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ANALYTICITY OF SOLUTIONS AND KOLMOGOROV'S DISSIPATION SCALE FOR 2D NAVIER–STOKES EQUATIONS

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1. Introduction. Let us consider the two-dimensional Navier–Stokes (NS) system on a torus:

$$\dot{u} - \nu\Delta u + (u,\nabla)u - \nabla p = \eta(t,x), \quad \operatorname{div} u = 0, \quad x = (x_1, x_2) \in \mathbf{T}^2. \tag{1}$$

Here $\mathbf{T}^2 = \mathbf{R}^2/2\pi\mathbf{Z}^2$, $\nu > 0$ is the viscosity, $u = u(t,x)$ is the velocity field, p is the pressure, and η is an external force. Equation (1) is supplemented with the initial condition

$$u(0,x) = u_0(x). \tag{2}$$

As is known [L], the problem (1), (2) is well-posed. Namely, for any right-hand side η and initial function u_0 that belong to appropriate functional classes there is a unique solution $u(t,x)$ for (1), (2). The aim of this article is to study analyticity of solutions regarded as functions of x and to find an asymptotic lower bound for the radius analyticity as $\nu \to 0$.

This problem is closely related to the Kolmogorov–Obukhov hypothesis on the behaviour of the energy spectrum of solutions in the turbulent regime. More precisely, let us expand a solution of (1) into the Fourier series,

$$u(t,x) = \sum_{j\in\mathbf{Z}^2} u_j(t)e^{ijx},$$

and define the energy corresponding to a wave number k by the formula

$$E_k = \sum_{k-\frac12\le|j|\le k+\frac12} |u_j|^2.$$

Roughly speaking, in the 2D case, the hypothesis is that there is a threshold $\lambda_{2\mathrm{D}} = \lambda_{2\mathrm{D}}(\nu)$ called Kolmogorov's dissipation scale such that

$$E_k \sim k^{-3} \quad \text{for} \quad k \le \lambda_{2\mathrm{D}}^{-1}, \qquad E_k \lesssim k^{-N} \quad \text{for} \quad k \ge \lambda_{2\mathrm{D}}^{-1},$$

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where $N > 0$ is an arbitrary constant. Furthermore, the Kolmogorov dissipation scale is of order $\nu^{1/2}$.

Let us consider a solution for (1) that admits analytic continuation to the domain $|\operatorname{Im} x_i| \le r\nu^\gamma$, $i = 1, 2$, where $r > 0$ is a constant. In this case, we have

$$|u_j| \le \operatorname{const} e^{-r\nu^\gamma |j|},$$

whence it follows that $E_k \lesssim e^{-r\nu^\gamma k}$. Therefore,

$$E_k \lesssim k^{-N} \quad \text{for} \quad \tfrac{k}{\ln k} \gtrsim \nu^{-\gamma}.$$

Hence, ignoring the logarithm, one can say that, if the Kolmogorov–Obukhov hypothesis is true, then $\lambda_{2D} \gtrsim \nu^\gamma$. Thus, *an asymptotic estimate for the radius of analyticity implies a lower bound for the Kolmogorov dissipation scale.* See [Fr, Ga, HKR, Ku] and references therein for a more detailed discussion.

The problem of analyticity of solutions for deterministic Navier–Stokes equations were studied in many papers (e.g., see [FT, HKR]). The aim of this article is to present some results in the case when the right-hand side η is a random process analytic in the space variables and white in time. This case was investigated earlier in [M, BKL]. In particular, as is shown in [BKL], for any initial function u_0 the solution of the problem (1), (2) is analytic in x for $t > 0$, and its radius of analyticity ρ_ν can be estimated asymptotically from below by $\nu^{3+\delta}$ for any $\delta > 0$. Our estimates for solutions of (1), (2) imply that $\rho_\nu \gtrsim \nu^{2+\delta}$ for any $\delta > 0$. We note that this assertion is true for any stationary solution of Eq. (1).

2. Preliminaries. In this section, we introduce necessary functional spaces, recall the definition of a solution for Eq. (1) and the notion of a stationary solution, and formulate some known results.

Let $H^s = H^s(\mathbf{T}^2, \mathbf{R}^2)$ be the space of vector functions $u = (u_1, u_2)$ on $\mathbf{T}^2$ whose components belong to the Sobolev space of order s. For $s = 0$, we obtain the usual space $L^2 = L^2(\mathbf{T}^2, \mathbf{R}^2)$ with natural norm $|\cdot|$. Let H be the subspace of $u \in L^2$ such that $\operatorname{div} u = 0$ and $\int_{\mathbf{T}^2} u(x)\,dx = 0$ and let $\Pi\colon L^2 \to H$ be the orthogonal projection onto H.

