correspond to the initial functions v and v', respectively.

Proof. Let u_t be the solution of (2.5), (2.6). It follows from (2.8) and the inequality $\|u\|^2 \ge \alpha_1|u|^2$ that

$$|u_t|^2 + 2\alpha_1\int_0^t |u_s|^2 ds \le |v|^2.$$

Applying the Gronwall inequality, we derive

$$|u_t|^2 \le e^{-2\alpha_1 t}|v|^2. \tag{2.25}$$

In particular, for $t = 1$, we obtain (2.22).

Inequality (2.23) follows from (2.17) with $t = 1$. Substituting (2.25) into (2.8) and setting $t = 1$, we get

$$2\int_0^1 \|u_s\|^2 ds \ge |u_0|^2(1 - e^{-2\alpha_1}) \ge c|u_0|^2,$$

where $c > 0$ is a constant. Combining this inequality and its analogue for u_t' with (2.10), we obtain (2.24). □

Finally, we establish an estimate for the difference between two solutions of Eq. (2.3) with different right-hand sides. Namely, for any integer $N \ge 1$, we denote by H_N the subspace in H spanned by the functions e_j, $j = 1, \dots, N$. Let P_N be the orthogonal projection onto H_N and let $\mathsf{Q}_N = I - \mathsf{P}_N$, where I is the identity operator.

Proposition 2.6. *Let u_t and u'_t be two solutions of Eq. (2.3) with right-hand sides*

$$\eta(t)=\sum_{k=1}^{\infty}\eta_k\delta(t-k),\quad \eta'(t)=\sum_{k=1}^{\infty}\eta'_k\delta(t-k),$$

respectively. Then there is a constant $C>0$ not depending on solutions and right-hand sides such that, for any integers $m<k$ and $N\ge 1$, we have

$$\begin{aligned}\bigl|\mathsf{Q}_N(u_k-u'_k)\bigr| &\le \bigl|\mathsf{Q}_N(\eta_k-\eta'_k)\bigr| + \bigl(C\alpha_N^{-\frac14}\bigr)^{k-m}D(m,k)|u_m-u'_m| + \\ &+\sum_{l=m+1}^{k-1}\bigl(C\alpha_N^{-\frac14}\bigr)^{k-l}D(l,k)\bigl(|\mathsf{P}_N(u_l-u'_l)|+|\mathsf{Q}_N(\eta_l-\eta'_l)|\bigr),\end{aligned}\tag{2.26}$$

where we set

$$D(m,k)=C\int_m^k\bigl(\|u_s\|^2+\|u'_s\|^2\bigr)\,ds.$$

Proof. Let us fix an arbitrary $N\ge 1$. In view of (2.7), (2.24), and the inequality $|\mathsf{Q}_Nv|\le\alpha_N^{-\frac14}\|v\|_{1/2}$, for any integer $l\ge 1$, we have

$$\begin{aligned}\bigl|\mathsf{Q}_N(u_l-u'_l)\bigr| &\le \bigl|\mathsf{Q}_N\bigl(S(u_{l-1})-S(u'_{l-1})\bigr)\bigr|+\bigl|\mathsf{Q}_N(\eta_l-\eta'_l)\bigr| \\ &\le \bigl\|S(u_{l-1})-S(u'_{l-1})\bigr\|_{1/2}+\bigl|\mathsf{Q}_N(\eta_l-\eta'_l)\bigr| \\ &\le C\alpha_N^{-\frac14}D(l-1,l)\,\bigl|u_{l-1}-u'_{l-1}\bigr|+\bigl|\mathsf{Q}_N(\eta_l-\eta'_l)\bigr|.\end{aligned}$$

Arguing by induction, we obtain (2.26). □

2.3 Markov chain associated with the NS system

From now on, we shall study the discrete-time random dynamical system (RDS)

$$u_k=S(u_{k-1})+\eta_k,\tag{2.27}$$

$$u_0=u,\tag{2.28}$$

where $\{\eta_k\}$ is a sequence of independent identically distributed (i.i.d.) H-valued random variables and $u=u(x)$ is an initial (random) function. We shall sometimes write $u_k(u)$ to indicate the dependence of the trajectory on the initial function u.

Theorem 2.7. *Let $\{u_k\}$ be a sequence defined by (2.27), (2.28), where u is an H-valued random variable independent of $\{\eta_k,k\ge 1\}$. Suppose that $m_1=\mathbb{E}\,|\eta_k|<\infty$. Then*

$$\mathbb{E}\,|u_k|\le q^k\mathbb{E}\,|u|+m_1\bigl(1+q+\dots+q^{k-1}\bigr),\tag{2.29}$$

where $q\in(0,1)$ is the constant in (2.22). Moreover, the sequence $\{u_k\}$ satisfies the Markov property. Namely, for any integers $k,n\ge 0$ and any bounded measurable function $f\colon H\to\mathbb{R}$, we have

$$\mathbb{E}\,\bigl(f(u_{k+n})\,|\,\mathcal{F}_k\bigr)=\bigl(\mathbb{E}\,f(u_n(v))\bigr)\bigr|_{v=u_k},\tag{2.30}$$

where $\mathcal{F}_k$ is the σ-algebra generated[4] *by $\eta_1, \dots, \eta_k$ and u, and the equality holds almost surely.*

Proof. To prove (2.29), we note that

$$\mathbb{E}\,|u_k| \le \mathbb{E}\,|S(u_{k-1})| + \mathbb{E}\,|\eta_k| \le q\mathbb{E}\,|u_{k-1}| + m_1,$$

where we used (2.22). Iteration of this inequality results in (2.29).

Let us prove the Markov property. We shall write $u_k(u) = u_k(u; \eta_1, \dots, \eta_k)$ to indicate the dependence of the trajectory of (2.27), (2.28) on the random variables $\{\eta_m\}$. We have

$$u_{k+n}(u; \eta_1, \dots, \eta_{k+n}) = u_n(u_k(u); \eta_{k+1}, \dots, \eta_{k+n}).$$

Since $u_k(u)$ is $\mathcal{F}_k$-measurable and $\{\eta_i,\, i \ge k+1\}$ is independent of $\mathcal{F}_k$, it follows from the above relation and Exercise 1.13 that

$$\mathbb{E}\,\bigl(f(u_{k+n}(u)) \,|\, \mathcal{F}_k\bigr) = \bigl(\mathbb{E}\,f(u_n(v; \eta_{k+1}, \dots, \eta_{k+n}))\bigr)\,\big|_{v=u_k(u)}. \tag{2.31}$$

Now note that the distributions of the vectors $(\eta_{k+1}, \dots, \eta_{k+n})$ and $(\eta_1, \dots, \eta_n)$ coincide. Therefore,

$$\mathbb{E}\,f(u_n(v; \eta_{k+1}, \dots, \eta_{k+n}) = \mathbb{E}\,f(u_n(v; \eta_1, \dots, \eta_n)),$$

where $v \in H$ is an arbitrary deterministic function. Substitution of the above relation into the right-hand side of (2.31) completes the proof of (2.30). $\square$

