A VERSION OF THE LAW OF LARGE NUMBERS AND APPLICATIONS

ARMEN SHIRIKYAN

Laboratoire de Mathématiques
Université de Paris-Sud XI, Bâtiment 425
91405 Orsay Cedex, France
E-mail: Armen.Shirikyan@math.u-psud.fr

We establish a version of the strong law of large numbers (SLLN) for mixing-type Markov chains and apply it to a class of random dynamical systems with additive noise. The result obtained implies the SLLN for solutions of the 2D Navier–Stokes system and the complex Ginzburg–Landau equation perturbed by a non-degenerate random force.

1. Introduction

We study the 2D Navier–Stokes (NS) system perturbed by an external random force:

$$\dot{u} - \Delta u + (u, \nabla)u + \nabla p = \eta(t, x), \quad \operatorname{div} u = 0, \quad x \in D, \tag{1}$$

$$u = 0, \quad x \in \partial D. \tag{2}$$

Here $D \subset \mathbb{R}^2$ is a bounded domain with smooth boundary ∂D and η is a random process of the form

$$\eta(t, x) = \sum_{k=1}^{\infty} \eta_k(x)\delta(t - k), \tag{3}$$

where η_k are i.i.d. random variables in $L^2(D, \mathbb{R}^2)$ and $\delta(t)$ is the Dirac measure concentrated at $t = 0$. It was established in [8, 1, 10, 11, 6, 13, 14] that, if the distribution of η_k is sufficiently non-degenerate, then the family of Markov chains associated with the problem (1), (2) has a unique stationary measure μ and possesses an exponential mixing property. Namely, for a large class of functionals f and any solution $u(t)$ of (1) – (3), the average of $f(u(k))$ converges exponentially, as $k \to \infty$, to the mean value of f with respect to μ:

$$\bigl|\mathbb{E}\, f(u(k)) - (f, \mu)\bigr| \le \operatorname{const} e^{-\beta k}, \quad k \ge 1, \tag{4}$$

where $\beta > 0$ is a constant not depending on $u(t)$. Moreover, as was shown in [7], the strong law of large numbers (SLLN) for stationary processes combined with the coupling of solutions constructed in [10] implies an SLLN for solutions of the problem (1)–(3): for any solution $u(t)$, with probability 1 we have

$$\frac{1}{k}\sum_{l=0}^{k-1} f(u(l)) \to (f,\mu) \quad \text{as} \quad k \to \infty. \tag{5}$$

We note that similar properties were established for perturbations of the NS system by a random force smooth in x and white in t (see $[5, 3, 4, 2, 12, 7]$). We refer the reader to $[12, 7]$ for a detailed discussion of the results obtained in this direction.

The aim of this article is to derive the SLLN (5) from the mixing property (4) without using the coupling of solutions and to estimate the rate of convergence. To this end, we establish a simple version of SLLN for a class of Markov chains (Section 2) and show that it applies to the problem in question (Section 3). We note that the result of this paper remains valid for the 2D NS system perturbed by a random force white in time and smooth in the space variables.

Notation

Let H be a real Hilbert space with norm $\|\cdot\|$. We shall use the following notation:
$B_H(R)$ is the ball in H of radius $R > 0$ centred at zero;
$\mathcal{B}(H)$ is the Borel σ-algebra in H;
$\mathcal{P}(H)$ is the family of probability measures on $(H, \mathcal{B}(H))$;
$C(H)$ is the space of continuous functions $f\colon H \to \mathbb{R}$;
$C_b(H)$ is the space of bounded functions $f \in C(H)$ endowed with the norm $\|f\|_\infty := \sup_{u\in H} |f(u)|$.
$\mathcal{L}(H)$ is the space of Lipschitz-continuous functions $f \in C_b(H)$ with norm

$$\|f\|_{\mathcal{L}} := \|f\|_\infty + \sup_{u\neq v} \frac{|f(u) - f(v)|}{\|u - v\|}.$$

If $f\colon H \to \mathbb{R}$ is a $\mathcal{B}(H)$-measurable function and $\mu \in \mathcal{P}(H)$, then we denote by (f, μ) the integral of f over H with respect to μ.