Applying Π to Eq. (1), we write it in the form (see [VF])

$$\dot u + \nu L u + B(u, u) = \eta(t), \tag{3}$$

where L is the restriction of the operator $-\Delta$ to H, $B(u, u) = \Pi(u, \nabla)u$ is the nonlinear term, and $\eta(t)$ is the projection of the external force. (To simplify the notation, we denote the external force and its projection to H by the same symbol.)

To describe the class of right-hand sides for (3), we introduce a trigonometric basis in H. Namely, let $\mathbf{Z}'$ be a subset of $\mathbf{Z}_0^2 = \mathbf{Z}^2 \setminus \{0\}$ such that $\mathbf{Z}' \cup (-\mathbf{Z}') = \mathbf{Z}_0^2$ and let

$$e_j(x) = \frac{\sin(jx)\, j^\perp}{\sqrt{2}\pi |j|}, \quad e_{-j}(x) = \frac{\cos(jx)\, j^\perp}{\sqrt{2}\pi |j|}, \quad j = (j_1, j_2) \in \mathbf{Z}',$$

where $j^\perp = (-j_2, j_1)$ and $|j| = (j_1^2 + j_2^2)^{1/2}$. It is clear that $Le_j = |j|^2 e_j$, $j \in \mathbf{Z}_0^2$, and that $\{e_j, j \in \mathbf{Z}_0^2\}$ is a basis in H.

We assume that η has the form

$$\eta(t, x) = \frac{\partial}{\partial t}\zeta(t, x), \quad \zeta(t, x) = \sum_{j \in \mathbf{Z}_0^2} b_j \beta_j(t) e_j(x), \tag{4}$$

where β_j are independent standard Brownian motions defined on a complete probability $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathcal{F}_t$ and b_j are real constants satisfying the condition

$$\sum_{j\in\mathbf{Z}_0^2} |j|^2 b_j^2 < \infty. \tag{5}$$

This assumption implies, in particular, that almost all sample paths $\zeta(t,\cdot)$ belong to the space $C(\mathbf{R}_+, V)$ of continuous functions on the half-line $\mathbf{R}_+ = [0, +\infty)$ with range in $V := H \cap H^1(\mathbf{T}^2, \mathbf{R}^2)$. Equation (3) is regarded as an Itô's stochastic PDE.

Let $L^2_{\rm loc}(\mathbf{R}_+, V)$ be the space of Bochner-measurable functions $u(t) : \mathbf{R}_+ \to V$ such that $\int_0^T \|f(t)\|^2 dt < \infty$ for any $T > 0$, where $\|u\| = |L^{1/2}u|$ is the norm in V.

Definition 1. A random process $u(t) = u(t,x)$ in H defined on the half-line $t \geq 0$ and progressively measurable with respect to $\mathcal{F}_t$ is called a *strong solution* of Eq. (3) if the following two conditions hold with probability 1.

(i) The function $u(t,x)$ belongs to $L^2_{\rm loc}(\mathbf{R}_+, V) \cap C(\mathbf{R}_+, H)$.

(ii) For any $t > 0$, we have

$$u(t) + \int_0^t \big(\nu L u + B(u,u)\big)\, ds = u(0) + \zeta(t),$$

where the left- and right-hand sides of this relation are regarded as elements of the space $H^{-1}(\mathbf{T}^2, \mathbf{R}^2)$.

A proof of the following result can be found in [VF, Chapter X] (also see [DaZ, Chapter 15]).

Proposition 2. *Suppose that condition (5) holds. Then for any $\mathcal{F}_0$-measurable random variable u_0 with range in H Eq. (3) has a unique solution on $[0,\infty)$ that satisfies the initial condition (2).*

Let $u(t)$ be a solution for Eq. (3) and let $\mu(t)$ be its distribution at time t. Thus, $\mu(t)$ is a probability Borel measure in the functional space H.

Definition 3. The solution $u(t)$ is said to be *stationary* if $\mu(t)$ does not depend on t. In this case, $\mu(t) \equiv \mu$ is called a *stationary measure* for Eq. (3).

A proof of the following theorem can be found in [DaZ, Chapter 15].