Exercise 2.8. In the notation of Theorem 2.7, show that, if $f\colon H \times \cdots \times H \to \mathbb{R}$ is a bounded measurable function of $n+1$ arguments, then

$$\mathbb{E}\,\bigl(f(u_k, u_{k+1}, \dots, u_{k+n}) \,|\, \mathcal{F}_k\bigr) = \bigl(\mathbb{E}\,f(v, u_1(v), \dots, u_n(v))\bigr)\,\big|_{v=u_k}.$$

The Markov property implies two important corollaries. To formulate them, we introduce the *transition function* for the RDS (2.27). Namely, for any deterministic function $v \in H$ and any integer $k \ge 0$, we denote by $P_k(v, \cdot)$ the distribution of $u_k(v)$:

$$P_k(v, \Gamma) = \mathbb{P}\{u_k(v) \in \Gamma\}, \quad \Gamma \in \mathcal{B}_H. \tag{2.32}$$

Corollary 2.9. *Let $u(x)$ be an H-valued random variable independent of $\{\eta_k\}$ and let μ be the distribution of u. Then the distribution of $u_k = u_k(u)$ is given by the formula*

$$\mathcal{D}(u_k)(\Gamma) = \int_H P_k(v, \Gamma)\mu(dv). \tag{2.33}$$

In particular, the measure $\mathcal{D}(u_k)$ depends only on μ (but not on the random variable u).

---

[4] We denote by $\mathcal{F}_0$ the σ-algebra generated by u.

Proof. Let us fix an arbitrary Borel set $\Gamma \subset H$. In view of relation (2.30) with $f(z) = I_\Gamma(z)$, we have

$$\mathbb{E}\, I_\Gamma(u_k) = \mathbb{E}\left\{\mathbb{E}\left(I_\Gamma(u_k)\,|\,\mathcal{F}_0\right)\right\} = \mathbb{E}\left\{\left(\mathbb{E}\, I_\Gamma(u_k(v))\right)\big|_{v=u_0}\right\}.$$

It remains to note that $\mathbb{E}\, I_\Gamma(u_k(v)) = \mathbb{P}\{u_k(v) \in \Gamma\} = P_k(u, \Gamma)$. □

Corollary 2.10. *The transition function $P_k(v, \Gamma)$ satisfies the Chapman–Kolmogorov relation. Namely, for any $k, n \geq 0$, $v \in H$, and $\Gamma \in \mathcal{B}_H$, we have*

$$P_{k+n}(v, \Gamma) = \int_H P_k(v, dz) P_n(z, \Gamma). \tag{2.34}$$

Proof. In view of (2.30), we have

$$\begin{aligned} P_{k+n}(v, \Gamma) &= \mathbb{E}\, I_\Gamma(u_{k+n}(v)) = \mathbb{E}\left\{\mathbb{E}\left(I_\Gamma(u_{k+n}(v))\,|\,\mathcal{F}_k\right)\right\} \\ &= \mathbb{E}\left\{\mathbb{E}\left(I_\Gamma(u_n(z))\right)\big|_{z=u_k(v)}\right\} = \mathbb{E}\left\{P_n(u_k(v), \Gamma)\right\}. \end{aligned}$$

This expression coincides with the integral on the right-hand side of (2.34). □

3 Stationary measures and exponential mixing

3.1 Existence of stationary measures

Let us recall that we denote by $P_k(v, \Gamma)$ the transition function associated with the RDS (2.27), (2.28) (see (2.32)). We now introduce the corresponding *Markov semigroups*:

$$\begin{aligned} &\mathfrak{P}_k : C_b(H) \to C_b(H), \qquad && \mathfrak{P}_k f(v) = \int_H P_k(v, dz) f(z), \\ &\mathfrak{P}_k^* : \mathcal{P}(H) \to \mathcal{P}(H), \qquad && \mathfrak{P}_k^* \mu(\Gamma) = \int_H P_k(v, \Gamma)\mu(dv). \end{aligned}$$

Exercise 3.1. Show that operators $\mathfrak{P}_k$ and $\mathfrak{P}_k^*$ are well defined. Show also that they form semigroups, that is, $\mathfrak{P}_0 = \mathrm{Id}$ and $\mathfrak{P}_{k+n} = \mathfrak{P}_n \circ \mathfrak{P}_k$, and similarly for $\mathfrak{P}_k^*$. *Hint:* Use the Chapman–Kolmogorov relation (2.34).

Exercise 3.2. Show that $\mathfrak{P}_k$ and $\mathfrak{P}_k^*$ are dual semigroups in the sense that

$$(\mathfrak{P}_k f, \mu) = (f, \mathfrak{P}_k^* \mu) \quad \text{for any } f \in C_b(H),\ \mu \in \mathcal{P}(H).$$

Definition 3.3. A measure $\mu \in \mathcal{P}(H)$ is said to be *stationary* for the RDS (2.27) if $\mathfrak{P}_1^* \mu = \mu$.

Let us note that, if $\mu \in \mathcal{P}(H)$ is a stationary measure and $u(x)$ is a random function in H with distribution μ, then the distribution of the trajectory u_k for (2.27), (2.28) coincides with μ for any $k \geq 1$. This assertion is a straightforward consequence of relation (2.33) and Definition 3.3.

Theorem 3.4. *Suppose that $\mathbb{E}\,|\eta_k| < \infty$. Then the RDS (2.27) has at least one stationary measure.*

Proof. We shall apply the classical Bogolyubov–Krylov argument (e.g., see Theorem 1.5.8 in [Arn98]).

Step 1. Let u_k be the trajectory of (2.27), (2.28) with $u \equiv 0$ and let μ_k be the distribution of u_k. We set

$$\bar\mu_k = \frac{1}{k}\sum_{l=0}^{k-1}\mu_l.$$

Suppose we have shown that the sequence $\{\mu_k\}$ is relatively compact in the space $\mathcal{P}(X)$ endowed with the metric $\|\cdot\|_{\mathcal{L}}^*$ (see (1.12)). Then there is a subsequence μ_{k_m} and a measure $\mu \in \mathcal{P}(H)$ such that $\mu_{k_m} \rightharpoonup \mu$ as $m \to +\infty$, where $\rightharpoonup$ stands for convergence with respect to the metric $\|\cdot\|_{\mathcal{L}}^*$. We claim that μ is a stationary measure. Indeed, for any $f \in \mathcal{L}(H)$, we have

$$\begin{aligned}(f, \mathfrak{P}_1^*\mu) &= \lim_{m\to\infty}\bigl(f, \mathfrak{P}_1^*\bar\mu_{k_m}\bigr) = \lim_{m\to\infty}\frac{1}{k_m}\sum_{l=0}^{k_m-1}(f, \mathfrak{P}_1^*\mu_l)\\ &= \lim_{m\to\infty}\Bigl\{\bigl(f,\bar\mu_{k_m}\bigr) - \frac{1}{k_m}\bigl(f,\mu_0\bigr) + \frac{1}{k_m}\bigl(f,\mu_{k_m}\bigr)\Bigr\} = (f,\mu). \qquad (3.1)\end{aligned}$$

Since this relation is true for any $f \in \mathcal{L}(H)$, we conclude that $\mathfrak{P}_1^*\mu = \mu$.