2. Strong law of large numbers for mixing-type Markov chains

2.1. *Formulation of the result*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let H be a real Hilbert space with norm $\|\cdot\|$. We consider a family of Markov chains $(u_k, \mathbb{P}_u)$ in H with transition function $P_k(u,\Gamma) = \mathbb{P}_u\{u_k \in \Gamma\}$, $u \in H$, $\Gamma \in \mathcal{B}(H)$. Recall that the corresponding Markov semi-groups are defined by the formulas

$$\mathfrak{P}_k : C_b(H) \to C_b(H), \quad \mathfrak{P}_k f(u) = \int_H P_k(u, dv) f(v),$$
$$\mathfrak{P}_k^* : \mathcal{P}(H) \to \mathcal{P}(H), \quad \mathfrak{P}_k^* \mu(\Gamma) = \int_H P_k(u, \Gamma)\mu(du).$$

A measure $\mu \in \mathcal{P}(H)$ is said to be *stationary* for the family $(u_k, \mathbb{P}_u)$ if $\mathfrak{P}_1 \mu = \mu$.

Definition 2.1. We shall say that the family $(u_k, \mathbb{P}_u)$ is *uniformly mixing* if it has a unique stationary measure $\mu \in \mathcal{P}(H)$ and there is a continuous function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ and a sequence $\{\gamma_k\}$ of positive numbers such that $\gamma_k \to 0$ as $k \to \infty$ and, for any $f \in \mathcal{L}(H)$ and $u \in H$, we have

$$\big|\mathfrak{P}_k f(u) - (f,\mu)\big| \le \gamma_k \rho(\|u\|) \|f\|_{\mathcal{L}}, \quad k \ge 0. \tag{6}$$

The following theorem shows that “sufficiently fast” mixing combined with a dissipation property implies an SLLN.

Theorem 2.1. *Let $(u_k, \mathbb{P}_u)$ be a uniformly mixing family of Markov chains in H such that*

$$C := \sum_{k=0}^{\infty} \gamma_k < \infty. \tag{7}$$

Suppose there is a continuous function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\mathfrak{P}_k \rho(u) := \mathbb{E}_u \rho(\|u_k\|) \le h(\|u\|) \quad \textit{for all} \quad k \ge 0, \tag{8}$$

where $\mathbb{E}_u$ is the expectation with respect to $\mathbb{P}_u$. Then there exists a constant $D > 0$ such that for any $f \in \mathcal{L}(H)$, $u \in H$, and $\delta > 0$ the following statements hold:

- ***(i)*** *There is a $\mathbb{P}_u$-a.s. finite random integer $K(\omega) \ge 1$ depending on f, u, and δ such that*

$$\left|\frac{1}{k}\sum_{l=0}^{k-1} f\big(u_l\big) - (f,\mu)\right| \le D\,\|f\|_\infty k^{-\frac{1}{3}+\delta} \quad \textit{for} \quad k \ge K(\omega). \tag{9}$$

(ii) *For* $0 < r < 3\delta$*, we have*

$$\mathbb{E}_u K^r \le 1 + \tfrac{D}{3\delta - r}\,\|f\|_{\mathcal{L}} h(\|u\|). \tag{10}$$

We note that Theorem 2.1 remains valid (with trivial modifications) for Markov processes with continuous time. Moreover, under some additional assumptions, one can take in (9) functionals f with polynomial growth at infinity.

We also note that inequality (9) immediately implies the following estimate:

$$\left|\frac{1}{k}\sum_{l=0}^{k-1} f\big(u_l\big) - (f,\mu)\right| \le M(\omega)\,\|f\|_\infty k^{-\frac{1}{3}+\delta}, \quad k \ge 1,$$

where $M(\omega) = D + 2K(\omega)^{\frac{1}{3}-\delta}$.

2.2. *Proof of Theorem 2.1*

Let us fix an arbitrary function $f \in \mathcal{L}(H)$ and set

$$S_k = \sum_{l=0}^{k-1} f(u_l), \quad s_k = S_k/k.$$

There is no loss of generality in assuming that $\|f\|_\infty \le 1$ and $(f,\mu) = 0$.