Proposition 4. *Suppose that condition (5) holds. Then Eq. (3) has a stationary measure.*

3. Main results. We begin with an estimate for a second exponential moment for stationary solutions.

Theorem 5. *Suppose that condition (5) holds. There are positive constants σ and C not depending on ν such that, if μ is a stationary measure for Eq. (3) with some $\nu \in (0,1]$, then*

$$\int_H \exp(\sigma\nu\|u\|^2)\, \mu(du) \leq C.$$

We now assume that the coefficients b_j entering the right-hand side of (3) (see (4)) satisfy the following inequality for some $\rho > 0$:

$$\sum_{j\in\mathbf{Z}_0^2} e^{2\rho|j|} b_j^2 < \infty. \tag{6}$$

Recall that, for a function $u(t,x)$ with range in H, we denote by $u_j(t)$ its Fourier coefficients. The following theorem establishes the analyticity of solutions for the Cauchy problem (3), (2) and gives an asymptotic lower bound for the radius of analyticity.

THEOREM 6. *Suppose that condition (6) holds. Let $u_0 = u_0^\nu(x)$ be a family of random initial functions that satisfy the inequality*

$$\mathbf{E}\,\exp\bigl(\sigma\nu\|u_0^\nu\|^2\bigr) \le R \quad \textit{for} \quad 0 < \nu \le 1,$$

where the positive constants σ and R do not depend on ν. Then for any $t_0 \ge 1$, $T > 0$ and $\delta \in (0,1]$ there are positive random variables $r_\nu = r_\nu(t_0,T,\delta)$ and $C_\nu = C_\nu(t_0,T,\delta)$ such that, with probability 1,

$$|u_j(t)| \le C_\nu e^{-r_\nu \nu^{2+\delta}|j|}, \quad t_0 \le t \le t_0+T, \quad j \in \mathbf{Z}_0^2.$$

Moreover, for any integer $m \ge 1$ there is a constant $K_m = K_m(\sigma,T,\delta,R) > 0$ not depending on t_0 and ν such that

$$\mathbf{E}\,r_\nu^{-m} \le K_m, \quad \mathbf{E}\,C_\nu^m \le K_m\nu^{-m/2} \quad \textit{for} \quad 0 < \nu \le 1.$$

In particular, any solution of Eq. (3) with deterministic initial function $u_0 \in V$ is analytic in x with probability 1, and its radius of analyticity can be estimated from below by $r_\nu\nu^{2+\delta}$. Theorem 5 implies that the above assertions are valid for any stationary solution of Eq. (3).

Proofs of Theorems 5 and 6 are given in [S].

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References

[BKL] J. BRICMONT, A. KUPIAINEN, and R. LEFEVERE, *Probabilistic estimates for the two-dimensional stochastic Navier–Stokes equations*, J. Statist. Phys. 100 (2000), no. 3–4, 743–756.

[DaZ] G. DA PRATO and J. ZABCZYK, *Ergodicity for Infinite-Dimensional Systems*, Cambridge University Press, Cambridge, 1996.

[FT] C. FOIAS and R. TEMAM, *Gevrey class regularity for the solutions of the Navier-Stokes equations*, J. Funct. Anal. 87 (1989), no. 2, 359–369.

[Fr] U. FRISCH, *Turbulence. The Legacy of A. N. Kolmogorov*, Cambridge University Press, Cambridge, 1995.

[Ga] G. GALLAVOTTI, *Foundations of Fluid Dynamics*, Springer, Berlin, 2001.

[HKR] W. D. HENSHAW, H.-O. KREISS, and L. G. REYNA, *Smallest scale estimates for the Navier-Stokes equations for incompressible fluids*, Arch. Rational Mech. Anal. 112 (1990), no. 1, 21–44.

[Ku] S. B. Kuksin, *Spectral properties of solutions for nonlinear PDEs in the turbulent regime*, Geom. Funct. Anal. 9 (1999), no. 1, 141–184.

[L] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Gauthier-Villars, Paris, 1969.

[M] J. C. Mattingly, *The Stochastic Navier-Stokes Equation: Energy Estimates and Phase Space Contraction*, Ph.D. thesis, Princeton University, 1998.

[S] A. Shirikyan, *Analyticity of solutions for randomly forced two-dimensional Navier–Stokes equations*, Russian Math. Surveys 57 (2002), no. 4, 785–799.

[VF] M. I. Vishik and A. V. Fursikov, *Mathematical Problems of Statistical Hydromechanics*, Kluwer Academic Publishers, Dordrecht, 1988.