Step 2. Let us show that $\{\mu_k\}$ is relatively compact. We resort to the following assertion due to Prokhorov (see Theorem 11.5.4 in [Dud02]).

Proposition 3.5. *A family $\{\mu_\alpha\}$ of probability Borel measures on a Polish space is relatively compact iff for any $\varepsilon > 0$ there is a compact subset K_ε such that $\mu_\alpha(K_\varepsilon) \ge 1 - \varepsilon$ for any α.*

We shall show that for any $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset H$ such that $\mu_k(K_\varepsilon) \ge 1-\varepsilon$ for any $k \ge 1$. This will imply that $\{\bar\mu_k\}$ is relatively compact.

Since $u_k = S(u_{k-1}) + \eta_k$, the required assertion will be established if we prove that

$$\mathbb{P}\bigl\{S(u_{k-1}) \notin K_\varepsilon^1\bigr\} \le \varepsilon/2, \quad \mathbb{P}\bigl\{\eta_k \notin K_\varepsilon^2\bigr\} \le \varepsilon/2. \qquad (3.2)$$

where K_ε^1 and K_ε^2 are compact sets in H. (We can take $K_\varepsilon = K_\varepsilon^1 + K_\varepsilon^2$.)

Step 3. It follows from (2.29) that $\mathbb{E}\,|u_k| \le m_1(1-q)^{-1}$ for all $k \ge 1$. Therefore we can choose $R_\varepsilon > 0$ so large that

$$\mathbb{P}\bigl\{|u_{k-1}| > R_\varepsilon\bigr\} \le R_\varepsilon^{-1}\mathbb{E}\,|u_{k-1}| \le \varepsilon/2. \qquad (3.3)$$

Furthermore, since the embedding $H^s \subset H$ is compact for $s > 0$, we conclude from inequality (2.24) with $v' = 0$ and relation (2.8) that the image under S of any bounded set in H is relatively compact. Hence, setting $K_\varepsilon^1 = S\bigl(B_H(R_\varepsilon)\bigr)$, from (3.3) we derive

$$\mathbb{P}\bigl\{S(u_{k-1}) \notin K_\varepsilon^1\bigr\} \le \mathbb{P}\bigl\{|u_{k-1}| > R_\varepsilon\bigr\} \le \varepsilon/2.$$

Finally, recall that, by Ulam's theorem, any probability Borel measure on a polish space is regular (see Theorem 7.1.4 in [Dud02]). Hence, if χ is the distribution of η_k, then there is a compact set $K_\varepsilon^2 \subset H$ such that $\chi(K_\varepsilon^2) \geq 1 - \varepsilon/2$. This is equivalent to the second inequality in (3.2). $\square$

Exercise 3.6. Justify the first relation in (3.1). *Hint:* Use the fact that $\mu_n \rightharpoonup \mu$ iff $(f, \mu_n) \to (f, \mu)$ for any $f \in C_b(H)$ (see Theorem 11.3.3 in [Dud02]).

Exercise 3.7. Show that any stationary measure μ has a finite moment, that is,

$$\mathfrak{m}(\mu) := \int_H |v| \mu(dv) < \infty.$$

Hint: Use inequality (2.29) and Fatou's lemma (see Lemma 4.3.3 in [Dud02] and Theorem 2.2 in [Shi02]).

3.2 Main theorem: uniqueness and mixing

In contrast to the existence of a stationary measure, which holds under rather general assumptions, to ensure its uniqueness, we have to impose some non-degeneracy conditions on the distribution of the random variables η_k. Namely, we shall assume that the following condition is fulfilled.

Hypothesis (H). The i.i.d. random variables η_k have the form

$$\eta_k(x) = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j(x), \tag{3.4}$$

where ξ_{jk} are independent scalar random variables and $b_j \geq 0$ are some constants such that

$$B := \sum_{j=1}^{\infty} b_j^2 < \infty. \tag{3.5}$$

Moreover, for any $j \geq 1$ the measure $\pi_j = \mathcal{D}(\xi_{jk})$ possesses a density $p_j(r)$ (with respect to the Lebesgue measure on $\mathbb{R}$) that is a function of bounded total variation such that

$$\operatorname{supp} p_j \subset [-1, 1], \qquad \int_{-\varepsilon}^{\varepsilon} p_j(r)\, dr > 0 \quad \text{for any } \varepsilon > 0. \tag{3.6}$$

Hypothesis (H) implies that, with probability 1, the random variables η_k are contained in the ball of radius $\sqrt{B}$ centered at zero. The following theorem, which is the main result of this course, is established in [KS01, KPS02].

Theorem 3.8. *Suppose that Condition* (H) *is satisfied. Then for any $B_0 > 0$ there is an integer $N \geq 1$ such that, if*

$$B \leq B_0, \qquad b_j \neq 0 \quad \textit{for } j = 1, \dots, N, \tag{3.7}$$

then the RDS (2.27) *has a unique stationary measure* $\mu \in \mathcal{P}(H)$. *Moreover, there are positive constants* C *and* β *such that, for any functional* $f \in \mathcal{L}(H)$ *and any* H*-valued random variable* $u(x)$ *that is independent of* $\{\eta_k\}$ *and has a finite mean value, we have*

$$\big|\mathbb{E}\, f(u_k) - (f,\mu)\big| \le C\,\|f\|_{\mathcal{L}}\big(1+\mathbb{E}\,|u|\big)\,e^{-\beta k}, \quad k \ge 0, \tag{3.8}$$

where $\{u_k\}$ *is the trajectory defined by* (2.27), (2.28).

A scheme of the proof of this theorem is given in the next subsection, and the details occupy Sections 4 and 5. Here we formulate two important corollaries of Theorem 3.8.

Corollary 3.9. *For any* $v \in H$, *we have*

$$\|P_k(v,\cdot) - \mu\|_{\mathcal{L}}^* \le C\left(1+|v|\right)e^{-\beta k}, \quad k \ge 0. \tag{3.9}$$

Moreover, for any initial measure $\lambda \in \mathcal{P}(H)$ *with finite moment* $\mathfrak{m}(\lambda) < \infty$, *we have*

$$\|\mathfrak{P}_k^*\lambda - \mu\|_{\mathcal{L}}^* \le C\left(1+\mathfrak{m}(\lambda)\right)e^{-\beta k}, \quad k \ge 0. \tag{3.10}$$

Corollary 3.10. *For any* $v \in H$ *and* $f \in \mathcal{L}(H)$, *we have*

$$\big|\mathfrak{P}_k f(v) - (f,\mu)\big| \le C\,\|f\|_{\mathcal{L}}\big(1+|v|\big)\,e^{-\beta k}, \quad k \ge 0. \tag{3.11}$$

Exercise 3.11. Show that inequalities (3.8), (3.9), (3.10), and (3.11) are pairwise equivalent.