Step 1. We first show that

$$\mathbb{E}_u s_k^2 \le C_1 \|f\|_{\mathcal{L}} h(\|u\|) k^{-1}, \quad k \ge 1. \tag{11}$$

Here and henceforth, we denote by C_i positive constants that do not depend on f, u, k and δ.

Let us note that

$$\mathbb{E}_u s_k^2 = \frac{1}{k^2}\sum_{l,m=0}^{k-1} \mathbb{E}_u\big(f(u_l)f(u_m)\big) \le \frac{2}{k^2}\sum_{l\le m}\big|\mathbb{E}_u\big(f(u_l)f(u_m)\big)\big|. \tag{12}$$

By the Markov property,

$$\mathbb{E}_u\big(f(u_l)f(u_m)\big) = \mathbb{E}_u\big(f(u_l)\mathbb{E}_u(f(u_m)\,\big|\,\mathcal{F}_l)\big) = \mathbb{E}_u\big(f(u_l)(\mathfrak{P}_{m-l}f)(u_l)\big),$$

where $\mathcal{F}_l$ is the σ-algebra generated by $\{u_r, 0 \le r \le l\}$. Furthermore, since $(f,\mu) = 0$, inequality (6) implies that

$$\big|\mathfrak{P}_{m-l}f(u_l)\big| \le \gamma_{m-l}\rho(\|u_l\|)\|f\|_{\mathcal{L}}.$$

Hence, using (8) and the inequality $|f| \le 1$, we derive

$$\big|\mathbb{E}_u\big(f(u_l)f(u_m)\big)\big| \le \gamma_{m-l}\|f\|_{\mathcal{L}}\mathbb{E}_u\big(|f(u_l)|\rho(\|u_l\|)\big) \le \gamma_{m-l}h(\|u\|)\|f\|_{\mathcal{L}}.$$

Substitution of this inequality into (12) results in (11) with $C_1 = 2C$, where C is the constant in (7).

Step 2. We now prove (9). To this end, we fix $\delta \in (0, \frac{1}{3})$ and set

$$k_n = [n^{3+\beta}], \quad \beta = \frac{9\delta}{1-3\delta},$$

where $[a]$ is the integer part of $a \geq 0$. Let us consider the events

$$G_n = \{\omega \in \Omega : |s_{k_n}| > n^{-1}\}, \quad n \geq 1.$$

Using (11) and the Chebyshev inequality, we derive

$$\mathbb{P}(G_n) \leq n^2 \mathbb{E}_u |s_{k_n}|^2 \leq C_2 \|f\|_{\mathcal{L}} h(\|u\|)\, n^{-1-\beta}. \tag{13}$$

Hence, by the Borel–Cantelli lemma, there is a $\mathbb{P}_u$-a.s. finite random integer $m(\omega) \geq 1$ such that

$$|s_{k_n}(\omega)| \leq n^{-1} \quad \text{for} \quad n \geq m(\omega). \tag{14}$$

We shall assume that $m(\omega) \geq 1$ is the smallest integer satisfying (14). In particular, if $m(\omega) \geq 2$, then

$$|s_{k_n}(\omega)| > n^{-1} \quad \text{for} \quad n = m(\omega) - 1. \tag{15}$$

To estimate $|s_k|$ for $k_{n-1} < k < k_n$, we note that

$$|s_k - s_{k_n}| \leq \left(\tfrac{1}{k} - \tfrac{1}{k_n}\right) |S_{k_n}| + \tfrac{1}{k} |S_k - S_{k_n}| \leq 2\, \tfrac{k_n - k_{n-1}}{k_{n-1}}. \tag{16}$$

Since $\frac{k_n - k_{n-1}}{k_{n-1}} \leq C_3 n^{-1}$ and $n^{-1} \leq k_n^{-\frac{1}{3+\beta}} = k_n^{-\frac{1}{3}+\delta}$, it follows from (14) and (16) that

$$\begin{aligned} |s_k| &\leq |s_k - s_{k_n}| + |s_{k_n}| \leq 2\, \tfrac{k_n - k_{n-1}}{k_{n-1}} + n^{-1} \leq (2C_3 + 1) n^{-1} \\ &\leq (2C_3 + 1) k_n^{-\frac{1}{3}+\delta} \leq (2C_3 + 1) k^{-\frac{1}{3}+\delta}, \end{aligned}$$

where $n \geq m(\omega)$ and $k_{n-1} < k < k_n$. Thus, inequality (9) holds with $K(\omega) = [m(\omega)^{3+\beta}]$.