3.3 Scheme of the proof of the main result

Step 1: Convergence implies uniqueness. We first note that it suffices to establish inequality (3.8), where μ is a probability measure in H. Indeed, if $\hat\mu \in \mathcal{P}(H)$ is stationary measure such that $\mathfrak{m}(\hat\mu) < \infty$, then, by (3.10), we have

$$\|\hat\mu - \mu\|_{\mathcal{L}}^* = \|\mathfrak{P}_k^*\hat\mu - \mu\|_{\mathcal{L}}^* \le C\left(1+\mathfrak{m}(\hat\mu)\right)e^{-\beta k}.$$

Passing to the limit as $k \to \infty$, we see that $\hat\mu = \mu$. Thus, it remains to show that any stationary measure has a finite moment. This follows from Exercise 3.7.

Step 2: Bounded absorbing invariant set. Since inequalities (3.8) and (3.11) are equivalent (see Exercise 3.11), we shall prove the latter. To this end, we note that the ball in H of radius $R = 2(1-q)^{-1}\sqrt{B}$ centred at zero is an invariant absorbing set for the RDS (2.27). Indeed, it follows from Hypothesis (H) that $\mathbb{P}\{|\eta_k| \le \sqrt{B}\} = 1$. Therefore, in view of (2.22), we have

$$|S(v) + \eta_k| \le q|v| + \sqrt{B} \quad \text{for any } v \in H,\ k \ge 1.$$

This inequality implies that $B_H(R)$ is invariant. Furthermore, its iteration results in

$$|u_k(v)| \le q^k|v| + R/2.$$

If $|v| \geq R$, it follows that

$$u_k(v) \in B_H(R) \quad \text{for} \quad k \geq \ell(v) := \left[\frac{\log\bigl(2|v|/R\bigr)}{\log q^{-1}}\right] + 1, \tag{3.12}$$

where $[a]$ is the integer part of $a \geq 0$.

Suppose now that we have proved inequality (3.11) for $v \in B_H(R)$ and assume that $|v| > R$. Then, applying the Markov property (see (2.30)), for $n \geq 0$ and $\ell = \ell(v)$, we derive

$$\mathfrak{P}_{n+\ell} f(v) = \mathbb{E}\left\{\mathbb{E}\left(f(u_{n+\ell}(v)) \,|\, \mathcal{F}_\ell\right)\right\} = \mathbb{E}\left\{\mathfrak{P}_n f(u_\ell(v))\right\}. \tag{3.13}$$

Since $u_\ell(v) \in B_H(R)$ (see (3.12)), we have

$$\left|\mathfrak{P}_n f(u_\ell(v)) - (f, \mu)\right| \leq C(1+R)e^{-\beta n}.$$

Taking the expectation, using (3.13), and setting $k = n + \ell$, we arrive at

$$\left|\mathfrak{P}_k f(v) - (f, \mu)\right| \leq C(1+R)e^{-\beta n} = C_1(1+|v|)e^{-\beta k},$$

where we assumed that $\beta > 0$ is sufficiently small.

From now on, we shall consider the restriction of the RDS (2.27), (2.28) to the invariant set $X := B_H(R)$. We retain the notation for the associated objects, such as transition function, Markov semigroups, etc.

Step 3: Comparison of the transition functions with different starting points. We wish to show that (3.11) holds for any $v \in X$. To this end, it suffices to prove that

$$\|P_k(v, \cdot) - P_k(v', \cdot)\|_{\mathcal{L}}^* \leq C\, e^{-\beta k} \quad \text{for any } v, v' \in X. \tag{3.14}$$

Indeed, if inequality (3.14) is established, then, by the Chapman–Kolmogorov relation (2.34), for any $l \geq k$, $v, v' \in X$, and $f \in \mathcal{L}(X)$ with $\|f\|_{\mathcal{L}} \leq 1$, we have

$$\begin{aligned}
&\bigl|(P_k(v, \cdot) - P_l(v', \cdot), f)\bigr| \\
&\quad = \left|\int_X P_{l-k}(v', dz) \int_X \bigl(P_k(v, dw) f(w) - P_k(z, dw) f(w)\bigr)\right| \\
&\quad \leq C\, e^{-\beta k} \int_X P_{l-k}(v', dz) = C\, e^{-\beta k}.
\end{aligned} \tag{3.15}$$

This implies that $\{P_k(v', \cdot)\}$ is a Cauchy sequence in $\mathcal{P}(X)$. In view of Theorem 1.19, it must have a limit $\mu \in \mathcal{P}(X)$. Passing to the limit in (3.15) as $l \to \infty$, we arrive at (3.14).

Step 4: Reduction to construction of coupled trajectories. To prove (3.14), let us fix arbitrary initial functions $v, v' \in X$. Suppose that for any integer $k \geq 1$ we have constructed a coupling (v_k, v_k') for the pair of measures $(P_k(v, \cdot), P_k(v', \cdot))$ such that

$$\mathbb{P}\bigl\{|v_k - v_k'| > C\, e^{-\beta k}\bigr\} \leq C\, e^{-\beta k}, \tag{3.16}$$

where C and β are positive constants not depending on the initial functions. Then, setting $Q_k = \{|v_k - v'_k| > C\,e^{-\beta k}\}$, for any $f \in \mathcal{L}(X)$ we derive

$$\begin{aligned}\big|(P_k(v,\cdot) - P_k(v',\cdot), f)\big| &= \big|\mathbb{E}\left(f(v_k) - f(v'_k)\right)\big| \\ &\le \mathbb{E}\left\{I_{Q_k}|f(v_k) - f(v'_k)|\right\} + \mathbb{E}\left\{I_{Q_k^c}|f(v_k) - f(v'_k)|\right\} \\ &\le 2C\,\|f\|_\infty e^{-\beta k} + C\,\|f\|_{\mathcal{L}} e^{-\beta k},\end{aligned}$$

whence follows (3.14) (with a different constant C).

Construction of the coupling (v_k, v'_k) is carried out in Section 4, and inequality (3.16) is established in Section 5.

4 Coupling operators

4.1 Construction and basic properties

Let us recall that we consider the RDS (2.27) in the ball $X = B_H(R)$, where the constant $R > 0$ is defined in Step 2 of Section 3.3. As before, we denote by P_N and Q_N the orthogonal projections onto the spaces $H_N = \operatorname{span}\{e_1, \dots, e_N\}$ and $H_N^\perp$, respectively.

Let $\chi \in \mathcal{P}(H)$ be the distribution of η_k and let $\chi_N = \mathsf{P}_N\chi$. Hypothesis (H) (see Section 3.2) implies that, if

$$b_j \ne 0 \quad \text{for } j = 1, \dots, N, \tag{4.1}$$

then χ_N has a density with respect to the Lebesgue measure in H_N. Namely, setting $y = (y_1, \dots, y_N) \in H_N$, we have

$$\chi_N(dy) = h(y)\,dy, \quad h(y) = \prod_{j=1}^N b_j^{-1} p_j\big(y_j/b_j\big). \tag{4.2}$$

Exercise 4.1. Prove relation (4.2).

For any $v \in H$, we denote by $\chi_N(v, dy)$ the distribution of the random variable $\mathsf{P}_N(S(v) + \eta_1)$. One easily shows that

$$\chi_N(v, dy) = h(y - \mathsf{P}_N S(v))\,dy. \tag{4.3}$$

Proposition 4.2. *Suppose that Hypothesis* (H) *is fulfilled and that* (4.1) *holds for some integer $N \ge 1$. Then there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for any pair of functions $v, v' \in H$ there are two H-valued random variables $\zeta = \zeta(v, v', \omega)$ and $\zeta' = \zeta'(v, v', \omega)$ possessing the following properties:*

(i) *The distributions of ζ and ζ' coincide with χ.*

(ii) *The random variables $(\mathsf{P}_N\zeta, \mathsf{P}_N\zeta')$ and $(\mathsf{Q}_N\zeta, \mathsf{Q}_N\zeta')$ are independent, and the projections $\mathsf{Q}_N\zeta$ and $\mathsf{Q}_N\zeta'$ coincide for all $\omega \in \Omega$ and do not depend on (v, v').*

(iii) *The pair*

$$V_N = \mathsf{P}_N\big(S(v)+\zeta\big), \quad V_N' = \mathsf{P}_N\big(S(v')+\zeta'\big)$$

is a maximal coupling for $(\chi_N(v,\cdot),\chi_N(v',\cdot))$. *Furthermore, there is a constant* $C_N \geq 1$ *depending only on* $\min\{b_j : 1\leq j\leq N\}$ *such that*

$$\mathbb{P}\{V_N \neq V_N'\} = \big\|\chi_N(v,\cdot)-\chi_N(v',\cdot)\big\|_{\mathrm{var}} \leq C_N\big|S(v)-S(v')\big|. \tag{4.4}$$

(iv) *The functions* ζ *and* ζ' *are measurable with respect to* (v,v',ω).

Taking this assertion for granted, let us complete the construction of the random sequences (v_k,v_k'). We define coupling operators by the formulas

$$\mathcal{R}(v,v',\omega)=S(v)+\zeta(v,v',\omega), \quad \mathcal{R}'(v,v',\omega)=S(v')+\zeta'(v,v',\omega). \tag{4.5}$$

Let $(\Omega_i,\mathcal{F}_i,\mathbb{P}_i)$, $i\geq 1$, be a sequence of countably many copies of the probability space constructed in Proposition 4.2 and let $(\Omega,\mathcal{F},\mathbb{P})$ be their direct product (see Exercise 1.7). The points of Ω will be denoted by $\omega=(\omega_1,\omega_2,\dots)$. We now fix $v,v'\in X$ and set

$$\begin{aligned} v_0 &= v, & v_0' &= v',\\ v_k &= \mathcal{R}(v_{k-1}(\omega),v_{k-1}'(\omega),\omega_k), & v_k' &= \mathcal{R}'(v_{k-1}(\omega),v_{k-1}'(\omega),\omega_k). \end{aligned} \tag{4.6}$$

Exercise 4.3. **(i)** Show that, for any $U=(v,v')$, the sequence $U_k=(v_k,v_k')$ constructed above is a Markov chain in the space $X\times X$ (cf. Theorem 2.7 and Exercise 2.8).

(ii) Show that

$$\mathcal{D}(v_k)=P_k(v,\cdot), \quad \mathcal{D}(v_k')=P_k(v',\cdot) \quad \text{for any } k\geq 0.$$

We shall investigate properties of the above Markov chain in the next two subsections. They will be used in Section 5.1 to prove Theorem 3.8.

Proof of Proposition 4.2. For any $v,v'\in H$, let (V_N,V_N') be a maximal coupling for the pair of measures $(\chi_N(v,\cdot),\chi_N(v',\cdot))$. By Theorem 1.23, we can assume that V_N and V_N' are defined on the same probability space $(\Omega_1,\mathcal{F}_1,\mathbb{P}_1)$ for all $v,v'\in H$ and are measurable functions of (v,v',ω_1). Let $(\Omega_2,\mathcal{F}_2,\mathbb{P}_2)$ be the probability space on which the random variables η_k are defined. We denote by $(\Omega,\mathcal{F},\mathbb{P})$ the direct product of these two probability spaces and, for any $\omega=(\omega_1,\omega_2)\in\Omega$, set

$$\begin{aligned} \zeta(v,v',\omega) &= V_N(v,v',\omega_1)-\mathsf{P}_NS(v)+\mathsf{Q}_N\eta_1(\omega_2),\\ \zeta'(v,v',\omega) &= V_N'(v,v',\omega_1)-\mathsf{P}_NS(v')+\mathsf{Q}_N\eta_1(\omega_2). \end{aligned} \tag{4.7}$$

Assertions (i), (ii), (iv) and the first part of (iii) are straightforward consequences of the construction. To prove inequality (4.4), we use Proposition 1.16 and formula (4.2). We have

$$\big\|\chi_N(v,\cdot)-\chi_N(v',\cdot)\big\|_{\mathrm{var}} = \frac{1}{2}\int_{H_N}\big|p(y-\mathsf{P}_NS(v))-p(y-\mathsf{P}_NS(v'))\big|\,dy. \tag{4.8}$$

Using relation (4.2) and the mean value theorem, we can show that

$$\int_{H_N} \bigl|p(y-z)-p(y-z')\bigr|\,dy \le C_N|z-z'|, \quad C_N=\sum_{j=1}^N b_j^{-1}\,\mathrm{Var}(p_j), \tag{4.9}$$

where $z, z' \in H_N$ are arbitrary points and $\mathrm{Var}(p_j)$ is the total variation of p_j. Substituting this inequality into (4.8), we arrive at (4.3). This completes the proof of Proposition 4.2. □

Exercise 4.4. Prove (4.9). *Hint:* See the proof of Lemma 3.2 in [KS01].

4.2 Squeezing

The following property of the coupling operators $\mathcal{R}$ and $\mathcal{R}'$ is the crucial point of the proof.

Lemma 4.5. *Suppose that Hypothesis* (H) *is satisfied. Then for any $B_0 > 0$ there is an integer $N \ge 1$ and a constant $K_N > 0$ such that, if* (3.7) *holds, then*

$$\mathbb{P}\bigl\{|\mathcal{R}(v,v',\cdot)-\mathcal{R}'(v,v',\cdot)| \le \tfrac{1}{2}|v-v'|\bigr\} \ge 1-K_N|v-v'|, \tag{4.10}$$

where $v, v' \in X$ are arbitrary functions.