Step 3. It remains to establish (10). To this end, we first note that, for $0 < q < \beta$,

$$\begin{aligned} \mathbb{E}_u m^q &= \sum_{l=1}^{\infty} \mathbb{P}_u\{m = l\} l^q \leq 1 + \sum_{l=2}^{\infty} \mathbb{P}_u(G_{l-1}) l^q \\ &\leq 1 + C_2\, h(\|u\|)\, \|f\|_{\mathcal{L}} \sum_{l=2}^{\infty} (l-1)^{-1-\beta} l^q \\ &\leq 1 + \tfrac{C_4}{\beta - q}\, h(\|u\|)\, \|f\|_{\mathcal{L}}, \end{aligned}$$

where we used inequalities (13), (15) and the definition of $m(\omega)$ and G_n. Since $K = [m^{3+\beta}]$, we see that, for $0 < r < 3\delta$,

$$\begin{aligned}\mathbb{E}_u K^r &\le \mathbb{E}_u m^{r(3+\beta)} \le 1 + \tfrac{C_4}{\beta - r(3+\beta)}\, h(\|u\|)\, \|f\|_{\mathcal{L}} \\ &\le 1 + \tfrac{C_4}{3(3\delta - r)}\, h(\|u\|)\, \|f\|_{\mathcal{L}}.\end{aligned}$$

The proof of Theorem 2.1 is complete.

3. Applications

3.1. *Dissipative PDE's perturbed by a bounded kick force*

Let H be a real Hilbert space with norm $\|\cdot\|$ and orthonormal base $\{e_j\}$. We consider the random dynamical system (RDS)

$$u_k = S(u_{k-1}) + \eta_k, \tag{17}$$

where $S\colon H \to H$ is a continuous operator such that $S(0) = 0$ and $\{\eta_k\}$ is a sequence of i.i.d. random variables. As was explained in [8, 9, 10], RDS of the form (17) naturally arise in the study of dissipative PDE's perturbed by the random force (3), and in this case S is the time-one shift along trajectories of the unperturbed equation. We assume that S satisfies the following three conditions introduced in [8, 10]:

(A) For any $R > r > 0$ there are positive constants $a = a(R, r) < 1$ and $C = C(R)$ and an integer $n_0 = n_0(R, r) \ge 1$ such that

$$\begin{aligned}\|S(u_1) - S(u_2)\| &\le C(R)\|u_1 - u_2\| \quad \text{for all } u_1, u_2 \in B_H(R), \\ \|S^n(u)\| &\le \max\{a\|u\|, r\} \quad \text{for } u \in B_H(R),\ n \ge n_0.\end{aligned}$$

(B) For any compact set $\mathcal{K} \subset H$ and any bounded set $B \subset H$ there is $R > 0$ such that the sets $\mathcal{A}_k(\mathcal{K}, B)$ defined recursively by the formulas $\mathcal{A}_0(\mathcal{K}, B) = B$ and $\mathcal{A}_k(\mathcal{K}, B) = S(\mathcal{A}_{k-1}(\mathcal{K}, B)) + \mathcal{K}$ are contained in the ball $B_H(R)$ for all $k \ge 0$.

(C) For any $R > 0$ there is an integer $N \ge 1$ such that

$$\big\|\mathsf{Q}_N(S(u_1) - S(u_2))\big\| \le \tfrac{1}{2}\|u_1 - u_2\| \quad \text{for all} \quad u_1, u_2 \in B_H(R),$$

where Q_N is the orthogonal projection onto the subspace spanned by $\{e_j, j \ge N+1\}$.