Proof. Step 1. We first note that, for any $v, v' \in X$,

$$\bigl|\mathsf{Q}_N(S(v)-S(v'))\bigr| \le C_1\alpha_N^{-\frac{1}{4}}|v-v'|, \tag{4.11}$$

where $C_1 > 0$ is a constant depending only on $B > 0$. Indeed, combining (2.24) and (2.8), we see that

$$\bigl\|S(v)-S(v')\bigr\|_{1/2} \le C\,\exp\bigl\{C\left(|v|^2+|v'|^2\right)\bigr\}|v-v'| \quad \text{for any } v, v' \in H.$$

Using now the Poincaré inequality $|\mathsf{Q}_N w| \ge \alpha_N^{-\frac{1}{4}}\|\mathsf{Q}_N w\|_{1/2}$, for any $v, v' \in X$ we derive

$$\begin{aligned}\bigl|\mathsf{Q}_N(S(v)-S(v'))\bigr| &\le \alpha_N^{-\frac{1}{4}}\bigl\|S(v)-S(v')\bigr\|_{1/2}\\ &\le C\,\alpha_N^{-\frac{1}{4}}\exp\bigl\{C\left(|v|^2+|v'|^2\right)\bigr\}|v-v'|.\end{aligned}$$

This inequality coincides with (4.11).

Step 2. The definition of $\mathcal{R}$ and $\mathcal{R}'$ implies that (see (4.5) and (4.7))

$$\bigl|\mathcal{R}(v,v',\omega)-\mathcal{R}'(v,v',\omega)\bigr| \le \bigl|V_N(v,v',\omega)-V_N'(v,v',\omega)\bigr| + \bigl|\mathsf{Q}_N(S(v)-S(v'))\bigr|. \tag{4.12}$$

It follows from (4.11) that, if $N \ge 1$ is sufficiently large, then

$$\bigl|\mathsf{Q}_N(S(v)-S(v'))\bigr| \le \frac{1}{2}|v-v'|. \tag{4.13}$$

Furthermore, by (4.4), we have

$$\mathbb{P}\big\{V_N = V_N'\big\} \geq 1 - C_N|S(v) - S(v')|.$$

Combining this with (4.12) and (4.13) and using the uniform Lipschitz continuity of S on bounded subsets of H (see (2.23)), we derive

$$\mathbb{P}\big\{|\mathcal{R} - \mathcal{R}'| \leq \tfrac{1}{2}|v - v'|\big\} \geq \mathbb{P}\big\{V_N = V_N'\big\} \geq 1 - K_N|v - v'|,$$

where $K_N > 0$ depends on B. This completes the proof of the lemma. ☐

Corollary 4.6. *Under the conditions of Lemma 4.5, for any $v, v' \in H$ the sequence $U_k = (v_k, v_k')$ constructed in Section 4.1 satisfies the inequality*

$$\mathbb{P}\big\{|v_k - v_k'| \leq 2^{-k}|v - v'| \text{ for all } k \geq 0\big\} \geq 1 - 2K_N|v - v'|. \tag{4.14}$$

Proof. For any $k \geq 0$, let us set

$$G_k = \big\{|v_k - v_k'| \leq \tfrac{1}{2}|v_{k-1} - v_{k-1}'|\big\}, \quad \overline{G}_k = \bigcap_{l=1}^{k} G_l.$$

Since $\overline{G}_1 \supset \overline{G}_2 \supset \cdots$, inequality (4.14) will be established once we show that

$$\mathbb{P}(\overline{G}_k) \geq 1 - K_N|v - v'|\sum_{l=0}^{k-1} 2^{-l}. \tag{4.15}$$

The proof of (4.15) is by induction on k. For $k = 1$, inequality (4.15) coincides with (4.10). Assume now that $k = m \geq 2$ and that (4.15) is established for $k \leq m - 1$. We have

$$\mathbb{P}(\overline{G}_m) = \mathbb{E}\big(I_{\overline{G}_{m-1}}\mathbb{E}\left(I_{G_m} \,|\, \mathcal{F}_{m-1}\right)\big), \tag{4.16}$$

where $\mathcal{F}_k$ is the σ-algebra generated by (v_l, v_l'), $l = 1, \dots, k$. The Markov property for $U_k = (v_k, v_k')$ (see Exercises 4.3 and 2.8) and inequality (4.10) imply that

$$\begin{aligned}\mathbb{E}\left(I_{G_m} \,|\, \mathcal{F}_{m-1}\right) &= \mathbb{P}\big\{|\mathcal{R}(z, z', \cdot) - \mathcal{R}'(z, z', \cdot)| \leq \tfrac{1}{2}|z - z'|\big\}\Big|_{(z,z')=U_{m-1}} \\ &\geq 1 - K_N|v_{m-1} - v_{m-1}'|.\end{aligned} \tag{4.17}$$

Now note that, on the set $\overline{G}_{m-1}$, we have

$$|v_{m-1} - v_{m-1}'| \leq 2^{-(m-1)}|v - v'|.$$

Combining this with (4.16) and (4.17) and taking into account the induction hypothesis, we obtain

$$\mathbb{P}(\overline{G}_m) \geq \big(1 - 2^{1-m}K_N|v - v'|\big)\mathbb{P}(\overline{G}_{m-1}) \geq 1 - K_N|v - v'|\sum_{l=0}^{m-1} 2^{-l},$$

which completes the proof of the corollary. ☐

4.3 Dissipation

Recall that we the constant $R > 0$ is defined in Section 3.3 (see Step 2). The following lemma shows that, with positive probability, the dynamics pushes the sequence U_k towards the origin.

Lemma 4.7. *Suppose that Hypothesis* (H) *is fulfilled and that* (4.1) *holds for some integer $N \geq 1$. Then for any $r \in (0, R]$ there is a constant $\varepsilon(r) > 0$ such that*

$$\mathbb{P}\big\{|\mathcal{R}(v, v', \cdot)| \leq \big(\gamma\,|v|\big) \vee r,\, |\mathcal{R}'(v, v', \cdot)| \leq \big(\gamma\,|v'|\big) \vee r\big\} \geq \varepsilon(r), \tag{4.18}$$

where $\gamma = \frac{1+q}{2}$, and $v, v' \in X$ are arbitrary functions.

Let us note that, in view of the inequality in (3.6), for any $\delta > 0$ there is $\nu_\delta > 0$ such that

$$\mathbb{P}\big\{|\zeta| \leq \delta\big\} \geq \nu_\delta, \quad \mathbb{P}\big\{|\zeta'| \leq \delta\big\} \geq \nu_\delta, \tag{4.19}$$

where we used the fact that the distribution of ζ and ζ' coincides with that of η_k. Choosing $\delta > 0$ sufficiently small and combining (4.19) and (2.22), for any $r > 0$ we can find $\varepsilon(r) > 0$ such that

$$\mathbb{P}\big\{|\mathcal{R}(v, v', \cdot)| \leq \big(\gamma\,|v|\big) \vee r\big\} \geq \varepsilon, \quad \mathbb{P}\big\{|\mathcal{R}'(v, v', \cdot)| \leq \big(\gamma\,|v'|\big) \vee r\big\} \geq \varepsilon.$$

If $\mathcal{R}$ and $\mathcal{R}'$ were independent, these inequalities would imply (4.18). However, this is not the case, and we have to proceed differently.

Proof of Lemma 4.7. Step 1. It suffices to show that for any $\delta > 0$ there is $\varepsilon_\delta > 0$ such that

$$P_\delta := \mathbb{P}\big\{|\mathcal{R}(v, v', \cdot)| \leq |S(v)| + 2\delta,\, |\mathcal{R}'(v, v', \cdot)| \leq |S(v')| + 2\delta\big\} \geq \varepsilon_\delta. \tag{4.20}$$

Indeed, suppose that (4.20) is already proved and fix an arbitrary $r > 0$. Setting $\delta = \frac{r(1-q)}{2(1+q)}$ and using (2.22), we derive

$$|S(v)| + 2\delta \leq q|v| + \tfrac{r(1-q)}{1+q} \leq \big(\gamma|v|\big) \vee r,$$

and a similar inequality holds for v'. It follows that the probability on the left-hand side of (4.18) is bounded from below by P_δ. Since δ depends only on r, this proves inequality (4.18).

Step 2. Let us recall that V_N and V'_N denote the projections of $\mathcal{R}$ and $\mathcal{R}'$ to the space H_N and set $W_N = \mathsf{Q}_N\mathcal{R}$ and $W'_N = \mathsf{Q}_N\mathcal{R}'$. We fix an arbitrary $\delta > 0$, define the events

$$G_\delta = \big\{|V_N| \leq |\mathsf{P}_N S(v)| + \delta\big\}, \quad F_\delta = \big\{|W_N| \leq |\mathsf{Q}_N S(v)| + \delta\big\},$$

and denote by G'_δ and F'_δ similar events for $\mathcal{R}'$. It is a matter of direct verification to show that, if $\omega \in G_\delta \cap F_\delta$, then $|\mathcal{R}(v, v', \omega)| \leq |S(v)| + 2\delta$, and similarly for $\mathcal{R}'$. In view of the independence of $(\mathsf{P}_N\zeta, \mathsf{P}_N\zeta')$ and $(\mathsf{Q}_N\zeta, \mathsf{Q}_N\zeta')$, we obtain

$$P_\delta \geq \mathbb{P}(G_\delta F_\delta G'_\delta F'_\delta) = \mathbb{P}(G_\delta G'_\delta)\,\mathbb{P}(F_\delta F'_\delta).$$

Hence, it suffices to find a constant $\varkappa_\delta > 0$ such that

$$\mathbb{P}(G_\delta G'_\delta) \geq \varkappa_\delta, \quad \mathbb{P}(F_\delta F'_\delta) \geq \varkappa_\delta. \tag{4.21}$$

Step 3. We first prove the second inequality in (4.21). Since $\mathsf{Q}_N\zeta = \mathsf{Q}_N\zeta'$, it follows from the inequality in (3.6) that, for any $\delta > 0$, we can find $\varkappa_\delta > 0$ such that

$$\mathbb{P}\big\{|\mathsf{Q}_N\zeta| = |\mathsf{Q}_N\zeta'| \leq \delta\big\} \geq \varkappa_\delta.$$

This implies the required estimate.

Step 4. It follows from (4.19) that

$$\mathbb{P}(G_\delta) \geq \nu_\delta, \quad \mathbb{P}(G'_\delta) \geq \nu_\delta. \tag{4.22}$$

We claim that the first inequality in (4.21) holds with $\varkappa_\delta = \nu_\delta^2/4$. Indeed, let us set $E = \{V_N = V'_N\}$ and assume that $|\mathsf{P}_N S(v)| \leq |\mathsf{P}_N S(v')|$. (The proof in the other case is similar.) Then $G_\delta E \subset G'_\delta E$ and $G_\delta G'_\delta E = G_\delta E$. Since the random variables V_N and V'_N conditioned on E^c are independent (see Definition 1.21), we conclude that

$$\begin{aligned}\mathbb{P}(G_\delta G'_\delta) = \mathbb{P}(G_\delta G'_\delta E) + \mathbb{P}(G_\delta G'_\delta E^c) &= \mathbb{P}(G_\delta E) + \frac{\mathbb{P}(G_\delta E^c)\,\mathbb{P}(G'_\delta E^c)}{\mathbb{P}(E^c)} \\ &\geq \mathbb{P}(G_\delta E) + \mathbb{P}(G_\delta E^c)\,\mathbb{P}(G'_\delta E^c).\end{aligned} \tag{4.23}$$

If $\mathbb{P}(G_\delta E) \geq \varkappa_\delta$, then the required inequality is obvious. In the opposite case, it follows from (4.22) that

$$4\varkappa_\delta \leq \mathbb{P}(G_\delta)\,\mathbb{P}(G'_\delta) \leq \mathbb{P}(G_\delta E^c)\,\mathbb{P}(G'_\delta E^c) + 3\varkappa_\delta,$$

whence we see that $\mathbb{P}(G_\delta E^c)\,\mathbb{P}(G'_\delta E^c) \geq \varkappa_\delta$. Comparing this with (4.23), we obtain the first inequality in (4.21). ☐

Corollary 4.8. *Under the conditions of Lemma 4.7, for any $d > 0$ there is an integer $\ell = \ell(d) \geq 1$ and a constant $p = p(d) > 0$ such that, for any initial functions $v, v' \in X$, we have*

$$\mathbb{P}\big\{|v_\ell| \vee |v'_\ell| \leq d\big\} \geq p \quad \textit{for any } v, v' \in X. \tag{4.24}$$

Exercise 4.9. Prove Corollary 4.8. *Hint:* Use the Markov property.

5 Proof of the exponential mixing

5.1 Decay of a Kantorovich type functional

Let us recall that we have reduced the proof of Theorem 3.8 to inequality (3.16), where (v_k, v'_k) is a coupling for $(P_k(v,\cdot), P_k(v',\cdot))$ (see Section 3.3). By Exercise 4.3 (ii), the random variable $U_k = (v_k, v'_k)$ constructed in Section 4.1 is a coupling for the above pair of measures. Thus, Theorem 3.8 will be established once we show that (3.16) holds.

The proof of (3.16) is based on the exponential decay of a Kantorovich type functional. To define it, we fix two integers $\ell \geq 1$ and $s \geq 0$ and a small constant $d > 0$ and introduce the events [5]

$$Q_{n,r} = \bigl\{d_r < |v_{s+n\ell} - v'_{s+n\ell}| \leq d_{r-1}\bigr\}, \quad n, r \geq 0,$$

where $d_r = 2^{-r}d$ for $r \geq 0$ and $d_{-1} = +\infty$. Let us set

$$F_n = \sum_{r=0}^{\infty} 2^{-r}\mathbb{P}(Q_{n,r}). \tag{5.1}$$

Theorem 5.1. *Let $d = (16K_N)^{-1}$, where K_N is the constant in (4.14), and let $\ell = \ell(d)$ be the integer constructed in Corollary 4.8. Then there is $\delta \in (0,1)$ not depending on the integer s and the initial functions $v, v' \in X$ such that*

$$F_n \leq \delta^n \quad \textit{for all } n \geq 1. \tag{5.2}$$

Proof. Step 1. For any $n \geq 0$, the sets $Q_{n,r}$, $r \geq 0$, are mutually disjoint, and therefore $F_0 \leq 1$. Hence, it suffices to show that $F_n \leq \delta F_{n-1}$ for any $n \geq 1$.