We note that the above conditions are satisfied for the resolving operators of the 2D Navier–Stokes system and the complex Ginzburg–Landau equation.

As for the i.i.d. random variables η_k, we assume that they have the form

$$\eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j, \tag{18}$$

where $b_j \geq 0$ are some constants such that

$$\sum_{j=1}^{\infty} b_j^2 < \infty, \tag{19}$$

and ξ_{jk} are independent scalar random variables whose distributions π_j satisfy the following condition:

(D) For any $j \geq 1$ there is a function of bounded variation $p_j(r)$ such that $\pi_j(dr) = p_j(r)\,dr$, where dr is the Lebesgue measure on $\mathbb{R}$. Moreover, $\operatorname{supp}\pi_j \subset [-1,1]$ and $\int_{|r|\leq\varepsilon} p_j(r)\,dr > 0$ for all $j \geq 1$ and $\varepsilon > 0$.

Let $(u_k, \mathbb{P}_u)$ be the family of Markov chains that is associated with the RDS (17) and is parametrized by the initial condition $u \in H$. We denote by $P_k(u,\Gamma)$ the corresponding transition function and by $\mathfrak{P}_k$ and $\mathfrak{P}_k^*$ the Markov operators generated by P_k. It was proved in [10, 11, 6] that, if conditions (A)–(D) are fulfilled and

$$b_j \neq 0 \quad \text{for} \quad j = 1, \dots, N, \tag{20}$$

where $N \geq 1$ is sufficiently large, then the RDS (17) has a unique stationary measure μ, and for any $f \in \mathcal{L}(H)$ we have

$$\big|\mathfrak{P}_k f(u) - (f,\mu)\big| \leq \rho(\|u\|)\|f\|_{\mathcal{L}} e^{-\beta k}, \quad k \geq 1, \tag{21}$$

where $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous increasing function and $\beta > 0$ is a constant not depending on f and u. Thus, the family $(u_k, \mathbb{P}_u)$ is uniformly mixing, and condition (7) is satisfied. We claim that (8) also holds. Indeed, let us define the compact set

$$\mathcal{K} = \Big\{ v = \sum_{j=1}^{\infty} v_j e_j : |v_j| \leq b_j \text{ for all } j \geq 1 \Big\},$$

where $b_j \geq 0$ are the constants in (18). It follows from condition (D) that the support of the distribution of η_k is contained in $\mathcal{K}$. Therefore, by assumption (B), there is a continuous increasing function $R = R(d)$, $d \geq 0$, such that

$$\mathbb{P}_u\big\{\|u_k\| \leq R(d)\big\} = 1 \quad \text{for} \quad \|u\| \leq d, \quad k \geq 0.$$

Hence, since ρ is increasing, for $\|u\| \le d$ we obtain

$$\mathfrak{P}_k\rho(u) = \mathbb{E}_u\rho(\|u_k\|) \le \rho\bigl(R(d)\bigr),$$

which means that (8) holds with $h(d) = \rho\bigl(R(d)\bigr)$.

Thus, Theorem 2.1 applies, and therefore inequalities (9) and (10) hold for the RDS (17).

3.2. *The Navier–Stokes system perturbed by an unbounded kick force*

We now consider the problem (1)–(3). It is assumed that η_k are i.i.d. random variables of the form (18), where $b_j \ge 0$ are some constants for which (19) holds, and ξ_{jk} are independent scalar random variables satisfying the following condition (cf. (D)):

(D′) For any $j \ge 1$ the distribution of ξ_{jk} possesses a density $p_j(r)$ (with respect to the Lebesgue measure) that is a function of bounded variation such that

$$\int_{\mathbb{R}} e^{r^2} p_j(r)\,dr \le Q, \qquad p_j(r) > 0 \quad \text{for all} \quad r \in \mathbb{R},$$

where $Q > 0$ is a constant not depending on j.