Setting $p_{n,r} = \mathbb{P}(Q_{n,r})$, we write

$$\mathbb{P}(Q_{n,r}) = \sum_{m=0}^{\infty} p_{n-1,m}\mathbb{P}(Q_{n,r}\,|\,Q_{n-1,m}).$$

Substituting this relation into (5.1) and changing the order of summation, we obtain

$$F_n = \sum_{r=0}^{\infty} 2^{-r} \sum_{m=0}^{\infty} p_{n-1,m}\mathbb{P}(Q_{n,r}\,|\,Q_{n-1,m}) \leq \sum_{m=0}^{\infty} p_{n-1,m}\Sigma(m,n), \tag{5.3}$$

where

$$\Sigma(m,n) = \sum_{r=0}^{m} \mathbb{P}(Q_{n,r}\,|\,Q_{n-1,m}) + 2^{-(m+1)} \sum_{r=m+1}^{\infty} \mathbb{P}(Q_{n,r}\,|\,Q_{n-1,m}).$$

The required inequality will be established if we show that

$$\Sigma(m,n) \leq 2^{-m}\delta \quad \text{for any } m \geq 0. \tag{5.4}$$

Step 2. To prove the above assertion, we first note that Corollaries 4.6 and 4.8 and the choice of the parameters d and ℓ imply the following inequalities:

$$\sum_{r=0}^{m} \mathbb{P}(Q_{n,r}\,|\,Q_{n-1,m}) \leq 2^{-(m+2)}, \quad m \geq 1, \tag{5.5}$$

$$\mathbb{P}(Q_{n,0}\,|\,Q_{n-1,0}) \leq 1 - p, \tag{5.6}$$

[5] We do not indicate the dependence on s, since it does not play any role in the estimates.

where $p = p(d) > 0$ is the constant in (4.24). Taking these estimates for granted, let us complete the induction step.

It follows from (5.5) that

$$\Sigma(n,m) \le \frac{3}{4}\, 2^{-m} \quad \text{for } m \ge 1. \tag{5.7}$$

Furthermore, using (5.6), we derive

$$\begin{aligned}\Sigma(n,0) &\le \mathbb{P}(Q_{n,0}\,|\,Q_{n-1,0}) + \frac{1}{2}\,\mathbb{P}(Q_{n,0}^c\,|\,Q_{n-1,0})\\ &= \frac{1}{2}\big(\mathbb{P}(Q_{n,0}\,|\,Q_{n-1,0}) + 1\big) \le 1 - \frac{p}{2}.\end{aligned} \tag{5.8}$$

Combining (5.7) and (5.8), we obtain (5.4) with $\delta = \big(1 - \frac{p}{2}\big) \vee \frac{3}{4}$. Thus, it only remains to establish inequalities (5.5) and (5.6).

Step 3. We shall confine ourselves to (5.5), since the proof of the other inequality is similar. To simplify notation, we shall assume that $s = 0$.

It follows from (4.14) that, if $|v - v'| < d_{m-1}$, then

$$\mathbb{P}\big\{|v_\ell - v'_\ell| > d_m\big\} \le 2K_N d_{m-1} = 2^{-m-2}. \tag{5.9}$$

We now set $B_{n,m} = \{|v_{n\ell} - v'_{n\ell}| > d_m\}$ and note that

$$\sum_{r=0}^{m} \mathbb{P}(Q_{n,r}\,|\,Q_{n-1,m}) = \mathbb{P}(B_{n,m}\,|\,Q_{n-1,m}) = p_{n-1,m}^{-1}\mathbb{P}(B_{n,m}Q_{n-1,m}). \tag{5.10}$$

Using the Markov property, we write

$$\begin{aligned}\mathbb{P}(B_{n,m}Q_{n-1,m}) &= \mathbb{E}\left\{I_{Q_{n-1,m}}\mathbb{E}\left(I_{B_{n,m}}\,|\,\mathcal{F}_{n-1}\right)\right\}\\ &= \mathbb{E}\left(I_{Q_{n-1,m}}\mathbb{P}\big\{|v_\ell(Z) - v'_\ell(Z)| > d_m\big\}\big|_{Z=U_{(n-1)\ell}}\right),\end{aligned} \tag{5.11}$$

where $Z = (z, z') \in H \times H$, and $(v_k(Z), v'_k(Z))$ denotes the trajectory of (4.6) with $v = z$ and $v' = z'$. Since $|v_{(n-1)\ell} - v'_{(n-1)\ell}| < d_{m-1}$ on the set $Q_{n-1,m}$, we conclude from (5.9) that the right-hand side of (5.11) does not exceed $2^{-m-2}\mathbb{P}(Q_{n-1,m})$. Substituting this expression into (5.10), we obtain (5.5). ☐

Exercise 5.2. Prove inequality (5.6). *Hint:* Repeat the argument used in the proof of (5.5)

5.2 Proof of Theorem 3.8

The theorem will be established if we show that inequality (3.16) holds for the random sequence $U_k = (v_k, v'_k)$. Let us fix an arbitrary integer $k \ge 1$ and represent it in the form $k = n\ell + s$, where $0 \le s < \ell$. Then, by Theorem 5.1, we have

$$F_n = \sum_{r=0}^{\infty} 2^{-r}\mathbb{P}(Q_r^k) \le \delta^{-n}, \tag{5.12}$$

where $Q_r^k = \{d_r < |v_k - v'_k| \le d_{r-1}\}$. Let $\alpha < 1$ be so small that $\gamma := 2^\alpha \delta < 1$. Consider the event

$$B_k = \bigcup_{r=0}^{[\alpha n]} Q_r^k = \bigl\{|v_k - v'_k| > d_{[\alpha n]}\bigr\},$$

where $[q]$ stands for the integer part of q. It follows from (5.12) that

$$\mathbb{P}(B_k) = \sum_{r=0}^{[\alpha n]} \mathbb{P}(Q_r^k) \le 2^{[\alpha n]} \sum_{r=0}^{[\alpha n]} 2^{-r} \mathbb{P}(Q_r^k) \le 2^{[\alpha n]} F_n \le \bigl(2^\alpha \delta\bigr)^n = \gamma^n. \quad (5.13)$$

Since

$$d_{[\alpha n]} = 2^{-[\alpha n]} d \le 2^{-\alpha n + 1} d \le 4d\, 2^{-\alpha k/\ell},$$

we conclude from (5.13) that

$$\mathbb{P}\bigl\{\{|v_k - v'_k| > 4d\, 2^{-\alpha k/\ell}\bigr\} \le \gamma^{k/\ell - 1}.$$

This implies inequality (3.16) with

$$C = (4d) \vee \gamma^{-1}, \quad \beta = \ell^{-1}\bigl\{(\alpha \ln 2) \wedge (\ln \gamma^{-1})\bigr\}.$$

The proof of Theorem 3.8 is complete.

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