The problem (1)–(3) reduces to an RDS of the form (17). Namely, let us introduce the Hilbert space (endowed with the L^2-norm)

$$H = \Bigl\{u \in L^2(D, \mathbb{R}^2) : \operatorname{div} u = 0,\ (u, \nu)\big|_{\partial D} = 0\Bigr\},$$

where ν is the unit normal to ∂D (see [15] for further details on the space H). Let $S : H \to H$ be the time-one shift along trajectories of the NS system (1), (2) with $\eta \equiv 0$. Setting $u_k = u(k, x)$, we obtain (17) (see [8, 10] for details).

Let $(u_k, \mathbb{P}_u)$ be the family of Markov chains associated with the RDS (17). As is shown in [9, 14], if the non-degeneracy condition (20) is satisfied for $N \gg 1$, then the family $(u_k, \mathbb{P}_u)$ has a unique stationary measure μ, and (21) holds with $\rho(d) = C_1(1 + d)$, where C_1 and β are positive constants not depending on f, u, and k. Moreover, by Theorem 1.3 in [9], we have

$$\mathbb{E}_u\rho(\|u_k\|) \le C_2\bigl(1 + \|u\|\bigr) \quad \text{for all} \quad k \ge 0.$$

Thus, the conditions of Theorem 2.1 are fulfilled, and we obtain the SLLN for solutions of the NS system (1)–(3).

References

1. J. Bricmont, A. Kupiainen, and R. Lefevere, Ergodicity of the 2D Navier-Stokes equations with random forcing, *Comm. Math. Phys.* **224** (2001), 65–81.
2. J. Bricmont, A. Kupiainen, and R. Lefevere, Exponential mixing for the 2D stochastic Navier–Stokes dynamics, *Comm. Math. Phys.* **230** (2002), no. 1, 87–132.
3. W. E, J. C. Mattingly, and Ya. G. Sinai, Gibbsian dynamics and ergodicity for the stochastically forced Navier–Stokes equation, *Comm. Math. Phys.* **224** (2001), 83–106.
4. J.-P. Eckmann and M. Hairer, Uniqueness of the invariant measure for a stochastic PDE driven by degenerate noise, *Comm. Math. Phys.* **219** (2001), 523–565.
5. F. Flandoli and B. Maslowski, Ergodicity of the 2-D Navier-Stokes equation under random perturbations, *Comm. Math. Phys.* **172** (1995), 119–141.
6. S. Kuksin, On exponential convergence to a stationary measure for nonlinear PDE's, perturbed by random kick-forces, and the turbulence-limit, The M. I. Vishik Moscow PDE seminar, AMS Translations, 2002.
7. S. Kuksin, Ergodic theorems for 2D statistical hydrodynamics, *Rev. Math. Physics* **14** (2002), no. 6, 585–600.
8. S. Kuksin and A. Shirikyan, Stochastic dissipative PDE's and Gibbs measures, Comm. Math. Phys. **213** (2000), 291–330.
9. S. Kuksin and A. Shirikyan, Ergodicity for the randomly forced 2D Navier–Stokes equations, *Math. Phys. Anal. Geom.* **4** (2001), no. 2, 147–195.
10. S. Kuksin and A. Shirikyan, A coupling approach to randomly forced nonlinear PDE's. I, *Comm. Math. Phys.* **221** (2001), no. 2, 351–366.
11. S. Kuksin, A. Piatnitski and A. Shirikyan, A coupling approach to randomly forced nonlinear PDE's. II, *Comm. Math. Phys.* **230** (2002), no. 1, 81–85.
12. S. Kuksin and A. Shirikyan, Coupling approach to white-forced nonlinear PDE's, *J. Math. Pures Appl.* **81** (2002), 567–602.
13. N. Masmoudi and L.-S. Young, Ergodic theory of infinite dimensional systems with applications to dissipative parabolic PDEs, *Comm. Math. Phys.* **227** (2002), 461–481.
14. A. Shirikyan, Exponential mixing for 2D Navier–Stokes equations perturbed by an unbounded noise, *J. Math. Fluid Mech.* **6** (2004), no. 2, 169–193.
15. R. Temam, *Navier–Stokes Equations. Theory and Numerical Analysis*, North-Holland, Amsterdam–New York–Oxford, 1977.